Lower semicontinuity of the feasible set mapping of linear systems relative to their domains *

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Abstract. This paper deals with stability properties of the feasible set of linear inequality systems having a finite number of variables and an arbitrary number of constraints. Several types of perturbations preserving consistency are considered, affecting respectively, all of the data, the left-hand side data, or the right-hand side coefficients.

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1 Introduction

In this work we deal with linear inequality systems in $\mathbb{R}^n$, with index set T (an arbitrary set, perhaps infinite), *i.e.* systems of the form

$$\sigma = \{a_t'x \leq b_t,\ t \in T\},$$

where $a : T \to \mathbb{R}^n$ and $b : T \to \mathbb{R}$. We shall identify σ with the data (a, b), so that the main *parametric space* is

$$\Theta = (\mathbb{R}^{n+1})^T.$$

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The *feasible set mapping* (or *feasible map*) $\mathcal{F} : \Theta \rightrightarrows \mathbb{R}^n$ is defined by

$$\mathcal{F}(\sigma) = \{x \in \mathbb{R}^n : a_t' x \leq b_t,\ t \in T\}, \tag{1}$$

with domain

$$\operatorname{dom} \mathcal{F} = \{\sigma \in \Theta : \mathcal{F}(\sigma) \neq \emptyset\}.$$

We shall also consider the inverse mapping $\mathcal{F}^{-1} : \mathbb{R}^n \rightrightarrows \Theta$, *i.e.* $\mathcal{F}^{-1}(x) := \{\sigma \in \Theta : x \in \mathcal{F}(\sigma)\}$.

Following [14], we call *feasible set mapping relative to its domain* the restriction of $\mathcal{F}$ to $\operatorname{dom} \mathcal{F}$, that we denote by $\mathcal{F}^R$. We hereby aim to establish properties of the mapping $\mathcal{F}^R$ for three different types of perturbations of the data, namely, for perturbations over all of the data, for perturbations carried out on the left-hand side data, and for perturbations carried out on the right-hand side data. Our analysis is focussed on (lower semi-)continuity properties of the feasible mapping, dimensional stability of the images and relations with Slater-type conditions. These properties are known to coincide at systems σ lying in the interior of the domain of the feasible mapping (see details in the sequel); thus it becomes particularly interesting to also study topological properties of this domain for each of the above cases.

Continuity properties of multivalued maps. Given a multivalued map between two topological spaces $\mathcal{M} : Y \rightrightarrows X$, we say that $\mathcal{M}$ is *lower semicontinuous* (in short, lsc) at a point $y \in Y$ if for every open set O of X such that $\mathcal{M}(y) \cap O \neq \emptyset$, there is an open set U of Y containing y such that $\mathcal{M}(z) \cap O \neq \emptyset$ for all $z \in U$. In a metric setting this can alternatively be expressed by saying that $\mathcal{M}(y) \subset \operatorname{Liminf} \mathcal{M}(y_k)$ for every sequence $\{y_k\} \subset Y$ such that $y_k \to y$, where $\operatorname{Liminf} A_k$ denotes the Kuratowski-Painlevé inner limit of a sequence of the closed sets A_k (see [12], [14]). We also say that a point $x \in \mathcal{M}(y)$ is *lower stable* for $\mathcal{M}$ at y if for every open set $O \ni x$ there is an open set U containing y such that $\mathcal{M}(z) \cap O \neq \emptyset$ for all $z \in U$, that is, $x \in \operatorname{Liminf} \mathcal{M}(y_k)$ for every sequence $\{y_k\} \subset Y$ converging to y. Clearly, $\mathcal{M}$ is lower semicontinuous at y if and only if every point in $\mathcal{M}(y)$ is lower stable.

If in addition Y is a linear space (more generally, if Y has an affine structure), the related notion of *lower hemicontinuity* is defined as follows: we say that $\mathcal{M}$ is *lower hemicontinuous* (in short, lhc) at a point $y \in Y$ if for every $d \in Y$ and for every open set O of X such that $\mathcal{M}(y) \cap O \neq \emptyset$, there is an open set U of Y containing y such that $\mathcal{M}(z) \cap O \neq \emptyset$ for all $z \in U \cap (y + \mathbb{R}d)$. In a metric setting, this simply means that $\mathcal{M}(y) \subset \operatorname{Liminf} \mathcal{M}(y_k)$ for every sequence $\{y_k\} \subset y + \mathbb{R}d$ such that $y_k \to y$.

Remark 1.1 (Lower semicontinuity along lines). It follows readily that if $\mathcal{M} : Y \rightrightarrows X$ is lower semicontinuous at y then it is also lower hemicontinuous there. If Y is a *topological vector space* (in short, tvs), then the inherited topology on $y + \mathbb{R}d$ coincides with the usual topology of the line, and the lower hemicontinuity of $\mathcal{M}$ at y is equivalent to the lower semicontinuity of all the restrictions of $\mathcal{M}$ to the lines $y + \mathbb{R}d$, for all $d \in Y \setminus \{0\}$. Should this be the case, assuming $\mathcal{M}$ lhc at y and $\mathcal{M}(y) \neq \emptyset$ yields readily that $y \in \operatorname{alg\,int} \operatorname{dom} \mathcal{M}$, the algebraic interior of $\operatorname{dom} \mathcal{M}$. (We recall that for $K \subset Y$, we have $y \in \operatorname{alg\,int} K$ if for every $d \in Y \setminus \{0\}$, there exists $r > 0$ such that $[y - rd, y + rd] \subset K$.) As we shall see later on, this might fail in general.

We say that $\mathcal{M} : Y \rightrightarrows X$ is *upper semicontinuous* at $y \in Y$ if for every open set O of X such that $\mathcal{M}(y) \subset O$, there is an open set U of Y containing y such that $\mathcal{M}(z) \subset O$ for all $z \in U$. If Y and X are metric spaces, $\mathcal{M}$ is said to be *pseudo-Lipschitz* (or *Aubin*) at $(\bar{y}, \bar{x}) \in \operatorname{gph} \mathcal{M}$

(the graph of $\mathcal{M}$) if there exist $L > 0$, and open neighborhoods O of $\bar{x}$ and U of $\bar{y}$ such that for all $y_1, y_2 \in U$ we have

$$\mathcal{M}(y_1) \cap O \subset \mathcal{M}(y_2) + Ld\,(y_1, y_2)\,\mathbb{B}$$

where $\mathbb{B}$ is the closed unit ball of X.

Notation. Given a subset X of some topological space, $\operatorname{int} X$ and $\operatorname{cl} X$ denote the *interior* and the *closure* of X, respectively. The zero vector in $\mathbb{R}^n$ is represented by 0_n. Given $X \subset \mathbb{R}^n$, we denote by $\operatorname{conv} X$, and $\operatorname{cone} X := \mathbb{R}_+ \operatorname{conv}\, X$, the *convex hull* of X, and the *convex conical hull* of X, respectively. If $X = \{x_s, s \in S\}$, denoting by $\mathbb{R}^{(S)}$ the linear space of mappings from S to $\mathbb{R}$ with finite support and by $\mathbb{R}^{(S)}_+$ its positive cone, we can write $\operatorname{cone} X = \left\{ \sum_{s\in S} \lambda_s x_s : \lambda \in \mathbb{R}^{(S)}_+ \right\}$ and $\operatorname{conv} X = \left\{ \sum_{s\in S} \lambda_s x_s : \lambda \in \mathbb{R}^{(S)}_+, \sum_{s\in S} \lambda_s = 1 \right\}$. If X is a convex set, $\operatorname{aff} X$ denotes the *affine hull* of X, *i.e.*, the smallest affine subspace containing X, while $\dim X$ denotes its *dimension*. Given $x \in X \subset \mathbb{R}^n$, X convex, the (convex) *cone of feasible directions* at x is

$$D\,(X; x) = \{y \in \mathbb{R}^n : \exists \mu > 0 \text{ such that } x + \mu y \in X\}\,.$$

The *linearity subspace* of a convex set X is the greatest linear subspace contained in the set X. The (positive) *polar cone* of a convex cone X is $X^\circ := \{y \in \mathbb{R}^n : x'y \geq 0,\ x \in X\}$. For more on these concepts and in general on convexity, see [6, 12].

Characteristic cone of a linear system. Let us revise briefly some known facts about linear systems that will be used in the sequel. Proofs can be found in [6] and references therein.

Given a system

$$\sigma = \left\{ a_t' x \leq b_t,\ t \in T \right\},$$

an important set connected to it is the so-called *characteristic cone* of σ, defined as follows:

$$K\,(\sigma) = \operatorname{cone}\,\{(a_t, b_t)\,, t \in T\} + \mathbb{R}_+\,(0_n, 1)\,. \tag{2}$$

A key tool for our analysis will be the so called *extended Farkas' lemma*, stating the following: suppose the system σ has at least one solution. Then the following are equivalent:

(i) $x \in \mathcal{F}\,(\sigma)$ (see (1)) $\Longrightarrow c'x \leq d$;

(ii) $(c, d) \in \operatorname{cl} K\,(\sigma)$.

To refer to this equivalence, we say that the linear inequality $c'x \leq d$ is a consequence of the consistent system $\sigma = \{a_t' x \leq b_t,\ t \in T\}$ if and only if the vector (c, d) is in the closure of the characteristic cone of σ. It follows readily that

$$\mathcal{F}(\sigma) \subset \left\{ x \in \mathbb{R}^n : c'x = d \right\} \Longleftrightarrow \pm\,(c, d) \in \operatorname{cl} K\,(\sigma)\,,$$

whence, denoting by L the linearity subspace of $\operatorname{cl} K\,(\sigma)$, we get

$$\operatorname{aff} \mathcal{F}(\sigma) = \left\{ x \in \mathbb{R}^n : c'x = d\ \forall\,(c, d) \in L \right\}.$$

The closure of the characteristic cone of σ allows to express the relevant properties of its feasible set $\mathcal{F}\,(\sigma)$ in terms of the data. The following properties hold (see [6, Chapter 5]):

1. $\mathcal{F}(\sigma) \neq \emptyset$ if and only if $(0_n, -1) \notin \operatorname{cl} K(\sigma)$.
   (This equivalence remains true even if we remove $(0_n, 1)$ from the definition of $K(\sigma)$.)
   In case $\mathcal{F}(\sigma) \neq \emptyset$ we have in particular:
2. $\mathcal{F}(\sigma)$ is full-dimensional if and only if $\operatorname{cl} K(\sigma)$ contains no line;
3. $\mathcal{F}(\sigma)$ is a singleton set if and only if $\operatorname{cl} K(\sigma)$ contains some hyperplane;
4. $\mathcal{F}(\sigma) = \{0_n\}$ if and only if $\operatorname{cl} K(\sigma) = \mathbb{R}^n \times \mathbb{R}_+$;
5. $0_n \in \mathcal{F}(\sigma)$ if and only if $\operatorname{cl} K(\sigma) \subset \mathbb{R}^n \times \mathbb{R}_+$;
6. $0_n \in \operatorname{aff} \mathcal{F}(\sigma)$ if and only if the linearity subspace L of $\operatorname{cl} K(\sigma)$ is orthogonal to $(0_n, 1)$.

The following definition highlights the existence of some remarkable points inside the feasible set.

Definition 1.2 (Strong Slater condition). We say that $\widehat{x}$ is a *Slater point* for the linear system σ if

$$a_t' \widehat{x} < b_t, \ \forall t \in T;$$

we say that $\widehat{x}$ is a *strong Slater point* for the linear system σ if there exists $\rho > 0$ such that

$$a_t' \widehat{x} + \rho \leq b_t, \ \forall t \in T.$$

If there is a strong Slater point for σ, we say that σ satisfies the strong Slater condition (in short *SSC*).

For any feasible system $\sigma \in \operatorname{dom} \mathcal{F}$ (*i.e.* $\mathcal{F}(\sigma) \neq \emptyset$) we have

$$\text{SSC} \iff 0_{n+1} \notin \operatorname{cl} \operatorname{conv} \{(a_t, b_t), \, t \in T\}. \tag{3}$$

If in addition T is finite, the SSC is equivalent to the fact that $\operatorname{int} \mathcal{F}(\sigma) \neq \emptyset$ (equivalently, $\dim \mathcal{F}(\sigma) = n$). The behavior of linear systems over an infinite set of constraints T may differ significantly and several properties (including the aforementioned one) fail to hold. Still a good compromise is achieved for certain subfamilies of infinite systems, introduced by the following definitions.

Definition 1.3 (Locally Farkas-Minkowski). A linear system $\sigma = \{a_t' x \leq b_t, t \in T\}$ is said to be *locally Farkas-Minkowski* (in short, *LFM*) if any linear consequent relation of σ determining a supporting hyperplane to $\mathcal{F}(\sigma)$ is also a consequence of a finite subsystem of σ ;
The system σ is called *Farkas-Minkowski* (in short *FM*) if the cone $K(\sigma)$ defined in (2) is closed, or equivalently, if any linear consequent relation of σ is also a consequence of some finite subsystem of σ.

Any FM system is LFM. Both properties hold for the so-called continuous systems, introduced by the following definition, provided they have Slater points.

Definition 1.4 (Continuous system). A system σ is said to be *continuous* whenever T is a compact Hausdorff topological space and $a : T \to \mathbb{R}^n$ and $b : T \to \mathbb{R}$ are continuous.

Any finite system $\sigma = \{a_t' x \leq b_t, t \in T\}$ is continuous (for the discrete topology on T) and any continuous system is *bounded* in the sense that its set of coefficients $\{(a_t, b_t), t \in T\}$ is bounded. Moreover, any continuous system satisfying the Slater condition satisfies obviously the strong Slater condition.

When studying the properties of a point $x \in \mathcal{F}(\sigma)$, a crucial role is played by the set of active constraints at x. A constraint a_t is *active at* x if $a_t' x = b_t$; we shall denote by T_x the *set of active indices at* x: $T_x := \{t \in T : a_t' x = b_t\}$. Finally, we shall denote by $A(x)$ the *active cone* at x: $A(x) := \operatorname{cone}\{a_t, t \in T_x\}$. A consistent system σ is LFM if and only if $D(F; x)^\circ = A(x)$ for all $x \in \mathcal{F}(\sigma)$ [6, Theorem 5.7].

Topology on the parametric space. If $\tilde{\sigma} = \{\tilde{a}_t' x \leq \tilde{b}_t, t \in T\}$ is the resulting system of perturbing $\sigma = \{a_t' x \leq b_t, t \in T\}$, the size of this perturbation is measured by means of the following uniform pseudometric on the *main parametric space* $\left(\mathbb{R}^{n+1}\right)^T$:

$$d(\sigma, \tilde{\sigma}) = \sup_{t \in T} \left\| (a_t, b_t) - \left(\tilde{a}_t, \tilde{b}_t\right) \right\|_\infty, \tag{4}$$

where $\|\cdot\|_\infty$ denotes the Chebyshev norm on $\mathbb{R}^{n+1}$ (the forthcoming results are independent of the chosen norm $\|\cdot\|_\infty$). Note that in full generality the above expression may take infinite values. An important linear subspace of $\left(\mathbb{R}^{n+1}\right)^T$ is the one formed by the bounded systems, $\ell^\infty(T, \mathbb{R}^{n+1})$. The topology induced by d on this subspace coincides with the one of the Chebyshev norm $\|f\|_\infty := \sup_{t \in T} \|f(t)\|_\infty$.

It is well-known that $\left(\ell^\infty(T, \mathbb{R}^{n+1}), \|\cdot\|_\infty\right)$ is a Banach space, whereas $\left(\left(\mathbb{R}^{n+1}\right)^T, d\right)$ is even not a tvs when T is infinite since for $\alpha \in \mathbb{R}$ and $\sigma \in \left(\mathbb{R}^{n+1}\right)^T$ the mapping $(\alpha, \sigma) \mapsto \alpha\sigma \in \left(\mathbb{R}^{n+1}\right)^T$ is not continuous: indeed, taking an unbounded system σ and denoting by θ the null system (the one formed by trivial inequalities), we have $d\left(\frac{1}{k}\sigma, \theta\right) = +\infty$ for all $k \in \mathbb{N}$, so that $\frac{1}{k}\sigma \nrightarrow 0\sigma = \theta$.

When T is a compact Hausdorff space, the linear space of continuous systems, $\mathcal{C}(T, \mathbb{R}^{n+1})$, is a closed subspace of $\ell^\infty(T, \mathbb{R}^{n+1})$, so that $\left(\mathcal{C}(T, \mathbb{R}^{n+1}), \|\cdot\|_\infty\right)$ is a Banach space too. If T is a finite set, $\mathcal{C}(T, \mathbb{R}^{n+1}) = \ell^\infty(T, \mathbb{R}^{n+1}) = \left(\mathbb{R}^{n+1}\right)^T$ is a Euclidean space. In the so-called *general setting,* arbitrary perturbations of the nominal system σ are allowed and the parametric space is $\Theta = \left(\mathbb{R}^{n+1}\right)^T$. Nontheless, in some applications the admissible perturbations are required to preserve either the boundeness or the continuity of σ. So in the *bounded* (respectively, *continuous*) *setting*, the parametric space Θ will be $\ell^\infty(T, \mathbb{R}^{n+1})$ (respectively, $\mathcal{C}(T, \mathbb{R}^{n+1})$). Unless otherwise indicated, the results of this paper are valid for all of the three mentioned settings.

On the other hand, under the topology induced by d, the feasible map $\mathcal{F} : \Theta \rightrightarrows \mathbb{R}^n$ has a closed graph in $\Theta \times \mathbb{R}^n$. Moreover, $\mathcal{F}$ is upper semicontinuous at those σ such that $\mathcal{F}(\sigma)$ is bounded ([6, Corollary 6.2.1]).

Domain of the feasible map. It follows readily that the domain $\operatorname{dom}\mathcal{F}$ of the feasible map has the structure of a cone in the parametric space Θ. However, this cone is not convex unless T is singleton. To see this in case $T = \{1, 2\}$ and $\Theta = (\mathbb{R}^2)^T$, consider the feasible systems $\sigma_1 = \{x \leq -1, 0x \leq 0\}$ and $\sigma_2 = \{0x \leq 0, -x \leq -1\}$. Then the mid-point

$$\frac{\sigma_1 + \sigma_2}{2} = \left\{\frac{1}{2}x \leq -\frac{1}{2}, \quad -\frac{1}{2}x \leq -\frac{1}{2}\right\}$$

is not consistent. This example can be easily adapted to the general case $\Theta = (\mathbb{R}^{n+1})^T$ for every $n \geq 1$ and set of constraints T containing more than one element. Finally, $\operatorname{dom}\mathcal{F}$ is neither open nor closed independently of n and T. Indeed, considering the systems $\sigma_\varepsilon := \{0_n' x \leq -\varepsilon, t \in T\}$, $\varepsilon \geq 0$, with $\sigma_\varepsilon \notin \operatorname{dom}\mathcal{F}$ for all $\varepsilon > 0$ and $\sigma_\varepsilon \to \sigma_0 = \theta \in \operatorname{dom}\mathcal{F}$ as $\varepsilon \searrow 0$, we deduce that $\operatorname{dom}\mathcal{F}$ is not open; similarly, taking an arbitrary vector $u \in \mathbb{R}^n \setminus \{0_n\}$, we define the feasible systems $\sigma_\varepsilon := \{\varepsilon u' x \leq -1, t \in T\}$, $\varepsilon > 0$, which tend to the inconsistent system $\sigma_0 \notin \operatorname{dom}\mathcal{F}$ as $\varepsilon \searrow 0$, showing that $\operatorname{dom}\mathcal{F}$ is not closed.

Remark 1.5 (Bounded representations). Any linear system $\sigma = \{a_t' x \leq b_t, t \in T\}$ in $\left(\mathbb{R}^{n+1}\right)^T$ can be replaced (through scaling) by an equivalent system $\hat{\sigma} = \{\hat{a}_t' x \leq \hat{b}_t, t \in T\}$ in $\ell^\infty(T, \mathbb{R}^{n+1})$. Indeed, it is sufficient to set $\hat{a}_t = \frac{a_t}{M_t}$ and $\hat{b}_t = \frac{b_t}{M_t}$, where $M_t > ||(a_t, b_t)||_\infty$ for all $t \in T$. Obviously both systems have the same feasible set, that is, $\mathcal{F}(\sigma) = \mathcal{F}(\hat{\sigma})$.
A natural question arises: why not replacing the parametric space $\left(\mathbb{R}^{n+1}\right)^T$ by its linear subspace $\ell^\infty(T, \mathbb{R}^{n+1})$?
Indeed, under the topology induced by (4), $\ell^\infty(T, \mathbb{R}^{n+1})$ is both open and closed in $\left(\mathbb{R}^{n+1}\right)^T$. Moreover, most of the $\ell^\infty(T, \mathbb{R}^{n+1})$ counterparts of our results essentially depend on the behavior of the feasible maps $\mathcal{F}$ and $\mathcal{F}^R$ in a neighborhood of a given system σ and remain valid in the bounded setting of $\ell^\infty(T, \mathbb{R}^{n+1})$.
In practice, studying unbounded systems may be imposed by the nature of the problem, and cannot be resumed in a mere study of the feasible set independently of its representation. In several cases the point of interest is precisely to study perturbations of certain unbounded data. The forthcoming Example 2.8 illustrates how different can be alternative representations of the same feasible set.

In this context of linear systems stability, the study of different properties of $\mathcal{F}$ is exhaustively carried out in [7, Theorem 3.1], [6, Theorems 6.1 and 6.9], together with [2, Corollary 5] (see also [1, Proposition 1]). In these works, the following result has been established for the general and the continuous settings, but it also holds for the bounded setting:

Theorem 1.6 (Basic equivalences). *Let $\sigma \in \operatorname{dom}\mathcal{F}$. The following assertions are equivalent:*

(i) σ has a strong Slater point.

(ii) $\mathcal{F}$ is lower semicontinuous at σ.

(iii) $\sigma \in \operatorname{int}\operatorname{dom}\mathcal{F}$.

(iv) $\mathcal{F}$ is dimensionally stable *at σ, i.e., all the feasible sets of systems in a certain neighborhood of σ have the same dimension.*

Theorem 1.6 implies that the domain $\operatorname{dom}\mathcal{F}$ has nonempty interior: as a matter of fact, any feasible system $\sigma = \{a_t' x \leq b_t, t \in T\} \in \operatorname{dom}\mathcal{F}$ is the limit of a sequence $\sigma_k := \left\{a_t' x \leq b_t + \frac{1}{k}, t \in T\right\}$ in $\operatorname{int}\operatorname{dom}\mathcal{F}$ as $k \to \infty$, in other words:

$$\operatorname{dom}\mathcal{F} \subset \operatorname{cl}\operatorname{int}\operatorname{dom}\mathcal{F}. \tag{5}$$

The next result shows that the lower hemicontinuity of $\mathcal{F}$ at σ can be aggregated to the above list of basic equivalences.

Corollary 1.7 (Lower hemicontinuity of $\mathcal{F}$). *The following are equivalent:*

(i) $\mathcal{F}$ *is lower semicontinuous at* σ *;*

(ii) $\mathcal{F}$ *is lower hemicontinuous at* σ*.*

Proof. Both assertions are true whenever $\sigma \notin \operatorname{dom} \mathcal{F}$. Further, we always have (i) $\Longrightarrow$ (ii). Let us now assume $\sigma \in \operatorname{dom} \mathcal{F}$ and $\mathcal{F}$ is lhc at σ. Taking $\tau = \{0_n' x \leq 1, t \in T\} \in \Theta$ (*i.e.* $a_t = 0_n$ and $b_t = 1$ for all $t \in T$), there exists $\delta > 0$ such that $\sigma + \alpha\tau \in \operatorname{dom} \mathcal{F}$ for all α such that $|\alpha| < \delta$. In particular, $\mathcal{F}\left(\sigma - \frac{\delta}{2}\tau\right) \neq \emptyset$, so that σ has a strong Slater point and Theorem 1.6 yields the conclusion. $\square$

The above proof also shows that $\operatorname{alg\,int} \operatorname{dom} \mathcal{F} \subset \operatorname{int} \operatorname{dom} \mathcal{F}$. The reverse inclusion holds trivially in any tvs, but it fails in our general setting (see also Remark 1.1 and the forthcoming Example 1.8).

Example 1.8 (Unbounded systems). Let us consider the unbounded system of one variable $\sigma = \left\{tx \leq \frac{1}{t}, t > 0\right\}$. Then $\sigma \in \operatorname{int} \operatorname{dom} \mathcal{F}$ (and $\mathcal{F}$ is lhs there) since $0_2 \notin \operatorname{cl} \operatorname{conv} \left\{\left(t, \frac{1}{t}\right), t > 0\right\}$. Consider $\tau = \left\{tx \leq -\frac{1}{t^2}, t > 0\right\}$. Given $\alpha > 0$, $\sigma + \alpha\tau = \left\{(1+\alpha)\, tx \leq \frac{1}{t} - \frac{\alpha}{t^2}, t > 0\right\}$, with $\lim_{t \searrow 0} \left(\frac{1}{t^2} - \frac{\alpha}{t^3}\right) = -\infty$. Since $\mathcal{F}(\sigma + \alpha\tau) = \emptyset$ for all $\alpha > 0$, it follows that $\sigma \notin \operatorname{alg\,int} \operatorname{dom} \mathcal{F}$.

Remark 1.9 (Pseudo-Lipschitz property). The properties (i)-(iv) in Theorem 1.6 are also equivalent to the fact that the feasible map $\mathcal{F}$ is pseudo-Lipschitz at $\sigma \in \operatorname{dom} \mathcal{F}$, for all $x \in \mathcal{F}(\sigma)$, see [1, Proposition 1] for details. It is also known that the pseudo-Lipschitz property of $\mathcal{F}$ at (σ, x) is equivalent to the metric regularity of $\mathcal{F}^{-1}$ at (x, σ). This latter is a fundamental notion in variational analysis studied by many authors in relation with stability properties as well as in the efficiency of certain algorithms. The reader is addressed to the classical works of Ioffe [9] and Dontchev, Lewis and Rockafellar [4] for the regularity modulus estimation of generic multivalued maps. See also the recent work [3] for genericity of these properties in case of semi-algebraic (tame) multivalued maps. The metric regularity of systems of convex inequalities and its relation with different constraint qualifications is dealt in Li [11] and Zheng and Ng [15]. See Henrion and Klatte [8] for the connections between metric regularity and the so-called extended Mangasarian-Fromovitz constraint qualification of certain parametrized semi-infinite systems with C^1 data and subject to Right Hand Side (in short, RHS) perturbations. In [13, Proposition 4.2], Mordukhovich and Nghia clarify the relationship between SSC and metric regularity of $\mathcal{F}^{-1}$ for infinite convex systems with equality and geometric constraints.

Structure and scope of this work. In this paper we aim to extend Theorem 1.6 along two directions: At a first stage we study properties of the restriction $\mathcal{F}^R$ of the feasible set map $\mathcal{F}$ to its domain and perturbations of the coefficients for which existence of feasible points is still guaranteed. This is motivated by the fact that in several practical situations we have a prior knowledge that the problem under investigation is feasible, before and after perturbations. At a second stage we handle the case when one of the parameters a or b is kept fixed: again motivation comes from applications where only the Left Hand Side (LHS) (respectively, RHS) of σ can be perturbed. A typical example stems from perturbed zero-sum games admitting only LHS perturbations of a specific type, when reformulated as equivalent Linear Programming problems.

To model the above situations in the general setting, we represent by $\Theta_a = \{a\} \times \mathbb{R}^T$, identified with $\mathbb{R}^T$ (respectively, $\Theta_b = (\mathbb{R}^n)^T \times \{b\}$, identified with $(\mathbb{R}^n)^T$) the closed subset of

Θ formed by all systems whose LHS (respectively, RHS) is a given fixed function $a : T \to \mathbb{R}^n$ (respectively, $b : T \to \mathbb{R}$). We denote by $\mathcal{F}_a$ (respectively, $\mathcal{F}_b$) the restriction of $\mathcal{F}$ to $\mathbb{R}^T$ (respectively, to $(\mathbb{R}^n)^T$), with domain $\mathcal{F}_a$ (respectively, $\operatorname{dom}\mathcal{F}_b$) and feasible set mapping relative to its domain $\mathcal{F}_a^R$ (respectively, $\mathcal{F}_b^R$). The notation is similar in the bounded (continuous) setting, just replacing $\mathbb{R}^T$ and $(\mathbb{R}^n)^T$ by $\ell^\infty(T,\mathbb{R})$ and $\ell^\infty(T,\mathbb{R}^n)$ ($\mathcal{C}(T,\mathbb{R})$ and $\mathcal{C}(T,\mathbb{R}^n)$, respectively).

Note that the domains of the aforementioned feasible set mappings are always nonempty: indeed, given $a \in (\mathbb{R}^n)^T$ ($\ell^\infty(T,\mathbb{R}^n), \mathcal{C}(T,\mathbb{R}^n)$) and $b \in \mathbb{R}^T$ ($\ell^\infty(T,\mathbb{R}), \mathcal{C}(T,\mathbb{R})$), any $\overline{x} \in \mathbb{R}^n \setminus \{0_n\}$ is solution of the systems

$$\left\{a_t' x \leq \tilde{b}_t, t \in T\right\} \text{ and } \left\{\tilde{a}_t' x \leq b_t, t \in T\right\}, \tag{6}$$

where $\tilde{b}_t := a_t'\overline{x}$, $t \in T$, and $\tilde{a}_t := \frac{b_t \overline{x}}{\|\overline{x}\|^2}$, $t \in T$. So, the first system in (6) belongs to $\operatorname{dom}\mathcal{F}_a$ and the second one to $\operatorname{dom}\mathcal{F}_b$.

It is natural to analyze topological properties of the domains of the feasible set mappings $\mathcal{F}$, $\mathcal{F}_a$ and $\mathcal{F}_b$ in their corresponding spaces of parameters Θ, Θ_a and Θ_b. Let us mention that when T is infinite, the domains of $\mathcal{F}_a$ and $\mathcal{F}_b$ might fail to be connected, due to the relative openness of their subsets formed by systems with bounded and unbounded coefficients. We also investigate connections similar to (ii)-(iv) of Theorem 1.6 relative to the maps $\mathcal{F}^R$, $\mathcal{F}_a^R$ and $\mathcal{F}_b^R$, with particular emphasis on finite or continuous systems. We complete our analysis by suitable counterexamples, showing that the converse of most valid connections is not true for arbitrary systems.

2 Perturbing all of the data

In this section we study the stability of the feasible map $\mathcal{F} : \Theta \rightrightarrows \mathbb{R}^n$ and its restriction $\mathcal{F}^R : \operatorname{dom}\mathcal{F} \rightrightarrows \mathbb{R}^n$ to its domain under simultaneous perturbations of the LHS and the RHS. For a feasible system $\sigma \in \operatorname{dom}\mathcal{F}$ such that $0_{n+1} \notin \operatorname{conv}\{(a_t, b_t), t \in T\}$, the neutral element 0_{n+1} may or may not belong to $\operatorname{cl}\operatorname{conv}\{(a_t, b_t), t \in T\}$ depending on whether we have $\sigma \in \operatorname{bd}\operatorname{dom}\mathcal{F}$ or $\sigma \in \operatorname{int}\operatorname{dom}\mathcal{F}$, respectively. In the latter case, since properties (ii)-(iv) of Theorem 1.6 are transmitted from the map $\mathcal{F}$ to its restriction $\mathcal{F}^R$ the analysis is straightforward. In particular, SSC (implying $\sigma \in \operatorname{int}\operatorname{dom}\mathcal{F}$) is obviously a sufficient condition for the lower semicontinuity of the function $\mathcal{F}^R : \operatorname{dom}\mathcal{F} \rightrightarrows \mathbb{R}^n$. But the general case $\sigma \in \operatorname{dom}\mathcal{F}$ requires, instead, a subtle analysis with nontrivial adjustments of the arguments.

Let us start with some results concerning lower semicontinuity of $\mathcal{F}^R$.

Lemma 2.1 (Lower semicontinuity of $\mathcal{F}^R$ at σ with $\mathcal{F}(\sigma)$ singleton)**.** *The mapping $\mathcal{F}^R$ is lower semicontinuous at every system σ whose feasible set is reduced to a singleton.*

Proof. Assume $\mathcal{F}^R(\sigma) = \mathcal{F}(\sigma) = \{\overline{x}\}$. Then the feasible map $\mathcal{F}$ is upper semicontinuous at σ, since it is bounded there. Let further O be any open set such that $\mathcal{F}(\sigma) \cap O \neq \emptyset$, which hereby means $\mathcal{F}(\sigma) = \{\overline{x}\} \subset O$. Then by upper semicontinuity, $\mathcal{F}(\sigma') \subset O$ for any $\sigma' \in \Theta$ sufficiently close to σ. In particular, taking $\sigma' \in \operatorname{dom}\mathcal{F}$, we conclude $\mathcal{F}(\sigma') = \mathcal{F}(\sigma') \cap O \neq \emptyset$. Thus $\mathcal{F}^R$ is lower semicontinuous at σ. $\square$

The next result reveals that whenever $\mathcal{F}(\sigma)$ contains more than one element, lower semicontinuity of $\mathcal{F}^R$ resumes to lower semicontinuity of $\mathcal{F}$, and this can only occur on $\operatorname{int}\operatorname{dom}\mathcal{F}$. In such case, σ cannot contain the trivial inequality $0_n' x \leq 0$ (this inequality is contained only by systems lying on the boundary of the domain).

Proposition 2.2 (Equivalence of lsc for $\mathcal{F}^R$ and $\mathcal{F}$ if $\dim \mathcal{F}(\sigma) \geq 1$)**.** *Assume that $\mathcal{F}(\sigma)$ has at least two elements (i.e. $\dim \mathcal{F}(\sigma) \geq 1$). Then the following are equivalent:*

(i) $\mathcal{F}^R$ is lower semicontinuous at σ ;

(ii) $\mathcal{F}$ is lower semicontinuous at σ ;

(iii) $\sigma \in \operatorname{int} \operatorname{dom} \mathcal{F}$.

Proof. It suffices to establish (i)$\Longrightarrow$(ii). Let us assume that $\mathcal{F}^R$ is lower semicontinuous at σ and that $\mathcal{F}(\sigma)$ is not a singleton. In view of Theorem 1.6, it is sufficient to prove that SSC holds. Reasoning by contradiction, if SSC fails, we deduce by (3) the existence of a sequence $\{\alpha^k\} \subset \mathbb{R}_+^{(T)}$ such that $\sum\limits_{t\in T} \alpha_t^k = 1,\ k = 1, 2, ...,$ and

$$(0_n, 0) = \lim_{k\to\infty} \sum_{t\in T} \alpha_t^k (a_t, b_t).$$

Consider now two distinct points $\widehat{x}, \bar{x} \in \mathcal{F}(\sigma)$, take $v \in \mathbb{R}^n$ and $w \in \mathbb{R}$ such that

$$v'\widehat{x} < w \text{ and } v'\bar{x} > w,$$

and define

$$\sigma_\varepsilon := \{(a_t + \varepsilon v)' x \leq b_t + \varepsilon w, t \in T\}.$$

Obviously, $\bar{x} \in \mathcal{F}(\sigma) \cap \{x \in \mathbb{R}^n : v'x > w\}$ and $\widehat{x} \in \mathcal{F}(\sigma_\varepsilon)$, while

$$\varepsilon(v, w) = \lim_{k\to\infty} \sum_{t\in T} \alpha_t^k (a_t + \varepsilon v, b_t + \varepsilon w) \in \operatorname{cl} K(\sigma_\varepsilon),$$

whence

$$\mathcal{F}(\sigma_\varepsilon) \subset \{x \in \mathbb{R}^n : v'x \leq w\}. \tag{7}$$

This contradicts lower semicontinuity of $\mathcal{F}^R$ and the assertion follows. $\square$

The next corollary shows that the lower hemicontinuity of $\mathcal{F}^R$ at σ can be aggregated to the equivalences above.

Corollary 2.3 (Lower hemicontinuity of $\mathcal{F}^R$)**.** *The following are equivalent:*

(i) $\mathcal{F}^R$ is lower semicontinuous at σ ;

(ii) $\mathcal{F}^R$ is lower hemicontinuous at σ.

Proof. We always have (i) $\Longrightarrow$ (ii). Assume $\mathcal{F}^R$ is lhc at σ and $\mathcal{F}(\sigma)$ has at least two elements (else Lemma 2.1 applies). A careful glance of the proof of Proposition 2.2 reveals that the convergence of the sequence $\sigma_\varepsilon \to \sigma$ occurs along the line $\sigma + \mathbb{R}\tau$, where τ is formed by $|T|$ copies of $v'x \leq w$. Thus $\mathcal{F}$ (and *a fortiori* $\mathcal{F}^R$) is lsc at σ. $\square$

Combining Lemma 2.1 with Proposition 2.2 we obtain the following result.

Corollary 2.4 (Characterization of lsc for $\mathcal{F}^R$)**.** *The mapping $\mathcal{F}^R$ is lower semicontinuous at $\sigma \in \operatorname{dom} \mathcal{F}$ if and only if either SSC holds or $\mathcal{F}(\sigma)$ is a singleton.*

The following lemma concerns LFM systems whose feasible set is not full-dimensional.

Lemma 2.5 (LFM systems with $\dim \mathcal{F}(\sigma) < n$)**.** *Let $\sigma \in \operatorname{dom} \mathcal{F}$ be an LFM system. Then*

$$\dim \mathcal{F}(\sigma) < n \implies \sigma \in \operatorname{bd} \operatorname{dom} \mathcal{F}$$

Equivalently, $\mathcal{F}(\sigma)$ is full-dimensional for any LFM system σ in $\operatorname{int} \operatorname{dom} \mathcal{F}$*.*

Proof. It is the result of combining [6, Theorem 5.9(iv)] with the equivalence (i)$\Longleftrightarrow$(iii) in Theorem 1.6. $\square$

The next corollary shows that, in the bounded and the continuous settings, the lower semicontinuity of $\mathcal{F}^R$ at σ implies that $\mathcal{F}(\sigma)$ is either a singleton or a full-dimensional set.

Corollary 2.6. *Let $\sigma \in \operatorname{dom} \mathcal{F}$ be either LFM or bounded. Then*

$$\mathcal{F}^R \textit{ is lsc at } \sigma \Longrightarrow \dim \mathcal{F}(\sigma) \in \{0, n\}.$$

Proof. Let $\mathcal{F}^R$ be lower semicontinuous at σ and assume $\dim \mathcal{F}(\sigma) \geq 1$. Then by Proposition 2.2 we get $\sigma \in \operatorname{int} \operatorname{dom} \mathcal{F}$ and SSC is satisfied. If σ bounded, then strong Slater points belongs to int $\mathcal{F}(\sigma)$ ([5, Proposition 1]), whence $\dim \mathcal{F}(\sigma) = n$. If instead, σ is assumed LFM, then Lemma 2.5 applies yielding $\dim \mathcal{F}(\sigma) = n$. $\square$

Example 2.7 (General setting). We have seen that

$$SSC \Longrightarrow \dim \mathcal{F}(\sigma) = n$$

if $\sigma \in \Theta$ is either a LFM system or a bounded system. However this fails for general systems even for $n = 1$, as shown by the system $\sigma = \{tx \leq 1,\ t \in \mathbb{R}\}$. Indeed, here 0 is a strong Slater point, while the feasible set is reduced to a singleton $\mathcal{F}(\sigma) = \{0\}$. Since $K(\sigma') = K(\sigma)$ for any other system such that $d(\sigma, \sigma') < 1$, we have $\mathcal{F}(\sigma') = \{0\}$ in this neighborhood and so dimensional stability at σ holds.

The following example shows that the chosen representation plays a crucial role in the study of stability. In particular, modifying the initial representation of a given feasible set, we can trivially fulfill with the conditions of Theorem 1.6.

Example 2.8 (Dependence on the representation). Let $F \subsetneq \mathbb{R}^n$ be an arbitrary nonempty closed convex set. By the separation theorem, we can represent F by means of the system of inequalities $\sigma_1 = \{a_s' x \leq b_s, s \in S\}$ for some infinite set S and mappings $a : S \to \mathbb{R}^n$ and $b : S \to \mathbb{R}$, in such a way that $\mathcal{F}(\sigma_1) = F$. We can further modify this representation by considering the equivalent one

$$\sigma_2 = \left\{ a_s' x \leq b_s + \frac{1}{k}, s \in S, k \in \mathbb{N} \right\}$$

which never satisfies LFM (there is no active index at any feasible point). Setting $T = S \times \mathbb{N}$, $t = (s, k)$, $a_t = k a_s$, and $b_t = k b_s + 1$ we can further consider the asymptotic representation $\sigma_3 = \{a_t' x \leq b_t, t \in T\}$, which in addition satisfies SSC (any $\overline{x} \in F$ is a strong Slater point of σ_3). Observe that the cardinality of the sets of indices is the same for the three systems, so that they belong to the same parametric space Θ.

Note also that $\mathcal{F}^R$ is lsc at σ_3 (in fact $\mathcal{F}$ is constant around σ_3) but $\mathcal{F}^R$ is not necessarily lsc at σ_1 and σ_2, despite the fact that $\mathcal{F}(\sigma_i) = F$, $i = 1, 2, 3$. This shows that, in contrast to Corollary 2.6, for infinite systems $\mathcal{F}^R$ can be lsc at σ and $\sigma \in \operatorname{int} \operatorname{dom} \mathcal{F}$ independently of $\dim \mathcal{F}(\sigma)$. Observe also that σ_2 and σ_3 are never LFM , even though σ_1 may enjoy this property.

The last result in this section applies to the continuous and finite settings.

Theorem 2.9 (Continuous systems)**.** *Let $\sigma \in \operatorname{dom} \mathcal{F}$ be a continuous system without trivial inequalities. Then, the following statements are equivalent to each other:*
(i) $\mathcal{F}^R$ is dimensionally stable at σ.
(ii) SSC holds (or, equivalently, $\sigma \in \operatorname{int} \operatorname{dom} \mathcal{F}$).
(iii) $\dim \mathcal{F}(\sigma) = n$.
Moreover, $\mathcal{F}^R$ is lsc at σ if and only if $\dim \mathcal{F}(\sigma) \in \{0, n\}$.

Proof. Let $\sigma = \{a_t' x \leq b_t, t \in T\}$. Recall that, for continuous systems, Slater and strong Slater conditions coincide. Moreover, by [6, Corollary 5.9.1], for continuous systems without trivial inequalities, the Slater condition is equivalent to the full dimensionality of the feasible set; therefore (ii) and (iii) are equivalent. Moreover, the equivalence with (i) is a consequence of the fact that (ii) implies that SSC also holds for any system in a certain neighborhood of σ. The last assertion stems from Corollary 2.4. $\square$

Both assumptions on σ in Theorem 2.9 are essential for the equivalence of statements (i)-(iii) and also for the characterization of the lower semicontinuity of $\mathcal{F}^R$ by means of $\dim \mathcal{F}(\sigma)$, as the next two examples show.

Example 2.10. The system $\sigma = \{tx \leq 0,\ t \in]0, +\infty[\}$ does not contain the trivial inequality but it is not continuous ($T = (0, +\infty)$ is not compact!). Since $\mathcal{F}(\sigma) =]-\infty, 0]$, $\mathcal{F}(\sigma)$ is full dimensional. Nevertheless, SSC fails and $\mathcal{F}^R$ is not dimensionally stable at σ. In fact, on one hand $0_2 \in \operatorname{cl}(]0, +\infty[\times \{0\})$ and, on the other hand, $\sigma = \lim_{\varepsilon \searrow 0} \sigma_\varepsilon$, where

$$\sigma_\varepsilon = \{tx \leq 0,\ t \in]0, +\infty[\setminus \{\varepsilon\}; -\varepsilon x \leq 0, t = \varepsilon\},$$

with $\dim \mathcal{F}(\sigma_\varepsilon) = 0$ for all $\varepsilon > 0$. Moreover, $\dim \mathcal{F}(\sigma) \in \{0, n\}$ although $\mathcal{F}^R$ is not lsc at σ because $\mathcal{F}(\sigma) \cap]-\infty, 0[\neq \emptyset$ while $\mathcal{F}(\sigma_\varepsilon) \cap]-\infty, 0[= \emptyset$ for all $\varepsilon > 0$.

Example 2.11. Consider the finite (thus, continuous) system $\sigma = \{x \leq 0, 0x \leq 0\}$. We have again that $\mathcal{F}(\sigma)$ has full dimension whereas SSC fails and $\mathcal{F}^R$ is neither dimensionally stable nor lsc at σ (the argument is the same as in Example 2.10, it suffices to take $\sigma_\varepsilon = \{x \leq 0, -\varepsilon x \leq 0\}$ for all $\varepsilon > 0$).

3 Perturbing the right-hand side data

Along this section the parametric space is, in the general setting, $\Theta_a = \{a\} \times \mathbb{R}^T$ (that can be identified to $\mathbb{R}^T$) equipped with the topology induced by d, which describes the uniform convergence of real-valued functions on T. Similarly, the parametric space is $\Theta_a = \ell^\infty(T, \mathbb{R})$ in the bounded setting and $\Theta_a = \mathcal{C}(T, \mathbb{R})$ in the continuous one.

The topological interior of $\operatorname{dom}\mathcal{F}_a$ in Θ, $\operatorname{int}\operatorname{dom}\mathcal{F}_a$, is always empty (as $\operatorname{int}\Theta_a = \emptyset$) but its relative interior in Θ_a, denoted by $\operatorname{int}_a \operatorname{dom}\mathcal{F}_a$, is certainly not since

$$\operatorname{dom}\mathcal{F}_a + \{f \in \Theta_a : \inf_{t\in T} f > 0\} \subset \operatorname{int}_a \operatorname{dom}\mathcal{F}_a.$$

Even more, in contrast with $\operatorname{dom}\mathcal{F}$ (which is neither open nor closed in Θ), $\operatorname{dom}\mathcal{F}_a$ can be open, closed or both, in Θ_a, depending on T and a. The following proposition sheds light on the topological nature of $\operatorname{dom}\mathcal{F}_a$.

Proposition 3.1 (Properties of $\operatorname{dom}\mathcal{F}_a$)**.** *The following statements hold true*:
(i) *If* $\operatorname{dom}\mathcal{F}_a = \Theta_a$, *then* $0_n \notin \operatorname{conv}\{a_t, t \in T\}$; *the converse holds whenever T is finite.*
(ii) $\operatorname{dom}\mathcal{F}_a$ *is open in* Θ_a *if and only if* $0_n \notin \operatorname{cl}\operatorname{conv}\{a_t, t \in T\}$.
(iii) *If T is finite, then* $\operatorname{dom}\mathcal{F}_a$ *is closed in* Θ_a *for any* $a : T \to \mathbb{R}^n$.

Proof. (i) Let $\operatorname{dom}\mathcal{F}_a = \Theta_a$. Assume that $0_n \in \operatorname{conv}\{a_t, t \in T\}$. Then,

$$(0_n, -1) \in \operatorname{conv}\{(a_t, -1), t \in T\},$$

so that $\{a_t' x \leq -1, t \in T\}$ is inconsistent, *i.e.*, $b \equiv -1 \notin \operatorname{dom}\mathcal{F}_a$ (contradiction).

Conversely, assume that T is finite and there exists $b : T \to \mathbb{R}$ such that $\{a_t' x \leq b_t, t \in T\}$ is inconsistent. Then,

$$(0_n, -1) \in \operatorname{cone}\{(a_t, b_t), t \in T; (0_n, 1)\}$$

and there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that

$$\sum_{t\in T} \lambda_t a_t = 0_n, \sum_{t\in T} \lambda_t b_t \leq -1, \rho := \sum_{t\in T} \lambda_t > 0. \tag{8}$$

Then, $\sum\limits_{t\in T} \rho^{-1}\lambda_t a_t = 0_n$, so that $0_n \in \operatorname{conv}\{a_t, t \in T\}$.

(ii) Let $0_n \notin \operatorname{cl}\operatorname{conv}\{a_t, t \in T\}$. Then

$$0_{n+1} \notin \operatorname{cl}\operatorname{conv}\{(a_t, b_t), t \in T\}$$

for any $b \in \Theta_a$. This means that any consistent system $\sigma = \{a_t' x \leq b_t, t \in T\}$ satisfies SSC, so $\mathcal{F}$ is lsc there and small perturbations of the RHS of σ preserve feasibility, *i.e.* $\sigma \in \operatorname{int}_a \operatorname{dom}\mathcal{F}_a$.

Conversely, if $0_n \in \operatorname{cl}\operatorname{conv}\{a_t, t \in T\}$, then

$$(0_n, -\varepsilon) \in \operatorname{cl}\operatorname{conv}\{(a_t, -\varepsilon), t \in T\}$$

for all $\varepsilon \geq 0$. Taking $\sigma_\varepsilon = \{a_t' x \leq -\varepsilon, t \in T\} \in \Theta_a$, $\varepsilon \geq 0$, we have $\sigma_\varepsilon \to \sigma_0 := \{a_t' x \leq 0, t \in T\}$ as $\varepsilon \searrow 0$. Then $\sigma_\varepsilon \notin \operatorname{dom}\mathcal{F}_a$ for every $\varepsilon > 0$, because $(0_n, -\varepsilon) \in \operatorname{cl} K(\sigma_\varepsilon)$, whereas $\sigma_0 \in \operatorname{dom}\mathcal{F}_a$, because $0_n \in \mathcal{F}(\sigma_0)$. Thus, $\sigma_0 \notin \operatorname{int}_a \operatorname{dom}\mathcal{F}_a$.

(iii) Assume that T is finite and let $\sigma = \{a_t' x \leq b_t, t \in T\} \notin \operatorname{dom}\mathcal{F}_a$. Then, $(0_n, -1) \in K(\sigma)$, *i.e.*, there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that (8) holds. If $d : T \to \mathbb{R}$ satisfies $\|d - b\|_\infty < \frac{1}{\rho}$, with $\rho < \sum_{t\in T} \lambda_t$, then

$$\sum_{t\in T} \lambda_t d_t < \sum_{t\in T} \lambda_t \left(b_t + \frac{1}{\rho}\right) \leq 0,$$

so that $\sigma' = \{a_t' x \leq d_t, t \in T\} \notin \operatorname{dom}\mathcal{F}_a$. $\square$

Corollary 3.2. *In the general setting,* $\operatorname{dom}\mathcal{F}_a = \Theta_a$ *if and only if* T *is finite and* $0_n \notin \operatorname{conv}\{a_t, t \in T\}$.

Proof. The "if" part follows from Proposition 3.1 (i). So, we assume $0_n \notin \operatorname{conv}\{a_t, t \in T\}$. Suppose also that T is infinite. Since $\|a_t\| \neq 0$ for all $t \in T$, we can choose a sequence $\{t_k\}$ of non-repeated elements of T and a function $b \in \mathbb{R}^T$ such that $b_{t_k} = -k\|a_{t_k}\|$ for all k. Since

$$\frac{1}{k\|a_{t_k}\|}(a_{t_k}, b_{t_k}) \to (0_n, -1),$$

$\{a_t'x \leq b_t, t \in T\}$ is inconsistent, *i.e.*, $b \notin \operatorname{dom}\mathcal{F}_a$ (contradiction). $\square$

By Proposition 3.1, $\operatorname{dom}\mathcal{F}_a$ is open and closed whenever T is a finite set and the polytope $\operatorname{conv}\{a_t, t \in T\}$ does not contain the origin. It is also open and closed when a is a constant nonzero mapping because, in that case, $\operatorname{dom}\mathcal{F}_a$ is the set of real valued mappings on T which are bounded from below. The latter example, with T infinite, shows that the converse of statement (iii) of Proposition 3.1 does not hold. Observe also that statement (iii) fails even for compact Hausdorff spaces:

Example 3.3. Let $n = 1$ and $T = [0,1]$. We associate with each $\varepsilon \geq 0$ the system $\sigma_\varepsilon = \left\{tx \leq \varepsilon - \sqrt{t}, t \in [0,1]\right\}$. We have $\sigma_\varepsilon \to \sigma_0$ as $\varepsilon \searrow 0$, $\sigma_0 \in \Theta_a \setminus \operatorname{dom}\mathcal{F}_a$, because

$$(0,-1) = \lim_{t \searrow 0} \frac{1}{\sqrt{t}}\left(t, -\sqrt{t}\right) \in \operatorname{cl} K(\sigma_0),$$

but $\sigma_0 \notin \operatorname{int}(\Theta_a \setminus \operatorname{dom}\mathcal{F}_a)$ because, given $\varepsilon > 0$, $\varepsilon - \sqrt{t} + \frac{t}{4\varepsilon} = \left(\sqrt{\varepsilon} - \sqrt{\frac{t}{4\varepsilon}}\right)^2 \geq 0$ for all $t \in \mathbb{R}$ implies that $-\frac{1}{4\varepsilon} \in \mathcal{F}(\sigma_\varepsilon)$. Thus, $\operatorname{dom}\mathcal{F}_a$ is not closed.

Let us now assume that $a : T \to \mathbb{R}^n$ is given. Since the proof of the equivalence between statements (i)-(vi) and (viii) in [6, Theorem 6.1] appeals to perturbations of the RHS exclusively, the following statements are equivalent to each other:

1. SSC holds;
2. $\mathcal{F}_a$ is lower semicontinuous at σ ;
3. $\sigma \in \operatorname{int}_a \operatorname{dom}\mathcal{F}_a$;
4. $\dim\mathcal{F}$ is constant around σ in Θ_a.

The following corollary asserts that the lower hemicontinuity of $\mathcal{F}_a^R$ at σ can be aggregated to the above list of equivalences in the bounded and the continuous settings.

Corollary 3.4 (Lower hemicontinuity of $\mathcal{F}_a$)**.** *The following assertions are equivalent:*

(i) $\mathcal{F}_a$ *is lower semicontinuous at* σ *;*

(ii) $\mathcal{F}_a$ *is lower hemicontinuous at* σ.

Proof. The proof is essentially the same as in Corollary 1.7, taking $b_t = 1$ for all $t \in T$ and $\tau = \{b_t, t \in T\}$. $\square$

We now study properties related to the results of Proposition 2.2 and Theorem 2.9 above, where only perturbations of the RHS are permitted. To this aim we denote by T_0 the set of zeros of a, *i.e.*,

$$T_0 := \{t \in T : a_t = 0_n\}.$$

Let us first treat the trivial case $T_0 = T$. In this case we have $\mathcal{F}_a(\sigma) = \mathbb{R}^n$ for all $\sigma = \{a_t'x \leq b_t, t \in T\}$ for which $b_t \geq 0$ for all $t \in T$, and $\mathcal{F}_a(\sigma) = \emptyset$ otherwise. Then $\mathcal{F}_a^R$ is identically equal to $\mathbb{R}^n$ on $\operatorname{dom} \mathcal{F}_a$, thus it is obviously lsc everywhere.

Thus, we may assume $T_0 \neq T$ from now on in this section.

Proposition 3.5 (Lower semicontinuity of $\mathcal{F}_a^R$)**.** *Let $\sigma = \{a_t'x \leq b_t, t \in T\} \in \operatorname{dom} \mathcal{F}_a$. If either*

$$0_{n+1} \notin \operatorname{cl} \operatorname{conv} \{(a_t, b_t), t \in T \backslash T_0\} \tag{9}$$

or $\mathcal{F}(\sigma)$ is a singleton set (i.e. $|\mathcal{F}(\sigma)| = 1$), then $\mathcal{F}_a^R$ is lsc at σ.

Proof. Condition (9) is equivalent to assert that the following subsystem of σ satisfies the SSC:

$$\sigma_0 := \left\{a_t'x \leq b_t, t \in T \backslash T_0\right\}.$$

Although the parametric spaces associated with σ and σ_0 are different as the index sets are T and $T\backslash T_0$, respectively, no confusion is possible if we represent by the same symbol $\mathcal{F}$ the corresponding feasible set mappings, so that we write $\mathcal{F}(\sigma_0) = \mathcal{F}(\sigma)$ as $\sigma \in \operatorname{dom} \mathcal{F}_a$.

Assuming that (9) holds, we deduce from Corollary 2.4 that $\mathcal{F}^R$ is lsc at σ. Let O be an open set such that $\mathcal{F}(\sigma) \cap O \neq \emptyset$. Then, $\mathcal{F}(\sigma_0') \cap O \neq \emptyset$ for any consistent system σ_0' sufficiently close to σ_0, in particular for those systems σ_0' of the form $\{a_t'x \leq d_t, t \in T\backslash T_0\}$. Let $\varepsilon > 0$ be such that $\mathcal{F}(\sigma_0') \cap O \neq \emptyset$ for any system $\sigma_0' = \{a_t'x \leq d_t, t \in T\backslash T_0\}$ such that $|d_t - b_t| < \varepsilon$ for all $t \in T\backslash T_0$. Now, consider a consistent system $\sigma' = \{a_t'x \leq d_t,\ t \in T\}$ such that $|d_t - b_t| < \varepsilon$ for all $t \in T$. Since $d_t \geq 0$ for all $t \in T_0$, $\mathcal{F}(\sigma') = \mathcal{F}(\sigma_0')$, so that $\mathcal{F}(\sigma') \cap O \neq \emptyset$. Thus, $\mathcal{F}_a^R$ is lsc at σ. The argument for

$$|\mathcal{F}(\sigma)| = 1 \Longrightarrow \mathcal{F}_a^R \text{ is lsc at } \sigma$$

is the same as in Lemma 2.1. $\square$

Observe that the first sufficient condition (9) in Proposition 3.5 is weaker that SSC (the latter precludes the existence of trivial inequalities) and independent from the second one, whose geometric meaning is that $\operatorname{cl} K(\sigma)$ contains a hyperplane, so that (9) does not characterize the lower semicontinuity of $\mathcal{F}_a^R$ at σ. The next example shows that the latter property does not imply, in this context, that $\dim \mathcal{F}(\sigma) \in \{0, n\}$ even though the system σ is finite and has no trivial inequality; in other words, the "only if" part in the last statement of Theorem 2.9 is no longer true for RHS perturbations.

Example 3.6. Let $n = 2$, $T = \{1, 2, 3\}$, $a_1 = (1, 0)$, $a_2 = (0, 1)$, $a_3 = -(0, 1)$, and consider the finite system $\sigma = \{x_1 \leq 0, x_2 \leq 0, -x_2 \leq 0\}$. It follows readily that $\mathcal{F}(\sigma) = (-\infty, 0] \times \{0\}$, whence $\dim\ \mathcal{F}(\sigma) \notin \{0, 2\}$. Obviously, $\operatorname{dom} \mathcal{F}_a^R = \left\{b \in \mathbb{R}^3 : b_2 + b_3 \geq 0\right\}$. Take an arbitrary $y \in \mathcal{F}(\sigma)$ and consider an open neighborhood of y of the form $O = \left\{x \in \mathbb{R}^2 : \|x - y\|_\infty < \rho\right\}$, with $\rho > 0$. Denote by U the Chebyshev open ball centered at 0_3 with radius $\varepsilon < \frac{\rho}{2}$ and take an arbitrary $b \in U$. Take a point $x = (x_1, x_2)$ such that $x_1 = y_1 - \frac{\rho}{2}$ and $x_2 \in [-b_3, b_2]$. Then, $|x_1 - y_1| = \frac{\rho}{2} < \rho$, $|x_2 - y_2| = |x_2| < \varepsilon < \rho$, $x_1 \leq -\frac{\rho}{2} < -\varepsilon \leq b_1$, $x_2 \leq b_2$, and $-x_2 \leq b_3$. Thus, there exists some $x \in O$ satisfying $\{x_1 \leq b_1, x_2 \leq b_2, -x_2 \leq b_3\}$. Since any point of $\mathcal{F}_a^R(\sigma)$ is lower stable for $\mathcal{F}_a^R$, we conclude that $\mathcal{F}_a^R$ is lsc at σ.

Theorem 3.7 (Continuous systems)**.** *Let $\sigma \in \operatorname{dom} \mathcal{F}_a$ be a continuous system without trivial inequalities. Then, the following statements are equivalent to each other*:
(i) *$\mathcal{F}_a^R$ is dimensionally stable at σ.*
(ii) *SSC holds (or, equivalently, $\sigma \in \operatorname{int}_a \operatorname{dom} \mathcal{F}_a$).*
(iii) *$\dim \mathcal{F}(\sigma) = n$.*
Consequently, if $\dim \mathcal{F}(\sigma) \in \{0, n\}$, then $\mathcal{F}_a^R$ is lsc at σ.

Proof. The proof of the equivalence between statements (i)-(iii) is identical to the one of Theorem 2.9 as only RHS perturbations have been considered.

Now assume that $\dim \mathcal{F}(\sigma) \in \{0, n\}$. If $\dim \mathcal{F}(\sigma) = 0$ we conclude that $\mathcal{F}^R$ is lsc at σ from Proposition 3.5. If, alternatively, $\dim \mathcal{F}(\sigma) = n$, then $\mathcal{F}^R$ is lsc at σ by Theorem 2.9 and, so, $\mathcal{F}_a^R$ is lsc at σ as well. $\square$

The next two examples show that Theorem 3.7 fails when σ is either non-continuous or it contains trivial inequalities.

Example 3.8. Let $n = 1$, $T =]0, +\infty[$, $a_t = t$ for all $t \in T$, and $\sigma = \{tx \leq 0,\ t > 0\}$, an unbounded system without trivial inequalities. Obviously, $\mathcal{F}(\sigma) =]-\infty, 0]$ and so $\dim \mathcal{F}(\sigma) = 1$. We associate with each $b \in \mathbb{R}^T$ the system $\{tx \leq b_t,\ t > 0\}$, whose feasible set is represented by $\mathcal{F}(b)$, so that $\mathcal{F}(\sigma) = \mathcal{F}(0)$, and let

$$\alpha_b := \inf \left\{ \frac{b_t}{t} : t > 0 \right\} \in \mathbb{R} \cup \{-\infty\}.$$

Then

$$\mathcal{F}(b) = \begin{cases}]-\infty, \alpha_b], & \text{if } \alpha_b > -\infty, \\ \emptyset, & \text{otherwise,} \end{cases}$$

so that $\operatorname{dom} \mathcal{F}_a = \left\{ b \in \mathbb{R}^T : \alpha_b > -\infty \right\}$ and $\dim \mathcal{F}(b) = 1$ for all $b \in \operatorname{dom} \mathcal{F}_a$. Concerning our nominal system $\sigma \in \operatorname{dom} \mathcal{F}_a$, (i) and (iii) hold while (ii) fails. We consider now the sequence $\left\{ b^k \right\} \subset \mathbb{R}^T$ such that

$$b_t^k = \begin{cases} -\frac{1}{\sqrt{k}}, & \text{if } t = \frac{1}{k}, \\ 0, & \text{otherwise.} \end{cases}$$

We have $\mathcal{F}\left(b^k\right) =]-\infty, -\sqrt{k}]$ for all k, so that $\operatorname{Liminf} \mathcal{F}\left(b^k\right) = \emptyset$. Thus, the last statement in Theorem 3.7 fails because $b^k \to 0$ whereas $\mathcal{F}(0) \nsubseteq \operatorname{Liminf} \mathcal{F}\left(b^k\right)$.

Example 3.9. Let $n = 1$, $T = [-1, 1]$, $a_t = t$ for all $t \in T$, and the continuous system $\sigma = \{tx \leq |t|,\ t \in [-1, 1]\}$, which contains the trivial inequality. Statement (iii) holds as $\mathcal{F}(\sigma) = [-1, 1]$, but (i) and (ii) fail. To see this, we associate with $\varepsilon > 0$ the perturbed system σ_ε which results from replacing the RHS coefficient $|t|$ by 0 whenever $|t| < \varepsilon$. Obviously, $\mathcal{F}(\sigma_\varepsilon) = \{0\}$ for all $\varepsilon > 0$ and $\sigma_\varepsilon \to \sigma$, so that $\mathcal{F}_a^R$ is not dimensionally stable. Moreover, $\mathcal{F}_a^R$ is not lsc at σ even though $\dim \mathcal{F}(\sigma) \in \{0, n\}$.

Proposition 3.10 (Finite systems)**.** *If T is finite, then $\mathcal{F}_a^R$ is lsc at any $\sigma \in \operatorname{dom} \mathcal{F}_a$.*

Proof. Let $\sigma = \{a_t'x \leq b_t, t \in T\}$, where T is finite. We can suppose that

$$T_0 := \{t \in T : a_t = 0_n\} = \emptyset$$

since otherwise, according to the proof of Proposition 3.5, $\mathcal{F}_a^R$ is lsc at any $\sigma = \{a_t'x \leq b_t, t \in T\}$ if and only if $\mathcal{F}_a^R$ is lsc at the subsystem $\sigma_0 := \{a_t'x \leq b_t, t \in T \backslash T_0\}$.

If $\dim \mathcal{F}(\sigma) = n$, then $\mathcal{F}_a^R$ is lsc at σ by Theorem 3.7 (we have assumed that $T_0 = \emptyset$, and so, σ does not contain the trivial inequality). Suppose now that $\dim \mathcal{F}(\sigma) < n$, and take an element $\bar{x}$ in the relative interior of $\mathcal{F}(\sigma)$, *i.e.*, $\bar{x} \in \operatorname{rint} \mathcal{F}(\sigma)$. We have that (see [6, Theorem 5.9(iii)])

$$\operatorname{aff} \mathcal{F}(\sigma) = \{u \in \mathbb{R}^n : a_t'u = b_t \ \forall t \in T_{\bar{x}}\} =: E,$$

where, as usual $T_{\bar{x}}$ is the set of active indices for $\bar{x}$. We shall write E_0 for the parallel vector subspace $E_0 = \{u : a_t'u = 0 \ \forall t \in T_{\bar{x}}\}$, and we shall decompose any element $x \in \mathbb{R}^n$ in the form $x = x^{\perp} + x^{E_0}$, with $x^{E_0} \in E_0$ and $x^{\perp}$ in the subspace orthogonal to E_0, *i.e.* $x^{\perp}$ belongs to the subspace spanned by $\{a_t, \ t \in T_{\bar{x}}\}$. Now take a sequence $\{\sigma_k\}$ in $\operatorname{dom} \mathcal{F}_a$

$$\sigma_k = \left\{a_t'x \leq b_t^k, t \in T\right\} \to \sigma,$$

i.e. a sequence $\{b^k\} \subset \mathbb{R}^T$ converging to $b \in \mathbb{R}^T$, and let $y_k \in \mathcal{F}(\sigma_k)$. Finally, set $x_k = y_k^{\perp} + \bar{x}^{E_0}$. We want to prove that $x_k \to \bar{x}$. To show convergence, first of all let us prove that the sequence $\{x_k\}$ is bounded, in other words, that $\left\{y_k^{\perp}\right\}$ is bounded. Take any $t_0 \in T_{\bar{x}}$. Then the inequality $-a_{t_0}'x \leq -b_{t_0}$ is a consequence of the system $\sigma = \{a_t'x \leq b_t, t \in T\}$ and this implies that there are non-negative scalars λ_t, $t \in T$, and μ such that

$$-(a_{t_0}, b_{t_0}) = \sum_{t \in T} \lambda_t (a_t, b_t) + \mu(0_n, 1),$$

i.e.

$$-(a_{t_0}, b_{t_0}) = \sum_{t \in T \backslash \{t_0\}} \frac{\lambda_t}{1 + \lambda_{t_0}} (a_t, b_t) + \frac{\mu}{1 + \lambda_{t_0}} (0_n, 1).$$

By defining $\alpha_t := \frac{\lambda_t}{1+\lambda_{t_0}}$, $t \in T \backslash \{t_0\}$, and $\nu := \frac{\mu}{1+\lambda_{t_0}}$, we write

$$-(a_{t_0}, b_{t_0}) = \sum_{t \in T \backslash \{t_0\}} \alpha_t (a_t, b_t) + \nu(0_n, 1),$$

and multiplying by $(\overline{x}, -1)$ yields $\alpha_t = 0$ for all $t \in T \backslash T_{\overline{x}}$ and therefore,

$$a_{t_0} = -\sum_{t \in T_{\overline{x}}, \ t \neq t_0} \alpha_t a_t.$$

Thus

$$-\sum_{t \in T_{\overline{x}}, \ t \neq t_0} \alpha_t b_t^k \leq a_{t_0}' y_k^{\perp} \leq b_{t_0}^k,$$

from which we conclude that $\left\{a_{t_0}' y_k^{\perp}\right\}$ is a bounded sequence, for every $t_0 \in T_{\bar{x}}$. But since $y_k^{\perp}$ is in the orthogonal of E_0, this implies that the sequence $\left\{y_k^{\perp}\right\}$ is bounded. So, let z be one of the cluster points of x_k. Since $a_t'x_k = a_t'y_k \leq b_t^k$ for all $t \in T_{\overline{x}}$, then $a_t'z \leq b_t$ for all $t \in T_{\overline{x}}$. This means that $\bar{x} - z \in (E_0^{\perp})^{\circ}$, but on the other hand by definition $\bar{x} - x_k \in E_0^{\perp}$ and thus $\bar{x} - z \in E_0^{\perp}$, and so $z - \overline{x} = 0_n$, and as it happens for every cluster point of the sequence $\{x_k\}$ we conclude that $x_k \to \bar{x}$. Now observe that $x_k \in \mathcal{F}(\sigma_k)$, eventually, since for all $t \notin T_{\bar{x}}$ it is $a_t'\overline{x} < b_t$ and thus $a_t'x_k < b_t^k$. This concludes this part of the proof, since it shows that every point in the relative interior of $\mathcal{F}(\sigma)$ is lower stable. $\square$

4 Perturbing the left-hand side data

In this section the parametric space is $\Theta_b = (\mathbb{R}^n)^T \times \{b\}$ (that we identify with $(\mathbb{R}^n)^T$) in the general setting, $\Theta_b = \ell^\infty(T, \mathbb{R}^n)$ in the bounded setting, and $\Theta_b = \mathcal{C}(T, \mathbb{R}^n)$ in the continuous setting, always equipped with the topology of the uniform convergence on the functions from T to $\mathbb{R}^n$. We have again that $\operatorname{dom} \mathcal{F}_b \neq \emptyset$, $\operatorname{int} \operatorname{dom} \mathcal{F}_b = \emptyset$ (as $\operatorname{int} \Theta_b = \emptyset$) but its interior in Θ_b, say $\operatorname{int}_b \operatorname{dom} \mathcal{F}_b$, is non-empty because

$$\left\{a_t' x \leq b_t, t \in T\right\} \in \operatorname{int}_b \operatorname{dom} \mathcal{F}_b$$

for all $b \in \Theta_b$ whenever $0_n \notin \operatorname{cl} \operatorname{conv} \{a_t, t \in T\}$ (as the latter condition guarantees that SSC holds). The next result shows that, in this context, $\operatorname{dom} \mathcal{F}_b$ can be open, closed or both, in Θ_b, depending on T and b.

Proposition 4.1 (Properties of $\operatorname{dom} \mathcal{F}_b$)**.** *The following statements hold true*:

(i) $\operatorname{dom} \mathcal{F}_b = \Theta_b$ *if and only if* $b_t \geq 0$ *for all* $t \in T$.

(ii) $\operatorname{dom} \mathcal{F}_b$ *is an open proper subset of* Θ_b *if and only if* $\sup_{t \in T} b_t < 0$.

(iii) $\operatorname{dom} \mathcal{F}_b$ *is closed in* Θ_b *if and only if* $\operatorname{dom} \mathcal{F}_b = \Theta_b$.

Proof. (i) If $b_t \geq 0$ for all $t \in T$, then $0_n \in \mathcal{F}(\sigma)$ for all $\sigma = \{a_t' x \leq b_t, t \in T\}$, so that $\operatorname{dom} \mathcal{F}_b$ is the whole space Θ_b. Conversely, if $\operatorname{dom} \mathcal{F}_b = \Theta_b$, the system $\{0_n' x \leq b_t, t \in T\}$ is consistent, so that $b_t \geq 0$ for all $t \in T$.

(ii) Assume that $\sup_{t \in T} b_t < 0$. Let $\delta > 0$ be such that $b_t \leq -\delta$ for all $t \in T$. Taking some $a \in \Theta_b$ vanishing at some $t \in T$, we get the inconsistent system $\{a_t' x \leq b_t, t \in T\}$, *i.e.* $(a_t, b_t) \notin \operatorname{dom} \mathcal{F}_b$, so that $\operatorname{dom} \mathcal{F}_b$ is a proper subset of Θ_b.

Consider now an arbitrary feasible system

$$\sigma = \left\{a_t' x \leq b_t, t \in T\right\} \in \operatorname{dom} \mathcal{F}_b.$$

Since

$$(0_n, -1) \notin \operatorname{cl} K(\sigma) = \operatorname{cl} \operatorname{cone} \{(a_t, b_t), t \in T; (0_n, 1)\},$$

by the separation theorem there exist $v \in \mathbb{R}^n$ and $w \in \mathbb{R}$ such that

$$(v, w) \neq (0_n, 0), \; w > 0 \text{ and } a_t' v + b_t w \geq 0 \; \forall t \in T. \tag{10}$$

Taking $y = w^{-1} v$, we have

$$a_t' y - \delta \geq a_t' y + b_t \geq 0 \text{ for all } t \in T.$$

So, if $x \in \mathcal{F}(\sigma)$, $x - y$ is a strong Slater point for σ. Since SSC holds, $\sigma \in \operatorname{int} \operatorname{dom} \mathcal{F}$ and so $\sigma \in \operatorname{int}_b \operatorname{dom} \mathcal{F}_b$.

We now assume that $\operatorname{dom} \mathcal{F}_b$ is an open proper subset of Θ_b. By (i), we can take some $t_2 \in T$ such that $b_{t_2} < 0$. Arguing by contradiction, let us assume that $\sup_{t \in T} b_t \geq 0$. Then for any $\varepsilon > 0$ such that $b_{t_2} < -\varepsilon$, there exist $t_\varepsilon \in T$ such that $b_{t_\varepsilon} > -\varepsilon$. Now, take an arbitrary vector $u \in \mathbb{R}^n \setminus \{0_n\}$ and consider the systems $\sigma_\varepsilon = ((b_t + \varepsilon) u, b_t)$, with $\varepsilon \geq 0$, and feasible set

$$\mathcal{F}_b((b_t + \varepsilon) u) = \left\{(b_t + \varepsilon) u' x \leq b_t, t \in T\right\}.$$

We have $\sigma_0 \in \operatorname{dom} \mathcal{F}_b$ because

$$(0_n, -1) \notin \operatorname{cl} K(\sigma_0) \subset \operatorname{cone}\{\pm(u,1), (0_n,1)\}.$$

We also have $\sigma_\varepsilon \notin \operatorname{dom} \mathcal{F}_b$ because $b_{t_\varepsilon} + \varepsilon > 0$ and $b_{t_2} + \varepsilon < 0$ imply that

$$\begin{gathered}(b_{t_\varepsilon} + \varepsilon)\left((b_{t_2} + \varepsilon) u, b_{t_2}\right) - (b_{t_2} + \varepsilon)\left((b_{t_\varepsilon} + \varepsilon) u, b_{t_\varepsilon}\right) \\ = \varepsilon\left(0_n, b_{t_2} - b_{t_\varepsilon}\right) \in K(\sigma_\varepsilon),\end{gathered}$$

with $b_{t_2} - b_{t_\varepsilon} < 0$. Since $\sigma_\varepsilon \to \sigma_0$, $\sigma_\varepsilon \notin \operatorname{dom} \mathcal{F}_b$ for all ε such that $b_{t_2} < -\varepsilon$, and $\sigma_0 \in \operatorname{dom} \mathcal{F}_b$, we conclude that $\operatorname{dom} \mathcal{F}_b$ is not open, and this is a contradiction.

(iii) The "if" part is obvious. For the "only if" part, let us assume that $\operatorname{dom} \mathcal{F}_b$ is a proper subset of Θ_b and show that $\operatorname{dom} \mathcal{F}_b$ is not closed. By (i), there exists $s \in T$ with $b_s < 0$. Define for $\varepsilon \geq 0$ the function $a^\varepsilon : T \to \mathbb{R}^n$ such that $a_t^\varepsilon = (\varepsilon + |b_t - b_s|, 0, ..., 0) \in \mathbb{R}^n$ and the system $\sigma_\varepsilon := \{(a_t^\varepsilon)' x \leq b_t,\ t \in T\}$. Observe that $a^\varepsilon \in \ell^\infty(T, \mathbb{R}^n)$ (respectively, $a^\varepsilon \in \mathcal{C}(T, \mathbb{R}^n)$) whenever $b \in \ell^\infty(T, \mathbb{R})$ (respectively, $b \in \mathcal{C}(T, \mathbb{R})$), so that $\sigma_\varepsilon \in \Theta_b$ for all $\varepsilon \geq 0$ in all three settings. In particular, given $\varepsilon > 0$, a simple calculation shows that

$$\inf_{z \in \mathbb{R}} \frac{z}{\varepsilon + |z - b_s|} = \min\left\{\frac{b_s}{\varepsilon}, -1\right\},$$

so that

$$x_\varepsilon := \left(\min\left\{\frac{b_s}{\varepsilon}, -1\right\}, 0, ..., 0\right) \in \mathcal{F}(\sigma_\varepsilon).$$

Then $\{\sigma_\varepsilon\}_{\varepsilon \in]0, +\infty[} \subset \operatorname{dom} \mathcal{F}_b$, with $\sigma_\varepsilon \to \sigma_0$ as $\varepsilon \searrow 0$ and

$$\sigma_0 = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : |b_t - b_s|\, x_1 \leq b_t, t \in T\} \notin \operatorname{dom} \mathcal{F}_b$$

because $b_s < 0$. Thus $\operatorname{dom} \mathcal{F}_b$ is not closed. $\square$

According to Proposition 4.1, if T is finite and $b : T \to \mathbb{R}$, then $\operatorname{dom} \mathcal{F}_b$ is open in Θ_b if and only if the images of b are either all negative or all non-negative numbers.

Theorem 4.2 (Basic equivalences for $\mathcal{F}_b$)**.** *Assume $0_n \notin \mathcal{F}_b(\sigma)$ (thus $b_s < 0$ for some $s \in T$). Then the following assertions are equivalent:*

(i) $\mathcal{F}_b$ *is lower semicontinuous at* σ *and* $\sigma \in \operatorname{dom} \mathcal{F}_b$ *;*

(ii) $\mathcal{F}_b$ *is lower hemicontinuous at* σ *and* $\sigma \in \operatorname{dom} \mathcal{F}_b$ *;*

(iii) $\sigma \in \operatorname{int}_b \operatorname{dom} \mathcal{F}_b$ *;*

(iv) $\sigma \in \operatorname{int} \operatorname{dom} \mathcal{F}$ *;*

(v) σ *satisfies SSC.*

Proof. By Theorem 1.6 we have (v) $\iff$ (iv) $\implies$ (i), while we obviously have (i) $\implies$ (ii) and (iv) $\implies$ (iii). It suffices to establish that each one of conditions (ii) and (iii) yields (v). To this end, let $\sigma = \{a_t' x \leq b_t, t \in T\} \in \operatorname{dom} \mathcal{F}_b$ be such that $0_n \notin \mathcal{F}_b(\sigma)$ and assume that

SSC fails, *i.e.* $0_{n+1} \in \operatorname{cl}\operatorname{conv}\{(a_t, b_t), t \in T\}$. Then there exists a sequence $\{y_k\}_k$ belonging to $\operatorname{conv}\{(a_t, b_t), t \in T\}$ such that $y_k \to (0_n, 0)$. For each $k \in \mathbb{N}$ there exists $\lambda^k \in \mathbb{R}_+^{(T)}$ such that

$$\sum_{t \in T} \lambda_t^k (a_t, b_t) = y_k \text{ and } \sum_{t \in T} \lambda_t^k = 1. \tag{11}$$

Let $s \in T$ be such that $b_s < 0$ and set for $\varepsilon \in]0, 1[$

$$\sigma_\varepsilon = \{(a_t - \varepsilon a_s)' x \leq b_t, \ t \in T\}.$$

Note that $d(\sigma_\varepsilon, \sigma) = \varepsilon |a_s|$, thus $\sigma_\varepsilon \to \sigma$ as $\varepsilon \searrow 0$. Note also that σ_ε, for $\varepsilon > 0$, lies in the line $\sigma + \mathbb{R}\tau$ where $\tau = \{a_s' x \leq 0, \ t \in T\}$, *i.e.* the coefficients of τ consist on repeating $(a_s, 0)$ $|T|$ times. (The reader should be aware of a slight abuse of notation here, since formally $\tau \in \Theta \setminus \Theta_b$. Nevertheless, Θ_b has an affine structure, and the line $\sigma + \mathbb{R}\tau$ is included in Θ_b.) Thus assuming either (ii) or (iii) we deduce that $\sigma_\varepsilon \in \operatorname{dom}\mathcal{F}_b$ for $\varepsilon > 0$ sufficiently small. On the other hand, a simple algebraic computation shows that

$$\begin{aligned} &\lim_{k\to\infty} \left\{ \sum_{t \in T \setminus \{s\}} (1-\varepsilon) \lambda_t^k (a_t - \varepsilon a_s, b_t) + \left[\varepsilon + (1-\varepsilon)\lambda_s^k\right] ((1-\varepsilon) a_s, b_s) \right\} \\ &\qquad = \lim_{k\to\infty} \{(1-\varepsilon) y_k + \varepsilon (0_n, b_s)\} = \varepsilon (0_n, b_s). \end{aligned} \tag{12}$$

Therefore, (12) yields that

$$(0_n, -1) \in \operatorname{cl} K(\sigma_\varepsilon)$$

and $\sigma_\varepsilon \notin \operatorname{dom}\mathcal{F}_b$, a contradiction. This shows that (ii) $\Longrightarrow$ (v) and (iii) $\Longrightarrow$ (v), and the result follows. $\square$

The next example shows that, in general, none of the conditions (i)-(iv) of the previous theorem implies (v), even though σ is a finite system without trivial inequalities.

Example 4.3. The finite system $\sigma = \{x \leq 0, -x \leq 0\}$ does not satisfy SSC. Nevertheless, $\mathcal{F}(\sigma') = \{0\}$ for all $\sigma' \in \Theta_b$ such that $d(\sigma', \sigma) < 1$, *i.e.*, $\mathcal{F}$ is constant around σ in Θ_b, so that $\mathcal{F}_b$ is lsc at σ, $\dim \mathcal{F}$ is constant around σ in Θ_b, and $\sigma \in \operatorname{int}_b \operatorname{dom}\mathcal{F}_b$.

The forthcoming Example 4.4 shows that the condition $\sigma \in \operatorname{int}_b \operatorname{dom}\mathcal{F}_b$ in Theorem 4.2 cannot be relaxed to $\sigma \in \operatorname{dom}\mathcal{F}_b$. Nevertheless, statement (ii) in forthcoming Proposition 4.5 shows that it is possible to replace $0_n \notin \mathcal{F}(\sigma)$ with the weaker condition that $\mathcal{F}(\sigma) \neq \{0_n\}$ whenever $\mathcal{F}_b^R$ is lsc at σ.

Example 4.4. Let $n = 1$, $T = \{1, 2\}$, $b = (1, -1) \in \mathbb{R}^2$, and consider the following systems:

$$\sigma_\varepsilon = \{(1+\varepsilon) x \leq 1, (\varepsilon - 1) x \leq -1\}, \ \varepsilon \geq 0.$$

We have $\mathcal{F}(\sigma_0) = \{1\}$ while $\sigma_\varepsilon \notin \operatorname{dom}\mathcal{F}_b$ for all ε such that $0 < \varepsilon < 1$. Since $\sigma_\varepsilon \to \sigma_0$ as $\varepsilon \searrow 0$, we get $\sigma_0 \in \operatorname{dom}\mathcal{F}_b \setminus \operatorname{int}_b \operatorname{dom}\mathcal{F}_b$, so that $\mathcal{F}_b$ is not lsc at σ_0. Let O be an open set such that $\mathcal{F}(\sigma_0) \cap O \neq \emptyset$, *i.e.*, $1 \in O$. Any perturbed system $\sigma' \in \Theta_b$ can be written in the form $\sigma' = \{a_1 x \leq 1, a_2 x \leq -1\}$ for some $a_1, a_2 \in \mathbb{R}$, with $\sigma' \in \operatorname{dom}\mathcal{F}_b$ if and only if either $a_2 < 0$ and $a_1 + a_2 \leq 0$ or $a_2 > 0$ and $a_1 + a_2 \geq 0$. Let $\sigma' \in \operatorname{dom}\mathcal{F}_b$ be such that $d(\sigma', \sigma_0) < \frac{1}{2}$. Then $a_1 > 0$, $a_2 < 0$, $-\frac{1}{a_2} \leq \frac{1}{a_1}$, and $\mathcal{F}(\sigma') = \left[-\frac{1}{a_2}, \frac{1}{a_1}\right]$. Observing that

$$\sigma' \to \sigma_0 \Longrightarrow (a_1, a_2) \to (1, -1) \Longrightarrow (-1/a_2, 1/a_1) \to (1, 1) \in O^2,$$

a continuity argument yields that $\mathcal{F}(\sigma') = \left[-\frac{1}{a_2}, \frac{1}{a_1}\right] \subset O$ for σ' sufficiently close to σ_0. Thus, $\mathcal{F}_b^R$ is lower semicontinuous at σ_0, $\sigma_0 \in \operatorname{dom} \mathcal{F}_b$ and $0_n \notin \mathcal{F}(\sigma)$, but σ_0 does not satisfy SSC.

We are now ready to state the following result.

Proposition 4.5 (Lower semicontinuity of $\mathcal{F}_b^R$)**.** *Given $\sigma \in \operatorname{dom} \mathcal{F}_b$, the following statements are true:*

(i) *If either σ satisfies SSC or $\mathcal{F}(\sigma)$ is a singleton, then $\mathcal{F}_b^R$ is lsc at σ.*

(ii) *If $\mathcal{F}_b^R$ is lsc at $\sigma \in \operatorname{int}_b \operatorname{dom} \mathcal{F}_b$ and $\mathcal{F}(\sigma) \neq \{0_n\}$, then σ satisfies SSC.*

(iii) *If $\mathcal{F}_b^R$ is lsc at σ and $\mathcal{F}(\sigma)$ is neither a singleton nor a subset of a ray, then σ satisfies SSC.*

Proof. Let $\sigma = \{a_t' x \leq b_t, t \in T\} \in \operatorname{dom} \mathcal{F}_b$.

(i) The proof is the same as in Lemma 2.1 and Corollary 2.4.

(ii) Since $\mathcal{F}(\sigma) \neq \{0_n\}$, we can take $x^0 \in \mathcal{F}(\sigma) \backslash \{0_n\}$ and an open set O defined by

$$O := \{x \in \mathbb{R}^n : \ u'x > \alpha\},$$

with $u \neq 0_n, \alpha > 0$, such that

$$x^0 \in \mathcal{F}(\sigma) \cap O.$$

Reasoning by contradiction, if σ has no Strong Slater point, the following implication holds for every $\varepsilon > 0$:

$$(a_t + \varepsilon u)' x \leq b_t, \ t \in T \Longrightarrow u'x \leq 0. \tag{13}$$

If $\varepsilon > 0$ is small enough, $\sigma \in \operatorname{int}_b \operatorname{dom} \mathcal{F}_b$ entails that the perturbed system

$$\sigma_\varepsilon := \{(a_t + \varepsilon u)' x \leq b_t, \ t \in T\}$$

is certainly consistent, but $\mathcal{F}(\sigma_\varepsilon) \cap O = \emptyset$ by (13), and this contradicts the assumed lower semicontinuity of $\mathcal{F}_b^R$ at σ.

(iii) If SSC fails, $0_{n+1} \in \operatorname{cl} \operatorname{conv} \{(a_t, b_t), t \in T\}$, and there must exist a sequence $\{\lambda^r\} \subset \mathbb{R}_+^{(T)}$ such that

$$\lim_{r \to \infty} \sum_{t \in T} \lambda_t^r (a_t, b_t) = 0_{n+1} \text{ and } \sum_{t \in T} \lambda_t^r = 1.$$

If $\mathcal{F}(\sigma)$ is neither a singleton set nor a subset of a ray, then there exist $x^0, x^1 \in C$ and $u \in \mathbb{R}^n$ such that

$$u'x^0 > 0 \geq u'x^1.$$

Then, for any $\varepsilon > 0$ the system

$$\sigma_\varepsilon := \{(a_t + \varepsilon u)' x \leq b_t, \ t \in T\}$$

is consistent since $x^1 \in \mathcal{F}(\sigma_\varepsilon)$, and $\sigma_\varepsilon \to \sigma$ as $\varepsilon \searrow 0$. It is also obvious that

$$\lim_{r \to \infty} \sum_{t \in T} \lambda_t^r (a_t + \varepsilon u, b_t) = \varepsilon(u, 0),$$

and by Farkas lemma, $u'x \leq 0$ is a consequence of σ_ε, precluding the lower semicontinuity of $\mathcal{F}_b^R$ at σ (take the open set $U := \{x \in \mathbb{R}^n : u'x > 0\}$). $\square$

Remark 4.6 (Lower hemicontinuity of $\mathcal{F}_b$). The proof of the previous proposition reveals that lower hemicontinuity can replace lower semicontinuity in conditions (ii) and (iii).

Example 4.4 shows that we cannot replace the assumption $\sigma \in \operatorname{int}_b \operatorname{dom} \mathcal{F}_b$ by $\sigma \in \operatorname{dom} \mathcal{F}_b$ in Proposition 4.5(ii). The next examples show that none of the remaining assumptions is superfluous.

Example 4.7. Consider the systems in one variable

$$\sigma_\varepsilon = \{(t-\varepsilon)x \leq 0,\ t \in]0,1]\}, \varepsilon \geq 0.$$

We have $\sigma_0 \in \operatorname{int}_b \operatorname{dom} \mathcal{F}_b$, because $\operatorname{dom} \mathcal{F}_b = \Theta_b$, and $\mathcal{F}(\sigma_0) =]-\infty, 0] \neq \{0\}$, but σ_0 has no Strong Slater point. Moreover, $\mathcal{F}(\sigma_\varepsilon) = \{0\}$, for all $\varepsilon > 0$, so that $\mathcal{F}_b$ and $\mathcal{F}_b^R$ are neither lsc nor dimensionally stable at σ_0.

Example 4.8. Consider now the system, once more in one variable,

$$\sigma = \{tx \leq 1,\ t \in \mathbb{R} \backslash \{0\}; 0x \leq 0\}.$$

Since $\mathcal{F}(\sigma') = \{0\}$ for any system $\sigma' \in \operatorname{dom} \mathcal{F}_b = \Theta_b$ such that $d(\sigma, \sigma') < +\infty$, we have $\sigma \in \operatorname{int}_b \operatorname{dom} \mathcal{F}_b$, and lower semicontinuity and dimensional stability of $\mathcal{F}_b$ and $\mathcal{F}_b^R$ at σ are fulfilled, despite the fact that SSC fails. So, the additional conditions guaranteeing that $\mathcal{F}_b^R$ lsc at σ implies SSC in Proposition 4.5, statements (ii) and (iii), are not superfluous neither.

Remark 4.9. From theorem 4.2 and Proposition 4.5(i), it is easy to see that, if $\sigma \in \operatorname{int}_b \operatorname{dom} \mathcal{F}_b$ and $0_n \notin \mathcal{F}(\sigma)$, then $\mathcal{F}_b^R$ is lsc at σ. Therefore, the sufficient condition for SSC in Theorem 4.2 is stronger than the one in Proposition 4.5 (ii), but this fact cannot be easily proved directly. These conditions are not equivalent (consider $\sigma = \{x \leq 0\}$), and the first one is also sufficient for $\mathcal{F}_b^R$ to be lsc at σ according to Proposition 4.5.

Theorem 4.10 (Continuous systems). *Let $\sigma \in \operatorname{dom} \mathcal{F}_b$ be a continuous system without trivial inequalities such that $0_n \notin \mathcal{F}(\sigma)$. Then, the following statements are equivalent to each other:*
(i) *$\mathcal{F}_b^R$ is dimensionally stable at σ;*
(ii) *SSC holds (or, equivalently, $\sigma \in \operatorname{int}_b \operatorname{dom} \mathcal{F}_b$);*
(iii) *$\dim \mathcal{F}(\sigma) = n$.*
Moreover, any of these properties implies that $\mathcal{F}_b^R$ is lsc at σ.

Proof. Let $\sigma = \{a_t' x \leq b_t, t \in T\}$ satisfy the assumptions. The equivalence claimed in (ii),

$$\text{SSC} \Longleftrightarrow \sigma \in \operatorname{int}_b \operatorname{dom} \mathcal{F}_b,$$

follows from Theorem 4.2. The equivalence (ii)$\Longleftrightarrow$(iii) comes from [6, Corollary 5.9.1].

(i)$\Longrightarrow$(ii) Assume that σ does not satisfy SSC, *i.e.*, that $\dim \mathcal{F}(\sigma) < n$. Since$(0_n, -1) \notin \operatorname{cl} K(\sigma)$, by the separation theorem, there exists $v \in \mathbb{R}^n$ and $w \in \mathbb{R}$ such that (10) holds. Since $0_n \notin \mathcal{F}(\sigma)$, *i.e.*, $b \notin \mathbb{R}_+^T$, we have $v \neq 0_n$ (otherwise $b_t \geq 0$ for all $t \in T$). Consider the continuous systems

$$\sigma_\varepsilon = \{(a_t + \varepsilon v)' x \leq b_t,\ t \in T\},\ \varepsilon > 0.$$

Obviously, $\sigma_\varepsilon \to \sigma$ as $\varepsilon \to 0$. If $0_{n+1} \in \operatorname{conv}\{(a_t + \varepsilon v, b_t), t \in T\}$, there will exist $\lambda \in \mathbb{R}_+^{(T)}$ such that

$$\sum_{t \in T} \lambda_t (a_t + \varepsilon v, b_t) = 0_{n+1} \tag{14}$$

and $\sum_{t\in T}\lambda_t = 1$. Multiplying both members of (14) by (v,w) one gets the following contradiction:

$$0=\sum_{t\in T}\lambda_t\left(a_t'v+\varepsilon\left\|v\right\|^2+b_tw\right)\geq\varepsilon\left\|v\right\|^2>0.$$

Since all systems σ_ε satisfy SSC, we have $\dim\mathcal{F}(\sigma_\varepsilon)=n$ for all $0<\varepsilon<\frac{\min_{t\in T}\|a_t\|}{\|v\|}$. Therefore, (i) fails.

(ii)$\Longrightarrow$(i) It is a consequence of the fact that (ii) implies that SSC also holds for any system σ' in Θ_b close to σ.

The last statement comes from Proposition 4.5 $\square$

Example 4.3 shows that the assumption $0_n\notin\mathcal{F}(\sigma)$ in Theorem 4.10 is not superfluous, even in the case of finite systems. The next proposition characterizes the lower semicontinuity of $\mathcal{F}_b^R$ for finite systems, independently of position of 0_n relative to $\mathcal{F}(\sigma)$ (or the sign of b_t, $t\in T$). The next example shows that (i)-(iii) are not equivalent to the lower semicontinuity of $\mathcal{F}_b^R$ at σ.

Example 4.11. The system

$$\sigma=\{x_1-x_2\leq 0,-x_1+x_2\leq 0,-x_1\leq -1\}$$

satisfies the assumptions of Theorem 4.10, with $\mathcal{F}(\sigma)=\{(\lambda,\lambda):\lambda\geq 1\}$. It can be realized that conditions (i) to (iii) fail whereas $\mathcal{F}_b^R$ *is lsc at* σ as it is shown in Proposition 4.12.

Proposition 4.12 (Finite systems)**.** *Assume that T is finite and that σ contains no trivial inequalities. Then $\mathcal{F}_b^R$ is lsc at $\sigma\in\operatorname{dom}\mathcal{F}$ if and only if one of the following alternatives holds:*

(i) $\dim\mathcal{F}(\sigma)=n$;

(ii) $\dim\mathcal{F}(\sigma)=0$ *(i.e. $\mathcal{F}(\sigma)$ is a singleton)*;

(iii) $\mathcal{F}(\sigma)$ *contains at least two elements and is contained in some open ray.*

Proof. Since σ does not contain trivial inequalities, we can apply Theorem 4.10.

About the necessity of one of the three conditions, simply observe that if (i) and (ii) fail, by Proposition 4.5, and because $\dim\ \mathcal{F}(\sigma)<n$, $\mathcal{F}(\sigma)$ is contained in some open ray. So, we only need to prove that if $0_n\in\mathcal{F}(\sigma)$ then $\mathcal{F}_b^R$ is not lower semicontinuous at σ. In fact, if we take a point $0_n\neq\bar{x}$ in the relative interior of $\mathcal{F}(\sigma)$, and an active constraint at $\bar{x}$, say $s\in T_{\bar{x}}$, we shall have $a_s'\overline{x}=b_s=0$. Now let

$$\sigma_\varepsilon:=\{\hat{a_t}'x\leq b_t:t\in T\},$$

with $\hat{a_t}=a_t$ for $t\neq s$, $\hat{a_s}=a_s+\varepsilon\bar{x}$. Then it is easy to see that

$$0_n\in\mathcal{F}(\sigma_\varepsilon)\subset\{x\in\mathbb{R}^n:\bar{x}'x\leq 0\},$$

and this concludes the argument (we have actually shown that no point of $\mathcal{F}(\sigma)$ is lower stable).

Concerning the sufficiency, it is straightforward that (i) and (ii) imply that $\mathcal{F}_b^R$ is lsc at σ (by Proposition 4.5 and Theorem 4.10, respectively). Now let us see that T implies lower semicontinuity of $\mathcal{F}_b^R$ at σ. Let $\bar{x}$ be in the relative interior of $\mathcal{F}(\sigma)$, and let $T_{\bar{x}}$ be the set of active indices at $\bar{x}$. It is well known that, for finite systems,

$$\operatorname{aff}\mathcal{F}(\sigma)=\left\{x\in\mathbb{R}^n:a_t'x=b_t\ \forall t\in T_{\bar{x}}\right\}. \tag{15}$$

Take $\eta > 0$ such that $a_t'\bar{x} \leq b_t - 2\eta$ for all $t \notin T_{\bar{x}}$. Now let $\varepsilon > 0$ be so small that for all $x \in B(\bar{x};\varepsilon)$ we have $a_t'x < b_t - \eta$ for all $t \notin T_{\bar{x}}$. On the other hand,

$$t \in T_{\bar{x}} \Longrightarrow a_t'\bar{x} = b_t = 0.$$

Moreover, as $0_n \notin \mathcal{F}(\sigma)$, there must exist $t_0 \in T \backslash T_{\bar{x}}$ such that

$$a_{t_0}'\bar{x} < b_{t_0} < 0.$$

Now take any sequence $\{\sigma_k\}$ such that

$$\operatorname{dom} \mathcal{F} \ni \sigma_k := \{(a_t^k)'x \leq b_t,\ t \in T\}$$

and $\sigma_k \to \sigma$; therefore $a_t^k \to a_t$ for all $t \in T$. Moreover, if $d(\sigma,\sigma_k) \leq \frac{\eta}{\sqrt{n}(\|\bar{x}\|+\varepsilon)}$ it can be verified that

$$(a_t^k)'x \leq b_t, \text{ for all } x \in B(\bar{x};\varepsilon) \text{ and all } t \notin T_{\bar{x}}; \tag{16}$$

now let K_1 be such that

$$d(\sigma_k,\sigma) \leq \frac{\eta}{\sqrt{n}(\|\bar{x}\|+\varepsilon)} \text{ for all } k \geq K_1. \tag{17}$$

Obviously

$$\emptyset \neq \mathcal{F}(\sigma_k) \subset \{x \in \mathbb{R}^n :\ (a_t^k)'x \leq 0,\ t \in T_{\bar{x}};\ (a_{t_0}^k)'x \leq b_{t_0}\},$$

and take x_k, $k = 1, 2, ...$, such that

$$(a_t^k)'x_k \leq 0,\ t \in T_{\bar{x}}, \text{ and } (a_{t_0}^k)'x_k \leq b_{t_0}. \tag{18}$$

Since $b_{t_0} < 0$, $x_k \neq 0_n$, $k = 1, 2, ...$, we can define

$$z_k := \frac{\|\bar{x}\|}{\|x_k\|}x_k,\ k = 1, 2, ...$$

Obviously,

$$(a_t^k)'z_k \leq 0, \text{ for all } t \in T_{\bar{x}} \text{ and all } k. \tag{19}$$

Suppose that $\{z_{k_r}\}_{r=1}^{\infty}$ is a subsequence converging to u. Taking limits in (19) for $r \to \infty$ one gets

$$a_t'u \leq 0, \text{ for all } t \in T_{\bar{x}}, \tag{20}$$

and this entails

$$a_t'u = 0, \text{ for all } t \in T_{\bar{x}}, \tag{21}$$

because if $\overline{u}$ is a solution of the system in (20) and there exists $t_1 \in T_{\bar{x}}$ such that $a_{t_1}'\overline{u} < 0$, for $\varepsilon > 0$ small enough we shall have

$$\overline{x} + \varepsilon\overline{u} \in \mathcal{F}(\sigma) \backslash \operatorname{aff} \mathcal{F}(\sigma),$$

and this is impossible by (15).

We see that (21) entails $u = \lambda\bar{x}$ for some $\lambda \in \mathbb{R}$, but $\|u\| = \|\bar{x}\|$ implies now that $|\lambda| = 1$, *i.e.*, $u = \bar{x}$. In fact, we cannot have $\lambda = -1$ because in this case (18) would yield the following contradiction

$$0 \geq \lim_{r\to\infty} (a_{t_0}^{k_r})'z_{k_r} = a_{t_0}'u = -a_{t_0}'\bar{x} > -b_{t_0} > 0.$$

As every converging subsequence of $\{z_k\}$ converges to $\bar{x}$, we conclude that

$$\lim_{k\to\infty} z_k = \bar{x},$$

and there will exist $K_0 \geq K_1$ such that

$$z_k \in B(\bar{x};\varepsilon), \text{ for all } k \geq K_0.$$

Since we have chosen $K_0 \geq K_1$, (16) and (17) yield

$$z_k \in \mathcal{F}(\sigma_k), \text{ for all } k \geq K_0,$$

and we have actually shown that every point of $\mathcal{F}(\sigma)$ is lower stable. $\square$

The above result has potential applications in the study of stability of the zero-sum games. This will be investigated in a future work.

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