Arrow-Barankin-Blackwell Theorems and Related Results in Cone Duality: A Survey

Aris Daniilidis

Laboratoire des Mathématiques Appliquées, Université de Pau et des Pays de l' Adour, Avenue de l' Université, 64000 PAU, France.
E-mail: Aris.Daniilidis@univ-pau.fr

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Abstract. We attempt a brief survey on the cone duality and on the density theorems of Arrow-Barankin & Blackwell's type. Concerning the latter aspect we show the equivalence of two recent and ostensibly different results. We follow a unified approach which provides in particular a simple way of surveying these results and their proofs.

Key words: Vector optimization, efficient points, positive functionals, denting points.

1 Introduction

In 1953, Arrow, Barankin and Blackwell stated an interesting density result in multicriteria optimization (see [1]) concerning the approximation of the Pareto efficient points of a compact convex subset of R^n by points that are maximizers of some strictly positive functional on this set. This theorem was extended to cover more general notions of efficiency that are defined via an abstract cone, see [2], [19] and was subsequently generalized to an infinite dimensional setting, involving either weakly or norm compact sets.

In this article we endeavour a survey on these density results of Arrow, Barankin and Blackwell's type. Our aim is to survey the state of the art and to set in detail the relations among ostensibly different results. To this end, we shall adopt a unified approach available nowadays and, in doing so, we shall slightly improve some norm approximation results concerning weakly compact subsets of a Banach space. Finally we shall show the equivalence of a recent result of Gong [16] with a well-known earlier one of Petschke [32].

2 Notation

Throughout this paper, X will always be a Banach space and X^* its (topological) dual. However for most of what follows this is not essential and one can also consider a more general setting (for instance that of a locally convex space). In the sequel, we shall focus our interest in the norm and the weak topology of X, which will be denoted respectively by $\|\cdot\|$-topology and w-topology.

If $\varepsilon > 0$ and $x \in X$, we denote by $B_\varepsilon(x)$ the closed ball centered at x with radious ε. For any $x, y \in X$, we define by $[x, y]$ the closed segment $\{tx + (1-t)y : 0 \leq t \leq 1\}$, while the segments (x, y), $(x, y]$ and $[x, y)$ are defined analogously. For any subset $A \subseteq X$, we denote by $int(A)$ the norm interior of the set A, by $cl(A)$ (resp. $w - cl(A)$) the norm (resp. the weak) closure of A and by $co(A)$ its convex hull. It is well known that for convex subsets of X the norm and the weak topological closures coincide (see [7] e.g.).

Let now K be a nonempty subset of X. A point $x_0 \in K$ is said to belong to the *algebraic interior* $\ell gint(K)$ of the set K, if for every $y \in X$, the intersection of the set K with the line joining x_0 and y, contains an open interval around the point x_0. It is easily seen that $int(K) \subseteq \ell gint(K)$. Moreover if $x_0 \in \ell gint(K)$, then one has $\bigcup_{\lambda>0} \lambda(K - \{x_0\}) = X$. If K is closed and convex, then using Baire's theorem one can deduce from the latter relation that $intK \neq \emptyset$ and $int(K) = \ell gint(K)$.

We further recall the definition of a *quasi-relative interior* point, see [5, Def. 2.3], or *inner* point, according to the terminology used in [18].

Definition 1. Let K be a nonempty closed convex subset of X and let $x_0 \in K$. The point x_0 is called a quasi-relative interior (or inner) point of the set K, if the set $cl(\bigcup_{\lambda>0} \lambda(K - \{x_0\}))$ is a subspace of X.

We shall keep the simple term 'inner point' in order to refer to this notion. We further denote by $innK$ the set of all inner points of K. The following proposition (see [5, Prop. 2.16]) reveals an interesting and characteristic property of these points. This property was actually used as the definition of inner points in [18].

Proposition 1. *Let K be a nonempty closed convex subset of X. Then $x_0 \in innK$ if and only if x_0 is a nonsupport point of K, in the sense that the following implication is true for every $x^* \in X^*$:*

$$(x^*, x - x_0) \leq 0, \forall x \in K \Longrightarrow (x^*, x - x_0) = 0, \forall x \in K \tag{2.1}$$

It is easy to see that $intK \subseteq \ell gintK \subseteq innK$. If K is closed and convex, each of the previous inequalities becomes equality whenever the smaller set is nonempty. We further recall from [18, Prop. 2.1] the following proposition:

Proposition 2. *If K is a (nonempty, closed, convex and) separable subset of X, then $innK \neq \emptyset$.*

Recently, inner points met important applications in variational inequality problems, see [18], [9] and [23]. In the following paragraph we shall see that this concept fits naturally also in the cone duality.

3 Order relations in Banach spaces

A nonempty subset C of a Banach space X is called a *cone,* if for every $x \in C$ the whole semiline $\{\lambda x : \lambda > 0\}$ is contained in C. A cone C is called *pointed*, if it does not contain whole lines, or equivalently if 0 is an extreme point of C. We recall here that a point x_0 is said to be an *extreme* point for the set A, if $x_0 \in A$ and x_0 is not contained in any open segment (x, y) lying in A. In the sequel we shall assume that the cone C is always closed, convex and pointed.

It is well known (see for instance [31] or [22]) that the cone C induces a partial order relation $\preceq$ on X by means of the following formula:

$$x \preceq y \Leftrightarrow y - x \in C \tag{3.2}$$

Setting $x = 0$ in the above formula (3.2) we see that the cone C itself corresponds to the set of nonnegative elements.

Let further A be a nonempty subset of X. The set A inherits naturally from X the aforementioned order relation $\preceq$. Consequently one can consider the set $Max(A, C)$ of *maximal* (or *efficient*) points of A (with respect to the cone C) as follows:

$$Max(A, C) = \{x_0 \in A : \{x_0\} = A \cap (x_0 + C)\} \tag{3.3}$$

The *dual cone* C^* of C is defined by

$$C^* = \{f \in X^* : f(x) \geq 0, \forall x \in C\} \tag{3.4}$$

The dual cone C^* corresponds to the set of all *positive functionals*. It is easily seen that C^* is always a nonempty closed convex cone of X^*.
We further denote by

$$C^\sharp = \{f \in Y^* : f(x) > 0, \forall x \in C, x \neq 0\} \tag{3.5}$$

the set of all *strictly positive functionals*. This set is also a cone; however in some cases it may be empty (see the example that follows Proposition 3). In fact one can show (see [5] e.g.) that $C^\sharp$ actually coincides with the set of inner points $innC^*$ of the closed convex set C^*, so its nonemptiness is assured if the space X^* is separable (see Proposition 2 above). The importance of the strictly positive functionals stems from the fact that they are closely related to the notion of a *cone base*. The definition of the latter is recalled below:

Definition 2. A closed convex subset V of C is said to be a (cone) base, if for every $x \in C$, $x \neq 0$, there exist unique $\lambda > 0$, $b \in V$ such that $y = \lambda b$.

The existence of a cone base for a given cone C is in fact equivalent to the nonemptiness of the set $C^\sharp = innC^*$, see also [22]. Indeed, if $C^\sharp \neq \emptyset$, then for any $f \in C^\sharp$ the set $\{x \in C : f(x) = 1\}$ defines a cone base on C. Conversely, if the cone C has a base V, then separating V from 0 (by the Hahn-Banach theorem), one immediately obtains a functional $f \in C^\sharp$.

It follows directly from Proposition 2 that if X is a separable Asplund space (i.e. X^* is separable), then every cone has a base. This result can be refined even further, as shows the following proposition in [5, Th. 2.19].

Proposition 3. *Assume that X is a separable Banach space. Then every (closed, convex pointed) cone C on X has a base.*

The separability assumption is indispensable in the statement of Proposition 3. Indeed, without this assumption nice cones may not have a base, as shows the following example taken from [18].

Example: Let I be any uncountable set and $Y = \ell^2(I)$ be the Hilbert space of all square integrable (with respect to the counting measure) functions $f : I \to R$. Consider the cone C of all non-negative real valued functions of Y. One easily sees that $C^* = C$. However this cone has no inner points, hence C has no base.

We further consider the interior $intC^*$ of the cone C^*, which is a (possibly empty) convex cone. One obviously has $intC^* \subseteq innC^* = C^\sharp$, the equality holding whenever $intC^* \neq \emptyset$. In particular, the latter is equivalent with the existence of a bounded base for the cone C, as states the following proposition, see [22].

Proposition 4. *Let C be a closed, convex, pointed cone of Y. The following are equivalent:*

(i) The dual cone C^* has a non-empty interior $intC^*$.

(ii) The cone C has a bounded base V.

However it is possible to have $intC^* = \emptyset$ and $innC^* \neq \emptyset$. In fact this is very often the case. To enlighten further the above situation we present below some standard examples of Banach spaces possessing a natural ordering structure.

Examples:

1. Let $X = R^n = X^*$, and $C = C^* = R^n_+$. In this case the cone has a bounded base, defined for instance by the strictly positive linear form $y = (1, 1, ..., 1) \in R^n$.

2. Let $X = \ell^1(N)$ be the space of all absolutely summable sequences and $C = \ell^1_+(N)$ be the corresponding cone of all nonnegative sequences of $\ell^1(N)$. One can easily see that the dual cone C^* (which is the set $\ell^\infty(N)_+$ of all nonnegative bounded sequences of the dual space $X = \ell^\infty(N)$) has a nonempty interior, which coincides with the set of all positive bounded sequences. We conclude from Proposition 4 that C has a bounded base.

3. Let $X = \ell^p(N)$, $X^* = \ell^q(N)$ where $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p, q < +\infty$. Consider the cone $C = \ell^p(N)_+$. It follows from Proposition 2 (or Proposition 3) that the dual cone $C^* = \ell^q(N)_+$ has inner points, hence C has a base. However since C^* has an empty interior, every base of C is unbounded.

4. Let X be the space $c_0(N)$ of all null sequences and let $C = c_0(N)_+$ be the cone of all nonnegative null sequences. Then $X^* = \ell^1(N)$ and $C^* = \ell^1(N)_+$. As in the previous case we conclude that the cone $c_0(N)_+$ has a base, but not a bounded base.

5. Let X be the space $C([0,1])$ of the real continuous functions equipped with the topology of the uniform convergence. Since X is separable, applying Proposition 3 we conclude that the cone $C([0,1])_+$ of the nonnegative valued functions has a base. However in this case X^* coincides with the set $BV([0,1])$ of all regural Borel (signed) measures on $[0,1]$ and C^* with the set $BV([0,1])_+$ of all regural Borel positive measures. Since the latter set has an empty interior, we conclude that the cone $C([0,1])_+$ has no bounded base.

From the previous examples it becomes clear that the existence of a cone base is a natural assumption in vector optimization, which is always fulfilled if X is separable. On the other hand this is not the case for the assumption of the boundedness of the base: Among the classical Banach spaces, this condition is fulfilled only in $\ell^1(N)$ (or in general in $L^1(\mu)$) and in the finite dimensional spaces. We summarize below our main conclusions from the above discussion:

Proposition 5. *Let C be a closed, convex, pointed cone of X. Then*
(i) C has a base iff $innC^* \neq \emptyset$
(ii) C has a bounded base iff $intC^* \neq \emptyset$
(iii) If X is separable, then $innC^* \neq \emptyset$

4 Positive (or proper efficient) points. Arrow-Barankin-Blackwell Theorem.

In the sequel we shall deal with a closed, convex pointed cone C with a base V in a Banach space X. In this case one has $C^\sharp = innC^* \neq \emptyset$, hence for any subset $A \subseteq X$ one can define the set of positive points $Pos(A, C)$ of A as follows:

$$Pos(A, C) = \{x_0 \in A : \exists f \in innC^*, f(x_0) = \sup f(A)\} \qquad (4.6)$$

where $\sup f(A)$ denotes the supremum of the functional f on the set A. We mention here that also other (more restrictive) notions of efficiency have been defined in the literature, as for instance the notion of “superefficiency” introduced in [6], see also [17] for a survey.

It is straightforward from relations (3.3), (4.6) and the definition of $C^\sharp$ (relation (3.5)) that $Pos(A,C) \subseteq Max(A,C)$. However simple examples even in two-dimensionsal spaces certify that in general this inclusion is strict.

In the special case $X = R^n$ and $C = R^n_+$ relations (3.3) and (4.6) have a certain interpretation in Economics in terms of the Pareto efficient commodity bundles and the supporting system of prices. This has motivated Arrow, Barankin and Blackwell in 1953 to show the following density result [1] (see also [30] for an alternative approach).

Theorem 1. *Let A be a compact convex subset of R^n and $C = R^n_+$. Then $Pos(A,C)$ is dense in $Max(A,C)$.*

In [19] and independently in [2] the preceding theorem has been extended to cover the case of more general cones C in R^n. Theorem 1 was also generalized to an infinite dimensional setting. The particular case of $\ell^\infty(N)$ has a certain significance in Economics involving models with an infinite horizon production, and has been studied in [33], [27], [29] and [12]. However the statement of Theorem 1 itself as a density result had an independent interest and generated pure mathematical extensions to arbitrary Banach spaces. Many authors have worked in this direction, see for instance [34], [4], [21], [10] etc.

In infinite dimensions there are two topologies that enter naturally into consideration, the weak and the norm topology. The result that follows was originally proved in [15]. Nowadays an easy and direct proof of it is available, that uses the technique of 'dilating cones' (see [20]). This technique is now classical and has already been repeated several times in density results of this kind in [35], [16], [28] and [13]; see also [14] for a more general approach in a locally convex setting. However we give here a sketch of this proof, since it will help the presentation of the forthcoming density results.

In the following statement one can consider $\Im$ to be either the norm or the weak topology of X.

Theorem 2. *Let X be a Banach space, X^* its dual and $\Im$ any topology of the dual system (X,X^*). Let A be a $\Im$-compact, convex subset of X and C a closed, convex, pointed cone with a base V. Then*

$$Max(A,C) \subseteq \overline{Pos(A,C)}^{\Im} \tag{4.7}$$

Proof. (Sketch) Let $x_0 \in Max(A,C)$, i.e. $\{x_0\} = A \cap (x_0 + C)$.
We first observe that $C = \overline{cone}(V)$, where $\overline{cone}(V)$ denotes the closed cone generating by V. Moreover, it is no loss of generality to assume that the distance $d(0,V)$ of the cone base V from 0, is greater than $1/2$.

Step 1: For every $n \geq 2$, consider the (closed, convex, pointed, based) cone $C_n = \overline{cone}(V + B_{\frac{1}{n}}(0))$. Then we obviously have $C = \bigcap_{n\geq 2} C_n$.
Note that in general x_0 does not remain a maximal point of A for the larger cone C_n.

Step 2: For each $n \geq 2$, choose a maximal (with respect to C_n) point $x_n \in Max(A, C_n)$, such that $x_n \in A_n := (x_0 + C_n) \cap A$. This is always possible (see for instance [26, Cor. 3.6]), since the set A_n is $\Im$-compact. Since the relation $C = \bigcap_{n \geq 2} C_n$ implies that $\{x_0\} = \bigcap_{n \geq 2} A_n$, we easily conclude that $x_n \to x_0$ in the $\Im$-topology.

Step 3: Since $\{x_n\} = A \cap (x_n + C_n)$ and the cone C_n has a nonempty interior, there exists a functional $x^* \in C_n^*$ that supports the set A at the point x_n. The proof now finishes by the observation that x^* is actually a strictly positive functional for the original cone C. □

A careful investigation of the previous proof leads easily to the forthcoming corollary. We will first need the following definition.

Definition 3. We say that $x_0 \in A$ is a *point of continuity* of the set A, if the identity mapping $id : (A, w) \to (A, \|\|)$ is continuous at x_0.

The proof of the following corollary is straighforward. However this result will be useful in the sequel. Let us recall from the proof of Theorem 2 that for $n \geq 2$, $C_n := \overline{cone}(V + B_{\frac{1}{n}}(0))$ and $A_n := (x_0 + C_n) \cap A$.

Corollary 1. *Let A be a w-compact, convex subset of X. Assume that $x_0 \in Max(A, C)$ and that for some $n_0 \geq 2$, x_0 is a point of continuity of the set A_{n_0}. Then $x_0 \in \overline{Pos(A, C)}^{\|\cdot\|}$.*

Proof. Repeating the proof of Theorem 2 we produce a sequence $(x_n)_n \subset Pos(A, C)$ that is weakly converging to x_0. We note that this sequence is eventually contained in A_{n_0}, hence in view of Definition 3, it is actually norm converging to x_0. □

Theorem 2 expresses simultaneously two different density results, one for the norm and one for the weak topology. However in the first case, the norm compactness assumption imposed on the convex set A is very restrictive in infinite dimensions. On the other hand the approximation result that we obtain in the second case is rather weak. It is desirable to obtain a strong approximation result involving weakly compact subsets of X, as for example does (in a local way) Corollary 1. To this end, Jahn [21] was the first to derive a norm approximation result for weakly compact subsets, by assuming that the cone C was of a 'Bishop-Phelps type'. Subsequently Petschke [32] (see also [15] for a different approach) refined Jahn's proof to conclude the same result, using - more general - any cone having a bounded base. We state below Petsche's result [32].

Theorem 3. *Let A be a w-compact convex subset of X and assume that C has a bounded base. Then*

$$Max(A, C) \subseteq \overline{Pos(A, C)}^{\|\cdot\|} \tag{4.8}$$

However, as we have already discussed in the previous section, the assumption of a bounded based cone is unpleasant. Recently Gong [16] tried to deal with this inconvenience by relaxing this assumption to an apparently weaker one. Before we proceed to this result, we shall need the following definition.

Definition 4. Let A be a closed convex subset of X and $x_0 \in A$.
(i) x_0 is called a *denting* point of A, if for every $\varepsilon > 0$, we have $x_0 \notin \overline{co}(A \setminus B_\varepsilon(x_0))$, where $\overline{co}(A \setminus B_\varepsilon(x_0))$ denotes the closed convex hull of the set $(A \setminus B_\varepsilon(x_0))$.
(ii) x_0 is called a *strongly exposed* point of A by the functional $x^* \in X^*$, if for every sequence $(x_n)_n \subset A$, the relation $x^*(x_n) \to x^*(x_0)$ implies the norm convergence of the sequence $(x_n)_n$ to x_0.

It follows easily from Definitions 3 and 4 that every denting point of A is a point of continuity for this set. Moreover every strongly exposed point of A is denting. It is worthmentioning that these last two notions coincide if $A = C$ and $x_0 = 0$, since in that case they are both equivalent to the boundedness of the cone base. This is the content of the following proposition in [22] (see also [16] for the equivalence of (ii) and (iii)).

Proposition 6. *The following statements are equivalent:*
(i) 0 *is a strongly exposed point of the cone* C.
(ii) 0 *is a denting point of* C.
(iii) C *has a bounded base*

We are now ready to state Gong's density result, see [16].

Theorem 4. *Let* A *be a* w*-compact convex subset of* X. *Assume that one of the following two conditions is fulfilled.*
(i) Every maximal point of A *is denting.*
(ii) 0 *is a point of continuity of the cone* C, *i.e.*

$$\forall \varepsilon > 0, \quad 0 \notin \overline{C \backslash B(0, \varepsilon)}^{w} \tag{4.9}$$

Then the following approximation result holds:

$$Max(A, C) \subseteq \overline{Pos(A, C)}^{\|\cdot\|} \tag{4.10}$$

Condition (i) of Theorem 4 is satisfied if for example A is taken to be the unit ball of ℓ^p, for $1 < p < +\infty$, see [16]. In the next section we shall see that this condition can be remplaced by a weaker one that would only require that every maximal point of A is a point of continuity. However even this latter condition remains undesirable, since it imposes an a priori assumption on the set of maximal points of A.

On the other hand, in view of Proposition 6 and of Definition 4(i) and its subsequent comments, it follows that condition (4.9) holds trivially whenever the cone C has a bounded base. In that sense the result of Theorem 4(ii) appears to be more general than the one in Theorem 3. In [16], the author queries (and states it as an open question) whether Theorem 4(ii) is indeed a real extension of Theorem 3. In next section we shall answer this question to the negative, by means of a characterization of the denting points of the closed convex subsets of a Banach space.

5 Equivalence of Petscke's and Gong's theorems.

In this section we show that if 0 is a point of continuity of a pointed cone C, then it is also a denting point of C. Consequently, it will follow that Theorems 3 and 4(ii) are equivalent.

Let K be a closed convex subset of X and $x_0 \in K$. As already partially seen in the previous section, every denting point is both an extreme and a point of continuity of K. In [24] (see also [25]) it has been proved that these two properties actually characterize denting points, in case of a closed convex and bounded subset K. The following proposition extends this result to the class of all closed convex subsets of X.

Proposition 7. *Let x_0 be a point of a closed convex subset K of a Banach space. Then x_0 is denting if and only if x_0 is an extreme point and a point of continuity.*

Proof. Let us assume that x_0 is both an extreme and a point of continuity of the set K. Take any $R > 0$ and consider the set $K_R = \{x \in K : \|x - x_0\| \le R\}$. Since $K_R \subseteq K$ and $x_0 \in K_R$, it follows easily that x_0 remains an extreme point and a point of continuity for the set K_R. Since the latter set is bounded, it follows from [25] that x_0 is a denting point of it. The following claim finishes the proof.

Claim: x_0 remains a denting point for the set K.

[Indeed, take any $\varepsilon > 0$. With no loss of generality we can assume that $R > \varepsilon$. Since x_0 is a denting point of the set K_R, we have $x_0 \notin \overline{co}(K_R \setminus B_\varepsilon(x_0))$, hence there exist $x^* \in X^*$ and $\alpha \in R$ such that $x^*(x_0) < \alpha < x^*(x')$, $\forall x' \in \overline{co}(K_R \setminus B_\varepsilon(x_0))$. Set $W = \{x \in X : x^*(x) < \alpha\}$ and observe that since W is a half-space and K is convex, we have $W \cap K \subset B_\varepsilon(x_0) \cap K$. Note now that $W \cap K$ is a neighborhood of x_0 for the (relative) weak topology of K. It now follows that $\overline{co}(K \setminus B_\varepsilon(x_0)) \subseteq K \setminus W$, hence in particular $x_0 \notin \overline{co}(K \setminus B_\varepsilon(x_0))$. The claim is proved.] □

Remark: It is interesting to observe that the previous result has the following interesting restatement:

$$\forall \varepsilon > 0, x \notin \overline{co}\,(K \backslash B(x,\varepsilon)) \Leftrightarrow \forall \varepsilon > 0, x \notin co(K \backslash B(x,\varepsilon)) \text{ and } x \notin \overline{K \backslash B(x,\varepsilon)}^{w}$$

i.e. the convex and the weak topological hull of the set $(K \backslash B(x,\varepsilon))$ can be considered separately.

In the special case of a closed, convex pointed cone C, since the point $x_0 = 0$ is extreme, we infer the following corollary.

Corollary 2. *Let C be a closed convex pointed cone of X. The following statements are equivalent:*
(i) 0 is a denting point of C
(ii) 0 is a point of continuity of C

The above corollary together with Proposition 6 shows in particular that Petschke's result (Theorem 3) and Gong's result (Theorem 4(ii)) are equivalent. Consequently, it remains widely open whether we can efficiently relax (or omit) the assumption of a bounded cone in Theorem 3, without giving up the norm approximation result theorem.

In the following theorem we survey the statements of Proposition 4, of Proposition 6 and of the previous corollary in the following theorem, see also [8]. The equivalence of (ii) and (iv) has also been observed in [16].

Theorem 5. *Let C be a closed convex pointed cone of Y. The following statements are equivalent:*
(i) 0 is a strongly exposed point of C
(ii) 0 is a denting point of C
(iii) 0 is a point of continuity of C
(iv) $\exists \varepsilon > 0, \quad 0 \notin \overline{co}(C \backslash B(0, \varepsilon))$
(v) C has a bounded base
(vi) $intC^* \neq \emptyset$

The following proposition is a local density result which extends in particular Theorem 4(i). The essence of this result comes actually from Corollary 1. We recall that a norm is said to have the *Kadec-Klee property* ([11] eg.), if the relative norm and the relative weak topologies on the unit ball B_X coincide at any point of the unit sphere $S_X := \{x \in X : \|x\| = 1\}$. We also recall that every reflexive Banach space admits a Kadec-Klee renorming.

Proposition 8. *Let A be a w-compact convex subset of X and $x_0 \in Max(A, C)$. Consider the following conditions:*
(i) x_0 is a point of continuity of the set A.
(ii) 0 is a point of continuity of the cone C
(iii) There exists $y \in X$, such that for some $n_0 \geq 2$, x_0 is the farthest point of y for the set $A_{n_0} := (x_0 + C_{n_0}) \cap A$, (i.e. $\|y - x_0\| \geq \|y - x\|$, for all $x \in A_{n_0}$), with respect to an equivalent norm $\|\cdot\|$ of X having the Kadec-Klee property.
(iv) For some $n_0 \geq 2$, x_0 is a point of continuity of the set A_{n_0}.
If any of the conditions (i)-(iv) holds, we have

$$x_0 \in \overline{Pos(A, C)}^{\|\cdot\|}$$

Proof. In virtue of the Corollary 1, it suffices to show that each of the conditions (i)-(iii) implies condition (iv).
Since $A_{n_0} \subseteq A$, it follows directly that condition (i) implies (iv).
Let us now assume that (ii) holds. Then from Theorem 5 it follows that C has a bounded base V. Following the construction of the proof of Theorem 2, we observe that the cones C_n also have a bounded base, hence applying again Theorem 5 we conclude that (iv) holds.
Let us finally assume that (iii) holds. Then x_0 is a boundary point of the closed ball $B_r(y)$ centered at y with radius $r = \|y - x_0\|$. Since the norm $\|\cdot\|$ has the Kadec-Klee property, it follows that x_0 is a point of continuity of the set $B_r(y)$. Since $A_{n_0} \subseteq B_r(y)$ it follows that (iv) holds. □

Remark: Since condition (ii) is equivalent to the existence of a bounded base (see Theorem 5), the above proposition gives in particular an alternative (and simpler) way to prove Theorem 3 of Petschke.

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