Metric and geometric relaxations of self-contracted curves

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Abstract Self-contractedness (or self-expandedness, depending on the orientation) is hereby extended in two natural ways giving rise, for any $\lambda \in [-1, 1)$, to the metric notion of λ-curve and the (weaker) geometric notion of λ-cone property (λ-eel). In the Euclidean space $\mathbb{R}^d$ it is established that for $\lambda \in [-1, 1/d)$ bounded λ-curves have finite length. For $\lambda \geq 1/\sqrt{5}$ it is always possible to construct bounded curves of infinite length in $\mathbb{R}^3$ which do satisfy the λ-cone property. This can never happen in $\mathbb{R}^2$ though: it is shown that all bounded planar curves with the λ-cone property have finite length.

Key words Self-contracted curve, self-expanded curve, rectifiability, length, λ-curve, λ-cone property.

AMS Subject Classification *Primary* 28A75, 52A38 ; *Secondary* 37N40, 53A04, 53B20, 52A41.

Contents

1. Introduction

Self-contracted curves have been introduced in [2]. They attract a lot of interest, since they are intimately linked to convex foliations ([1], [7], [8]), to the proximal algorithm of a convex function and the gradient flow of a quasiconvex potential in a Euclidean space ([2], [3]) and recently to generalized flows in CAT(0) spaces ([9]). The main feature of this notion is its simple purely metric definition, which inspires developments in more general settings:

Definition 1.1. Let (M, d) be a metric space and $I \subset \mathbb{R}$ be an interval. A curve $\gamma : I \to M$ is called self-contracted, if for all $\tau \in I$, the map $t \mapsto d(\gamma(t), \gamma(\tau))$ is non-increasing on $I \cap (-\infty, \tau]$.

The length of a curve γ is defined as

$$\ell(\gamma) := \sup \Big\{ \sum_{i=0}^{m-1} d(\gamma(t_i), \gamma(t_{i+1})) \Big\},$$

where the supremum is taken over all finite increasing sequences $t_0 < t_1 < \cdots < t_m$ lying in I. The curve γ is called rectifiable, if its total variation is locally bounded around any $t \in I$, that is, its length is locally finite.

Rectifiability and asymptotic behaviour are central questions in the study of self-contracted curves. It is shown in [2] that self-contracted curves (are rectifiable and) have finite length whenever M is a bounded subset of the 2-dimensional Euclidean space. Based on ideas of [8], the aforementionned result was extended in [3], and independently in [7], to any finite dimensional Euclidean space. In [4] a further extension has been established encompassing the case where M is a compact subset of a Riemannian manifold. In [6] the result of [2] has been generalized for 2-dimensional spaces equipped with other (smooth) norms. This has been the first result of this type outside a Euclidean/Riemannian setting. An important breakthrough is eventually achieved in [10] by establishing (rectifiability and) finite length for all self-contracted curves contained on a bounded subset of any finite dimensional normed space. Finally, rectifiability of self-contracted curves in Hadamard manifolds and CAT(0) spaces is established in [9].

The aforementioned results remain valid if we replace the assumption "γ self-contracted" by the assumption "γ self-expanded". A curve γ is called *self-expanded* if for all $\tau \in I$, the map $t \mapsto d(\gamma(t), \gamma(\tau))$ is non decreasing on $I \cap [\tau, +\infty)$, or equivalently, when the curve $\overline{\gamma} : -I \to M$ given by $\overline{\gamma}(t) = \gamma(-t)$ is self-contracted. Thus, $\gamma : I \to \mathbb{R}^d$ is self-expanded if for every $t_1 \leq t_2 \leq t_3$ in I we have

$$d(\gamma(t_1), \gamma(t_2)) \leq d(\gamma(t_1), \gamma(t_3)).$$

In the Euclidean setting, there is a nice geometric interpretation of self-expandedness (see [3, Lemma 2.8]). A differentiable curve is self-expanded if and only if

$$\langle \gamma'(t), \gamma(u) - \gamma(t) \rangle \leq 0 \quad \text{for all } u \in I \text{ such that } u < t,$$

which geometrically means that the tail of the curve (the past) is always contained in half-space (cone of aperture π). The notion of self-expandedness therefore admits the following two natural generalizations. Let us fix $-1 \leq \lambda < 1$. A curve $\gamma : I \to \mathbb{R}^d$ is called λ-curve if for every $t_1 \leq t_2 \leq t_3$ in I we have

$$d(\gamma(t_1), \gamma(t_2)) \leq d(\gamma(t_1), \gamma(t_3)) + \lambda d(\gamma(t_2), \gamma(t_3)). \tag{1.1}$$

If γ is continuous and admits right derivative at each point, we say that γ has the λ-cone-property if, for every $t < \tau$ in I, we have, denoting $\gamma'(\tau)$ the right derivative,

$$\langle \gamma'(\tau), \gamma(t) - \gamma(\tau) \rangle \leq \lambda \, ||\gamma'(\tau)|| \; ||\gamma(t) - \gamma(\tau)|| \, .$$

As a matter of the fact, the λ-cone property will be defined more generally, for merely continuous curves using (forward) secants, see Definition 2.5 and it will be shown that every λ-curve has the λ-cone property (*c.f.* Proposition 2.6). However there exist smooth curves satisfying the latter property for some $\lambda_0 < 1$ without being λ-curves for any $\lambda \in [-1, 1)$ (*c.f.* Example 2.7).

In this work we establish the following results:

- if $|\cdot|$ is an equivalent norm to the Euclidean norm $||\cdot||$, then there exists $\lambda \in [0, 1)$ such that every $|\cdot|$-self-expanded curve is a $||\cdot||$-λ-curve (Proposition 2.2);
- for $\lambda < 1/d$ every bounded λ-curve (is rectifiable and) has finite length (Theorem 3.5);
- for $\lambda \geq 1/\sqrt{5}$ there exists a bounded curve in $\mathbb{R}^3$ with infinite length satisfying the λ-cone property (Theorem 4.2).

Nonetheless due to topological obstructions for $d = 2$ we have:

- for any $\lambda < 1$, bounded planar curves with the λ-cone property (and a fortiori λ-curves) have finite length (Theorem 5.3).

Combining the first and the last statement, we readily obtain that all bounded planar self-contracted curves (under any norm) are rectifiable and have finite length. This clearly generalizes the result of [6], but it is contained in the result of [10] that asserts that the same holds in any dimension. Notice that the asymptotic behaviour of both λ-curves and curves with the λ-cone property remains unknown in $\mathbb{R}^d$ for $d \geq 3$ and $\lambda \in [1/d, 1/\sqrt{5})$.

Notation. Let us fix our notation. Throughout this work $\mathbb{R}^d$ will denote the d-dimensional Euclidean space endowed with the Euclidean norm $||\cdot||$ and the scalar product $\langle \cdot, \cdot \rangle$. We denote by $\mathbb{S}^{d-1}$ the unit sphere of $\mathbb{R}^d$, and by $B(x,r)$ (respectively, $\overline{B}(x,r)$) the open (respectively, closed) ball of radius $r > 0$ and center $x \in \mathbb{R}^d$. A (convex) subset C of $\mathbb{R}^d$ is called a (convex) cone, if for every $x \in C$ and $r > 0$ it holds $rx \in C$. If A is a nonempty subset of $\mathbb{R}^d$, we denote by $\mathrm{int}(A)$ its interior, by $\mathrm{conv}\,(A)$ its convex hull and by $\mathrm{diam}\, A := \sup\{d(x,y) : x, y \in A\}$ its diameter.

Given a closed convex subset K of $\mathbb{R}^d$, the normal cone $N_K(u_0)$ of K at $u_0 \in K$ is the following closed convex cone (see [11] *e.g.*):

$$N_K(u_0) = \{v \in \mathbb{R}^n : \langle v, u - u_0 \rangle \leq 0, \forall u \in K\}.$$

Notice that $u_0 \in K$ is the projection onto K of all elements of the form $u_0 + tv$, where $t \geq 0$ and $v \in N_K(u_0)$. In the particular case that K is a closed convex pointed cone (that is, K contains no lines), then its polar (or dual) cone

$$K^o := N_K(0) = \{v \in \mathbb{R}^n : \langle v, u \rangle \leq 0, \forall u \in K\}$$

has nonempty interior and the bipolar theorem holds: $K^{oo} = K$. For $\delta > 0$ sufficiently small, we denote by K_δ the δ-enlargement of the cone K, that is, the closed convex cone generated by the set $\left(K \cap \mathbb{S}^{d-1}\right) + B_\delta$, where $B_\delta := B(0, \delta)$. Notice that

$$\Big((K_\delta)^o \cap \mathbb{S}^{d-1}\Big) + B_\delta \ \subset \ K^o. \tag{1.2}$$

We define the aperture $A(S)$ of a nonempty subset $S \subset \mathbb{S}^{d-1}$ by

$$A(S) := \inf \{ \langle u_1, u_2 \rangle \ : \ u_1, u_2 \in S \}. \tag{1.3}$$

Based on the above notion, we define the aperture $\mathcal{A}(C)$ of a nontrivial convex pointed cone C as follows:

$$\mathcal{A}(C) = \arccos \Big(A(C \cap \mathbb{S}^{d-1}) \Big).$$

Given $v \in \mathbb{S}^{d-1}$ and $\alpha \in [0, \pi)$, we define the "open" cone directed by v as follows:

$$C(v, \alpha) = \Big\{ u \in \mathbb{R}^d : \ \langle u, v \rangle > \ ||u|| \, \cos\alpha \Big\} \cup \{0\}. \tag{1.4}$$

Notice that if $\alpha < \pi/2$, the above cone is convex and has aperture 2α. Given $x \in \mathbb{R}^d$, we adopt the notation

$$C_x(v, \alpha) := x + C(v, \alpha). \tag{1.5}$$

A mapping $\gamma : I = [0, T_\infty) \to \mathbb{R}^d$, where $T_\infty \in \mathbb{R} \cup \{+\infty\}$ is referred in the sequel as a curve. Although the usual definition of a curve comes along with continuity and injectivity requirements for the map γ, we do not make these prior assumptions here. By the term continuous (respectively, absolutely continuous, Lipschitz, smooth) curve we shall refer to the corresponding properties of the mapping $\gamma : I \to \mathbb{R}^d$. A curve γ is said to be bounded if its image, denoted by $\Gamma = \gamma(I)$, is a bounded set of $\mathbb{R}^d$.

For $t \in I$ we denote by $\Gamma(t) := \{\gamma(t') \in \Gamma : t' \leq t\}$ the initial part of the curve and by

$$K(t) = \overline{\mathrm{cone}}\, (\Gamma(t) - \gamma(t)) \tag{1.6}$$

the closed convex cone generated by $\Gamma(t)$. In particular

$$\Gamma(t) \subset \gamma(t) + K(t) \tag{1.7}$$

Notice further that $K(t)$ contains the set $\sec^-(t)$ of (all possible limits of) backward secants at $\gamma(\tau)$ which is defined as follows (see [3]):

$$\sec^-(t) := \left\{ q \in \mathbb{S}^{d-1} : q = \lim_{t_k \nearrow t^-} \frac{\gamma(t_k) - \gamma(t)}{||\gamma(t_k) - \gamma(t)||} \right\},$$

where the notation $\{t_k\}_k \nearrow t^-$ indicates that $\{t_k\}_k \to t$ and $t_k < t$ for all k.

The set $\sec^+(t)$ of all possible limits of forward secants at $\gamma(t)$ is defined analogously:

$$\sec^+(t) := \left\{ q \in \mathbb{S}^{d-1} : q = \lim_{t_k \searrow t^+} \frac{\gamma(t_k) - \gamma(t)}{||\gamma(t_k) - \gamma(t)||} \right\},$$

where the notation $\{t_k\}_k \searrow t^+$ indicates that $\{t_k\}_k \to t$ and $t < t_k$ for all k. Compactness of $\mathbb{S}^{d-1}$ guarantees that both $\sec^-(t)$ and $\sec^+(t)$ are nonempty. If $\gamma : I \to \mathbb{R}^d$ is differentiable at $t \in I$ and $\gamma'(t) \neq 0$, then $\sec^+(t) = \left\{ \frac{\gamma'(t)}{||\gamma'(t)||} \right\}$.

In this work we introduce two new notions, depending on a parameter $\lambda \in [-1, 1)$. For each value of λ we obtain the class of λ-curves and the class of curves with the λ-cone property. We associate to these classes an angle $\alpha \in (0, \pi]$ via the relation

$$\alpha = \arccos(\lambda). \tag{1.8}$$

As we shall see, the above classes enjoy interesting geometric properties which can be described in terms of the angle α. (For $\lambda = 0$, which corresponds to the angle $\alpha = \pi/2$, the above classes coincide and yield the class of self-expanded curves.)

2. λ-curves and curves with the λ-cone property

Definition 2.1 (λ-curve)**.** A curve $\gamma : I \to \mathbb{R}^d$ is called λ-curve ($-1 \leq \lambda < 1$) if for every $t_1 \leq t_2 \leq t_3$ in I we have

$$d(\gamma(t_1), \gamma(t_2)) \leq d(\gamma(t_1), \gamma(t_3)) + \lambda d(\gamma(t_2), \gamma(t_3)). \tag{2.1}$$

The above definition yields that every λ-curve is necessarily injective and cannot admit more than one accumulation point. Based on this, one can easily see that every λ-curve has at most countable discontinuities. Setting $\lambda = 1$ to (2.1) yields the triangle inequality of the distance (hence no restriction) while $\lambda = -1$ corresponds to segments. On the other hand, for $\lambda = 0$ we recover the definition of a self-expanded curve. The following result shows that the study of self-contracted/self-expanded curves with respect to a non-Euclidean norm can be shifted to the study of λ-curves in the Euclidean setting.

Proposition 2.2 (self-expanded vs λ-curve)**.** *Let $\|\cdot\|$ be an Euclidean norm in $\mathbb{R}^d$ and $|\cdot|$ be another norm in $\mathbb{R}^d$. Then there exists $\lambda < 1$ (depending on the equivalence constant of the norms) such that every $|\cdot|$-self-expanded curve is a $\|\cdot\|$-λ-curve.*

Proof. Since the norms $|\cdot|$ and $\|\cdot\|$ are equivalent and since the properties of being self-expanded or being a λ-curve are invariant by homothetic transformation, we may assume that there exists $\delta > 0$ such that for all $x \in \mathbb{R}^d$, $\delta\|x\| \leq |x| \leq \|x\|$. Let $t_0 < t_1 < t_2$ in I and set $x_0 = \gamma(t_0)$, $x = \gamma(t_1)$ and $y = \gamma(t_2)$. It follows by assumption that

$$|x - x_0| \leq |y - x_0|. \tag{2.2}$$

To establish the result it is sufficient to prove that there exists $\lambda < 1$ such that for all choices of x_0, x, y satisfying (2.2), we have

$$\|x - x_0\| \leq \|y - x_0\| + \lambda\|x - y\|.$$

By translation, we may, and do assume, that $x_0 = 0$. Moreover, by homogeneity, we can assume $|y| = 1$. Set

$$B = \{z \in \mathbb{R}^d;\, |z| \leq \delta\} \quad \text{and} \quad C_y = \{y + t(y - z);\, \|z\| < \delta,\, t > 0\}.$$

We claim that $B \cap C_y = \emptyset$. Indeed, fix z such that $\|z\| < \delta$. The function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by $\varphi(t) = |y + t(y - z)|$ is convex, $\varphi(-1) < 1$ and $\varphi(0) = 1$, hence $\varphi(t) > 1$ whenever $t > 0$, that is, $y + t(y - z) \notin B$. Since this is true for all z satisfying $\|z\| < \delta$, the claim is proved.

Therefore Proposition 2.2 is a consequence of the following lemma.

Lemma 2.3. *There exists $\lambda < 1$ such that, whenever $1 \leq \|y\| \leq 1/\delta$, $\|x\| \leq 1/\delta$ and $x \notin C_y$, then $\|x\| - \|y\| \leq \lambda\|x - y\|$.*

Proof. Set $u = x - y$ and $\Gamma_y := \{t(y - z);\, ||z|| < \delta,\, t > 0\}$. We claim that there exists $\rho < 1$ such that, whenever $1 \leq ||y|| \leq 1/\delta$ and $u \in \mathbb{R}^d \diagdown \Gamma_y$, then

$$\langle u, y\rangle \leq \rho\, ||u|| \cdot ||y||. \tag{2.3}$$

Indeed, since $\mathbb{R}^d \diagdown \Gamma_y$ is a cone, it is enough to establish (2.3) when $\|u\| = \|y\|$. Let us denote by $c(u, y)$ the cosine of the angle of the two vectors u and y. The condition $u \notin \Gamma_y$ yields $\|u - y\| \geq \delta$. Then we obtain $\|u - y\|^2 = \|y\|^2(2 - 2c(u, y)) \geq 1$, which yields

$$c(u, y) \;\leq\; 1 - \frac{\delta^2}{2\|y\|^2} \;\leq\; 1 - \frac{\delta^4}{2} := \rho \;<\; 1.$$

This proves the claim.

Since $||y + u||^2 \leqslant ||y||^2 + ||u||^2 + 2||y||\,||u||\,\rho$, we deduce from (2.3) that

$$||x|| - ||y|| = ||y + u|| - ||y|| \;\leq\; ||y|| \left(\sqrt{1 + \frac{2\rho||u||}{||y||} + \frac{||u||^2}{||y||^2}} - 1 \right).$$

Since $||u|| = ||x - y|| \leq ||x|| + ||y|| \leq 2/\delta$ and $||y|| \geq |y| = 1$, we have $t = \frac{||u||}{||y||} \in [0, 2/\delta]$. Notice that taking $\lambda < 1$ sufficiently close to 1, we ensure that for all $t \in [0, 2/\delta]$ it holds

$$\sqrt{1 + 2\rho t + t^2} - 1 \leq \lambda t.$$

Therefore we conclude that

$$||x|| - ||y|| \leq \lambda||u|| = \lambda||x - y||.$$

The proof is complete. □

From now on we consider exclusively a Euclidean setting. An important feature of the notion of λ-curve is the following property:

Proposition 2.4 (uniform non-collinearity)**.** *Let $\gamma : I \to \mathbb{R}^d$ be a λ-curve. Then, γ is λ-uniformly non-collinear, that is, for every $s, u, t \in I$ such that $s, u \leq t$ we have*

$$\left\langle \frac{\gamma(u) - \gamma(t)}{\|\gamma(u) - \gamma(t)\|}, \frac{\gamma(s) - \gamma(t)}{\|\gamma(s) - \gamma(t)\|} \right\rangle \;>\; -\lambda \quad (> -1)\,. \tag{2.4}$$

Proof. Assume that $u < s < t$. Because γ is λ-curve we have that

$$d(\gamma(u), \gamma(s)) \;\leq\; d(\gamma(u), \gamma(t)) \;+\; \lambda\, d(\gamma(s), \gamma(t))$$

Consider the triangle of vertices $\gamma(t)$, $\gamma(u)$ and $\gamma(s)$ and set $c = d(\gamma(u), \gamma(s))$, $a = d(\gamma(u), \gamma(t))$ and $b = d(\gamma(s), \gamma(t))$. The previous equation now reads $c \leq a + \lambda b$, and after squaring both sides we get

$$c^2 \leq a^2 + \lambda^2 b^2 + 2\lambda ab. \tag{2.5}$$

Evoking the law of cosine $c^2 = a^2 + b^2 - 2ab\cos\varphi$ we deduce

$$\cos\varphi = \frac{a^2 + b^2 - c^2}{2ab} \overset{(2.5)}{\geq} \frac{(1-\lambda^2)b^2 - 2\lambda ab}{2ab} > -\lambda,$$

that is, the angle φ between the vectors

$$\frac{\gamma(u) - \gamma(t)}{\|\gamma(u) - \gamma(t)\|} \quad \text{and} \quad \frac{\gamma(s) - \gamma(t)}{\|\gamma(s) - \gamma(t)\|}$$

is strictly less than $\pi - \alpha$ ($\alpha = \arccos(\lambda)$ is given by (1.8)). □

Before we proceed, we give the following definition.

Definition 2.5 (λ-cone property)**.** Let $\lambda \in [-1, 1)$ and $\alpha = \arccos(\lambda)$. We say that a continuous curve $\gamma : I \to \mathbb{R}^d$ satisfies the λ-cone property if for every $t \in I$ and for every $q_t^+ \in \sec^+(t)$ it holds

$$\left\langle q_t^+, \frac{\gamma(u) - \gamma(t)}{\|\gamma(u) - \gamma(t)\|} \right\rangle \leq \lambda, \quad \text{for all } u < t. \tag{2.6}$$

In other words, recalling (1.4), the set $\Gamma(t) - \gamma(t)$ does not intersect the cone $C\left(q_t^+, \alpha\right)$ directed by q_t^+ and of aperture 2α expect at 0, that is, for every $t \in I$

$$\left(\gamma(t) + \bigcup_{q_t^+ \in \sec^+(t)} C\left(q_t^+, \alpha\right)\right) \bigcap \; \Gamma(t) \;=\; \{\gamma(t)\}\,. \tag{2.7}$$

We shall now consider a second important feature of the class of (continuous) λ-curves.

Proposition 2.6 (λ-curve $\Longrightarrow$ λ-cone property)**.** *Every continuous λ-curve has the λ-cone property.*

Proof. Fix $t \in I$, let $u < t$, $q_t^+ \in \sec^+(t)$ and choose $\{t_k\}_k \searrow t$ such that

$$\frac{\gamma(t_k) - \gamma(t)}{\|\gamma(t_k) - \gamma(t)\|} \longrightarrow q_t^+.$$

Since γ is a λ-curve we have

$$\|\gamma(t) - \gamma(u)\| \leq \|\gamma(t_k) - \gamma(u)\| + \lambda\|\gamma(t_k) - \gamma(t)\|,$$

yielding

$$\frac{\|\gamma(t_k) - \gamma(u)\| - \|\gamma(t) - \gamma(u)\|}{||\gamma(t_k) - \gamma(t)||} \geq -\lambda$$

Set $\Phi(X) = ||X||$, $X_k = \gamma(t_k) - \gamma(u)$ and $X = \gamma(t) - \gamma(u)$. Then the above inequality reads

$$\frac{\Phi(X_k) - \Phi(X)}{||X_k - X||} \geq -\lambda.$$

Since the norm is differentiable around the segment $[X, X_k] := \{tX + (1-t)X_k : t \in [0,1]\}$, applying the Mean Value theorem we obtain $\theta_k \in [0,1)$ such that

$$\Phi(X_k) - \Phi(X) = D\Phi(X + \theta_k(X_k - X))(X_k - X) = \left\langle \frac{X + \theta_k(X_k - X)}{\|X + \theta_k(X_k - X)\|}, X_k - X \right\rangle.$$

Combining the above formulas and taking the limit as $k \to \infty$ we get

$$\left\langle \frac{\gamma(t) - \gamma(u)}{\|\gamma(t) - \gamma(u)\|}, q_t^+ \right\rangle \geqslant -\lambda.$$

The above is equivalent to (2.6) and the proof is complete. □

The following example reveals that there exist C^1curves satisfying the λ-cone property but failing to satisfy the non-collinearity property. Therefore these curves cannot be λ-curves for any value of the parameter $\lambda \in [-1, 1)$.

Example 2.7. Let $\gamma : [-3\pi/2, 1+\pi] \to \mathbb{R}^3$ be defined by

$$\gamma(t) = \begin{cases} (0, -\sin t, -\cos t), & \text{if } t \in [-3\pi/2, -\pi/2], \\ (-\frac{1}{2}(1+\cos 2t), 1, \frac{1}{2}\sin 2t), & \text{if } t \in [-\pi/2, 0], \\ (-1, 1, t), & \text{if } t \in [0, 1], \\ (-1, \frac{1}{2}(1+\cos 2(t-1)), 1 + \frac{1}{2}\sin 2(t-1)), & \text{if } t \in [1, 1+\pi/2], \\ (-\sin(t-1), 0, 1+\cos(t-1)), & \text{if } t \in [1+\pi/2, 1+\pi]. \end{cases}$$

It is easy to check that γ is C^1-smooth. Moreover, γ fails to satisfy the non-collinearity property: indeed, $\gamma(1+\pi) = (0,0,0)$ is the midpoint of the segment $[\gamma(-3\pi/2), \gamma(-\pi/2)]$. Hence, by Proposition 2.4, γ cannot be a λ-curve for any value of the parameter $\lambda < 1$. On the other hand, any tangent line

$$\{\gamma(t) + s\gamma'(t);\ s \in \mathbb{R}\}$$

meets the curve $\{\gamma(\tau);\ \tau \in [-3\pi/2, 1+\pi]\}$ only at the point $\gamma(t)$. Therefore, by a simple compactness argument, there exists $\lambda_0 < 1$ for which γ satisfies the λ_0-cone property.

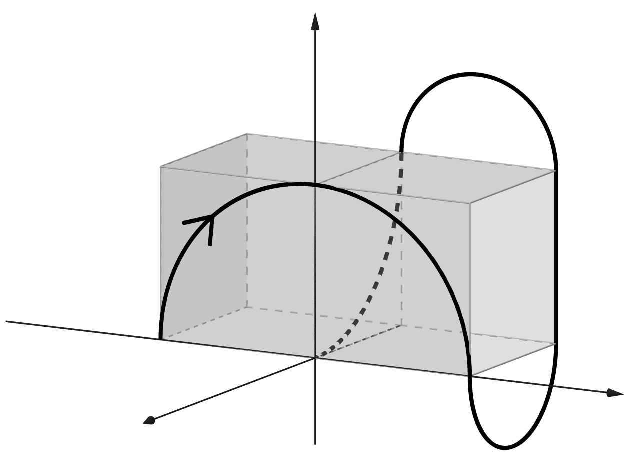

FIGURE 1. Example of a curve with the λ_0-cone property, failing to be λ-curve for any $\lambda < 1$.

3. Length of λ-curves

Before we proceed we recall from [3] the following result (we provide a proof for completeness).

Lemma 3.1. *Let $\Sigma \subset \mathbb{S}^{d-1}$ (the unit sphere of $\mathbb{R}^d$, $d > 1$) and assume that for $\lambda < 1/d$ it holds*

$$\langle x, x' \rangle \geq -\lambda, \quad \text{for all } x, x' \in \Sigma.$$

Then Σ is contained in a half-sphere (therefore it generates a closed convex pointed cone).

Proof. Notice that the conclusion holds if and only if $0 \notin \operatorname{conv}(\Sigma)$. Let us assume that $0 \in \operatorname{conv}(\Sigma)$. Then by Caratheodory theorem, there exist $\alpha_0, \alpha_1, \cdots, \alpha_d \geq 0$ and $x_0, x_1, \cdots, x_d \in \mathbb{S}^{d-1}$ such that

$$\sum_{i=0}^{d} \alpha_i = 1 \quad \text{and} \quad \sum_{i=0}^{d} \alpha_i x_i = \mathbf{0}.$$

It follows that for $j \in \{0, 1, \ldots, d\}$,

$$0 = \langle \mathbf{0}, x_j \rangle = \sum_{i=0}^{d} \alpha_i \langle x_i, x_j \rangle \geq \alpha_j - \lambda \sum_{i \neq j} \alpha_i = \alpha_j - \lambda(1 - \alpha_j).$$

Summing up for all $j \in \{0, 1, \ldots, d\}$ we get $0 \geq 1 - \lambda(d+1-1)$, which contradicts the assumption $\lambda < 1/d$. ☐

Recalling the notation of (1.3), (1.7) and (1.8), and assuming $\lambda < 1/d$ we obtain the following result (as a straightforward combination of Lemma 3.1 with Proposition 2.4).

Corollary 3.2 (conical control of the initial part)**.** *Let $-1 \leq \lambda < 1/d$ and $\alpha = \arccos(\lambda)$. Then for every $t \in I$, the initial part $\Gamma(t)$ of a λ-curve γ is contained in a closed convex cone $K(t)$ of aperture at most $\pi - \alpha$ centered at $\gamma(t)$. In other words,*

$$\Gamma(t) \subset \gamma(t) + K(t) \quad \text{and} \quad \mathcal{A}(K(t)) \leq \pi - \alpha. \tag{3.1}$$

To sum up, given a continuous λ-curve γ, Proposition 2.6 ensures that its initial part $\Gamma(t)$ avoids the union of all cones centered at $\gamma(t)$ and directed by forward secants of γ at t, see (2.7), while Corollary 3.2 asserts that, provided $\lambda < 1/d$, the initial part of the curve $\Gamma(t)$ is itself contained in the closed convex pointed cone $\gamma(t) + K(t)$, centered at $\gamma(t)$. The following proposition asserts that an even stronger property is satisfied.

Proposition 3.3 (conical split at each t)**.** *Let $\gamma : I \to \mathbb{R}^d$ be a continuous λ-curve, with $\lambda \in [-1, 1/d)$ and $\alpha = \arccos(\lambda)$. Then it holds:*

$$\left(\bigcup_{q_t^+ \in \sec^+(t)} C\left(q_t^+, \alpha\right) \right) \bigcap \; K(t) \;=\; \{0\}, \qquad \text{for all } \; t \in I. \tag{3.2}$$

Proof. Assume towards a contradiction that for some $q_t^+ \in \sec^+(t)$ there exists $q \in C\left(q_t^+, \alpha\right) \cap K(t)$, $q \neq 0$. This yields, in view of Proposition 2.6, that $\operatorname{int} K(t)$ is nonempty. Therefore, since q satisfies the open condition

$$\langle q_t^+, q \rangle \;>\; \lambda = \cos \alpha,$$

there is no loss of generality to assume that $q \in \operatorname{int} K(t)$. Therefore, there exist $t_1 < t_2 < \ldots < t_d < t$ and $\{\mu_i\}_{i=1}^d \subset \mathbb{R}_+$ such that

$$u_i := \frac{\gamma(t_i) - \gamma(t)}{\|\gamma(t_i) - \gamma(t)\|} \quad \text{and} \quad q = \sum_{i=1}^{d} \mu_i \, u_i.$$

Fix $\varepsilon > 0$ such that $\langle q_t^+, q\rangle > \lambda + 3\varepsilon$. By continuity, there exists $\delta > 0$ such that for all $s \in (t, t+\delta)$ the vectors

$$\tilde{u}_i := \frac{\gamma(t_i) - \gamma(s)}{\|\gamma(t_i) - \gamma(s)\|}, \quad i \in \{1, \ldots, d\},$$

are sufficiently close to $\{u_i\}_{i=1}^d$ to ensure that

$$\langle q_t^+, \tilde{q}\rangle > \lambda + 2\varepsilon, \qquad \text{where} \quad \tilde{q} = \sum_{i=1}^{n} \mu_i \, \tilde{u}_i.$$

Take now $s \in (t, t+\delta)$ in a way that the vector $\hat{q} = (\|\gamma(s) - \gamma(t)\|)^{-1} (\gamma(s) - \gamma(t))$ is sufficiently close to the secant q_t^+ so that $\langle \hat{q}, \tilde{q}\rangle > \lambda + \varepsilon$ or equivalently, $\langle -\hat{q}, \tilde{q}\rangle < -\lambda - \varepsilon$. Since $\tilde{q}, -\hat{q} \in K(s) \cap \mathbb{S}^{d-1}$, we deduce that $\mathcal{A}(K(s)) > \pi - \alpha$, which contradicts Corollary 3.2 for $s = t$. □

We shall finally need the following lemma.

Lemma 3.4. *Let $\gamma : I \to \mathbb{R}^d$ be a continuous λ-curve, with $\lambda \in [-1, 1/d)$ and $\alpha = \arccos(\lambda)$. Then there exists $\rho > 0$ such that for every $t \in I$ and $q_t^+ \in \sec^+(t)$, there exists $\xi_t \in \mathbb{S}^{d-1}$ satisfying*

$$\langle \xi_t, u\rangle \leq -\rho < 0, \quad \textit{for all } u \in K(t) \tag{3.3}$$

and

$$\langle \xi_t, q_t^+\rangle \geq \rho > 0\,. \tag{3.4}$$

Proof. Let $\delta \leq \sqrt{2(1-\lambda)}$ and $\rho = \delta/2$. Then for every $t \in I$ and $q_t^+ \in \sec^+(t)$, we have $\mathbb{S}^{d-1} \cap B(q_t^+, \delta) \subset C(q_t^+, \alpha)$. We deduce from Proposition 3.3 that the δ-enlargement of the cone $K(t)$ satisfies:

$$K(t)_\delta \cap \sec^+(t) = \emptyset.$$

Setting $\tilde{N}(t) = (K(t)_\delta)^o$ and $N(t) = K(t)^o$ (the polar of $K(t)_\delta$ and $K(t)$ respectively), we deduce by (1.2) that

$$\bar{B}(\xi, \delta) \cap \mathbb{S}^{d-1} \subset N(t), \quad \text{for every } \xi \in \tilde{N}(t) \cap \mathbb{S}^{d-1}. \tag{3.5}$$

Let us now fix $q_t^+ \in \sec^+(t)$. Then by the bipolar theorem we get $q_t^+ \notin \tilde{N}(t)^o = K(t)_\delta$, that is, there exists $\tilde{\xi} \in \tilde{N}(t) \cap \mathbb{S}^{d-1}$ such that $\langle \tilde{\xi}, q_t^+\rangle > 0$. Maximizing the functional q_t^+ over the closed ball $\bar{B}(\tilde{\xi}, \rho)$ we obtain $\xi_t \in \mathbb{S}^{d-1}$ such that (3.4) holds. Since $B(\xi_t, \rho) \subset B(\tilde{\xi}, \delta) \subset N(t)$, we easily deduce that (3.3) also holds. □

We are now ready to prove the main result of this section.

Theorem 3.5 (rectifiability)**.** *Every continuous λ-curve $\gamma : I \to \mathbb{R}^d$ with $\lambda < 1/d$ is rectifiable. In particular, bounded λ-curves with $\lambda < 1/d$ have finite length.*

Proof. We may assume that $I = [0, +\infty)$ and that γ is bounded. Set $\eta = \rho/3$, where ρ is given by Lemma 3.4. Since $\mathbb{S}^{d-1}$ is compact, there exists an η-net $\mathcal{F} := \{\xi_1, \cdots, \xi_N\}$, satisfying that for every $v \in \mathbb{S}^{d-1}$, there exists $i \in \{1, \cdots, N\}$ such that $\langle v, \xi_i\rangle > \eta$ (that is, v is η-close to some $\xi_i \in \mathcal{F}$). Then we deduce from Lemma 3.4 that for every $t \in I$ and $q_t^+ \in \sec^+(t)$, there exists $\xi_i \in \mathcal{F}$ such that

$$\langle \xi_i, q^+\rangle > 2\eta \qquad \text{and} \qquad \langle \xi_i, u\rangle \leq -2\eta < 0, \quad \text{for all } u \in K(t). \tag{3.6}$$

Reasoning by contradiction we can prove the existence of some $\delta_t > 0$ such that for every $s \in [t, t+\delta_t)$ there exists $q_{t,s}^+ \in \sec^+(t)$ such that

$$\left\| \frac{\gamma(s) - \gamma(t)}{||\gamma(s) - \gamma(t)||} - q_{t,s}^+ \right\| < \eta. \tag{3.7}$$

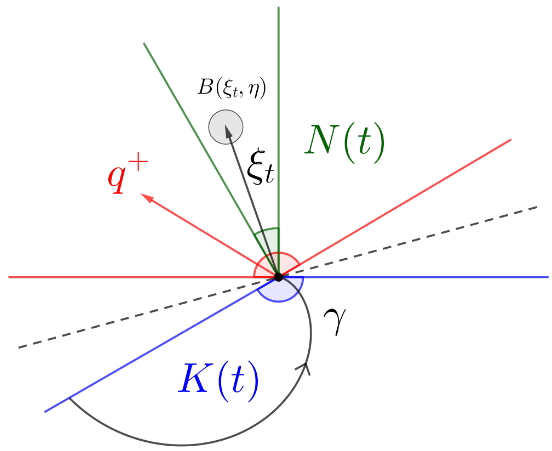

FIGURE 2. The initial part of the curve generates the cone $K(t)$ (in blue) with aperture $\mathcal{A}(K(t)) \leq \pi - \alpha$ and avoids the cone generated by the positive secants (in red).

Combining the above we deduce that for every $t \in I$ and $s \in [t, t+\delta_t)$, there exists $\xi_i \in \mathcal{F}$ such that

$$\langle \xi_i, \gamma(s) - \gamma(t) \rangle \geq \eta \, \|\gamma(s) - \gamma(t)\|. \tag{3.8}$$

On the other hand, it follows directly from (3.6) that for every $\tau \in [0, t)$

$$\langle \xi_i, \gamma(t) - \gamma(\tau) \rangle \geq \eta \, \|\gamma(t) - \gamma(\tau)\|. \tag{3.9}$$

Considering for $i \in \{1, \ldots, N\}$ the projection operator

$$\begin{cases} \pi_i : \mathbb{R}^d \to \mathbb{R}\xi_i \\ \pi_i(x) = \langle \xi_i, x \rangle \, \xi_i \end{cases}$$

we define $W_i(t)$ to be the width of the projection of the initial part of the curve $\Gamma(t)$ onto $\mathbb{R}\xi_i$, that is,

$$W_i(t) := \mathcal{H}^1(\pi_i(\Gamma(t))), \quad t \in I,$$

where $\mathcal{H}^1$ denotes the 1-dimensional Lebesgue measure. Notice that $\mathcal{H}^1(\pi_i(\Gamma(t)))$ is simply the length of the bounded interval $\pi_i(\Gamma(t))$ of $\mathbb{R}\xi_i$. It follows readily that for every $i \in \{1, \ldots, N\}$ the function $t \mapsto W_i(\tau)$ is non-decreasing on $[0, T_\infty)$ and bounded above by $r := \operatorname{diam}(\gamma(I))$. Therefore, the function

$$W_{\mathcal{F}}(t) := \sum_{i=1}^{N} W_i(t),$$

is non-decreasing on I and bounded above by Nr. We now deduce from (3.8) and (3.9) that for every $t \in I$ there exists $\delta_t > 0$ such that for all $s \in [t, t+\delta_t)$ we have

$$W_{\mathcal{F}}(s) - W_{\mathcal{F}}(t) \geq \eta \, \|\gamma(s) - \gamma(t)\|. \tag{3.10}$$

The result follows via a standard argument if we establish that for any $a, b \in I$ with $a < b$ it holds:

$$W_{\mathcal{F}}(b) - W_{\mathcal{F}}(a) \geq \eta \, \|\gamma(b) - \gamma(a)\|. \tag{3.11}$$

Let us assume, towards a contradiction, that (3.11) does not hold, that is,

$$W_{\mathcal{F}}(b) - W_{\mathcal{F}}(a) + \varepsilon < \eta \, \|\gamma(b) - \gamma(a)\|, \quad \text{for some } \varepsilon > 0.$$

Set $\sigma(t) = \sup\{s > t : \text{ (3.10) holds}\}$, for $t \in [a, b)$. Then our assumption yields that for every $t \in [a, b)$ we have $a \le t + \delta_t \le \sigma(t) < b$. Using transfinite induction we construct a (necessarily) countable set $\Lambda = \{t_\mu\}_{\mu \le \hat{\varsigma}}$ by setting $t_1 = a$, $t_\mu = \sigma(t_{\mu^-})$ if $\mu = \mu^- + 1$ is a successor ordinal, and $t_\mu = \sup\{t_\nu : \nu < \mu\}$ if μ is a limit ordinal and we stop when $t_{\hat{\varsigma}} = b$. Let now $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0, \varepsilon)$ with $\sum_{n\in N} \varepsilon_n = \varepsilon\eta^{-1}$. Let $i : \Lambda \to \mathbb{N}$ be an injection of Λ into $\mathbb{N}$. Then denoting by μ^+ the successor of μ, we obtain by continuity, that for each ordinal μ there exists $t_\mu \le s_\mu < \sigma(t_\mu) := t_{\mu^+}$ such that $||\gamma(s_\mu) - \gamma(t_{\mu^+})\| < \varepsilon_{i(\mu)}$. We deduce by (3.10):

$$\begin{aligned}\|\gamma(b) - \gamma(a)\| &\le \sum_{\mu\in\Lambda} \|\gamma(t_{\mu^+}) - \gamma(t_\mu)\| \le \sum_{\mu\in\Lambda} \left(\|\gamma(s_\mu) - \gamma(t_\mu)\| + \varepsilon_{i(\mu)}\right)\\ &\le \frac{1}{\eta}\left(\sum_{\mu\in\Lambda} \left(W_{\mathcal{F}}(s_\mu) - W_{\mathcal{F}}(t_\mu)\right) + \varepsilon\right) \le \frac{1}{\eta}\left(W_{\mathcal{F}}(b) - W_{\mathcal{F}}(a) + \varepsilon\right),\end{aligned}$$

which contradicts (3.11). □

Remark 3.6 (universal constant)**.** The above proof reveals that the length $\ell(\gamma)$ of any λ-curve lying in a set of diameter r is bounded by the quantity $N \cdot \eta^{-1} \cdot r$. Since the constant $\eta > 0$ is determined in Lemma 3.4, it only depends on λ and the dimension d of the space (in particular, it is independent of the specific λ-curve γ). Since N (the cardinality of the net $\mathcal{F}$) also depends exclusively on η and the dimension d, we conclude that for a given $\lambda \in [-1, 1/d)$ there exists a prior bound for the lengths of all λ-curves γ lying inside a prescribed bounded subset of $\mathbb{R}^d$.

Remark 3.7 (Double cone property)**.** A close inspection of Theorem 3.5 shows that the proof depends exclusively on (3.3)–(3.4) which in turn depend on (3.2). Therefore, every bounded continuous curve γ satisfying (3.2) has finite length.

4. A bounded curve with the λ-cone property and infinite length

In this section we consider continuous right differentiable curves $\gamma : I \to \mathbb{R}^d$ satisfying the λ-cone property (Definition 2.5). In the sequel we denote by $\gamma'(\tau)$ the right derivative of γ at the point τ and we assume this derivative is nonzero. Observe that in this case we have

$$\sec^+(t) = \left\{\frac{\gamma'(t)}{\|\gamma'(t)\|}\right\}.$$

So γ satisfies the λ-cone property if, for all $t, \tau \in I$ with $t < \tau$, (2.6) holds, or equivalently:

$$\langle \gamma'(\tau), \gamma(t) - \gamma(\tau)\rangle \le \lambda\, ||\gamma'(\tau)||\, ||\gamma(t) - \gamma(\tau)||.$$

This means that the angle between the vectors $\gamma'(\tau)$ and $\gamma(t) - \gamma(\tau)$ is greater or equal to α, where $\alpha = \arccos(\lambda)$. We simplify the notation by setting

$$C(t, \alpha) := \gamma(t) + C\left(\frac{\gamma'(t)}{\|\gamma'(t)\|}, \alpha\right). \tag{4.1}$$

A curve γ satisfying the above property will be also called a λ-eel. The reason is as follows: the set $\Gamma(\tau) := \{\gamma(t);\, t \in I,\, t < \tau\}$ is the apparent body (or tail) of a λ-eel at time τ going out of a hole. The cone $C(\tau, \alpha)$ represents what the λ-eel can see at time τ. The λ-cone property just says that the λ-eel never sees its apparent tail. Notice that $\pi/2$-eels correspond to self-expanded curves. Therefore, if the range of γ is bounded and γ is a $\pi/2$-eel, then its length is finite ([3], [7]).

Recall from the introduction that a curve γ is self-expanded if for all $\tau \in I$, the map $t \mapsto d(\gamma(t), \gamma(\tau))$ is non decreasing on $I \cap [\tau, +\infty)$. The following lemma illustrates that one can also associate a Lyapunov function to λ-eels.

Lemma 4.1. *If $\gamma : I \to \mathbb{R}^d$ is a $\lambda-$eel, then the function*

$$t \mapsto \|\gamma(t_1) - \gamma(t)\| + \lambda \ell(\gamma_{|[t_1,t]})$$

is non-decreasing on $I \cap [t_1, \infty)$.

Proof. By definition,

$$\frac{d}{d\tau}(\|\gamma(\tau) - \gamma(t)\|) = \left\langle \gamma'(\tau), \frac{\gamma(\tau) - \gamma(t)}{\|\gamma(\tau) - \gamma(t)\|} \right\rangle \geq -\lambda \|\gamma'(\tau)\| \quad \forall\, t < \tau.$$

For $t < t_1 < t_2$, integrating for $\tau \in [t_2, t_3]$ we obtain

$$\int_{t_2}^{t_3} \frac{d}{d\tau}(\|\gamma(\tau) - \gamma(t)\|) d\tau \geq -\lambda \int_{t_2}^{t_3} \|\gamma'(s)\| ds \quad \forall\, t < \tau,$$

which implies

$$\|\gamma(t_3) - \gamma(t)\| - \|\gamma(t_2) - \gamma(t)\| \geq -\lambda \ell(\gamma_{|[t_2,t_3]}).$$

Since $\ell(\gamma_{|[t_2,t_3]}) = \ell(\gamma_{|[t_1,t_3]}) - \ell(\gamma_{|[t_1,t_2]})$ the conclusion follows. $\square$

Our main aim now is to prove the following result.

Theorem 4.2 (λ-eel of infinite length)**.** *Assume $\lambda = \frac{1}{\sqrt{5}}$ (i.e. $\alpha = \arccos \frac{1}{\sqrt{5}}$), and let $B = \overline{B}(0,1)$ the unit ball of $\mathbb{R}^3$. Then, there exists a λ-eel $\gamma : [0, +\infty) \to B$ of infinite length. Moreover $\lim_{t \to \infty} \gamma(t)$ exists.*

The proof of Theorem 4.2 is constructive: the construction will be carried out in three steps organized in subsections. Let us mention that the result remains true if we require γ to be $\mathcal{C}^1$-smooth (and probably even $\mathcal{C}^\infty$-smooth), but the construction would then become less transparent. Before we proceed, let us make the following remark.

Remark 4.3. Let us denote by λ_* the infimum of all λ for which there exists a bounded λ-eel of infinite length inside the unit ball of $\mathbb{R}^3$. Since for $\lambda = 0$ we obtain a self-expanded curve, it follows from the above theorem that $0 \leq \lambda_* \leq \frac{1}{\sqrt{5}}$. Notice that we cannot readily conclude that λ_* is strictly greater than 0. (Nonetheless, according to [8] or [3], for $\lambda = 0$ bounded λ-eels have finite length.)

4.1. **Helicoidal maps.** Let us start by constructing a helicoidal curve along the z-axis, which is self-expanded.

Lemma 4.4. *There exists a positive constant $\mu < 1/2$ such that, if $\gamma : \mathbb{R} \to \mathbb{R}^3$ is a spiral of the form*

$$\gamma(t) = (r \cos t,\, r \sin t,\, \mu r t)\,, \quad t \in \mathbb{R}, \tag{4.2}$$

then γ is self-expanded (hence γ satisfies the λ-cone property for all $\lambda \in [0, 1)$).

Proof. Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a spiral along a cylinder of radius $r > 0$ of the form (4.2) and let us show that γ is a self-expanded curve. By symmetry, this amounts to verify that

$$a(t) := \langle \gamma'(0), \gamma(t) - \gamma(0) \rangle \leq 0, \quad \text{for all } t < 0.$$

We check easily that $\gamma(0) = (r, 0, 0)$ and $\dot{\gamma}(0) = (0, r, \mu r)$, so that $a(t) = r^2(\sin t + \mu^2 t)$. Since $\sup\{-t^{-1}\sin t : t < 0\} < 1/4$, we deduce that there exists $\mu < \frac{1}{2}$ such that the curve γ is self-expanded. □

Notation. Throughout this subsection, γ will refer to the curve given in Lemma 4.4 and $\mu < 1/2$ will be the constant fixed there.

The following lemma says that the curve γ constructed in the previous lemma satisfies that for each τ, the associated cone $C(t, \alpha)$, $\alpha = \arccos(1/\sqrt{5})$ does not meet the z-axis, that is, the axis of evolution of the spiral curve.

Lemma 4.5. *Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a spiral of the form (4.2). If $\lambda = 1/\sqrt{5}$ and $\alpha = \arccos(\lambda)$, then the cone $C(t, \alpha)$ does not intersect the line parametrized by $\ell(z) = (0, 0, z)$.*

Proof. Under the notation of the previous lemma, it is enough to verify that for all $z \in \mathbb{R}$

$$\langle \gamma'(0), \ell(z) - \gamma(0)\rangle \leq \frac{1}{\sqrt{5}} \|\dot{\gamma}(0)\| \, \|\ell(z) - \gamma(0)\| .$$

The above condition reads

$$\mu r z \leq \frac{1}{\sqrt{5}} \sqrt{r^2(1+\mu^2)}\, \sqrt{r^2 + z^2}, \qquad \text{for all } z \in \mathbb{R}, \tag{4.3}$$

or equivalently,

$$\frac{(z/r)}{\sqrt{1 + (z/r)^2}} \leq \frac{\sqrt{(1+\mu^2)}}{\mu\sqrt{5}}, \qquad \text{for all } z \in \mathbb{R}. \tag{4.4}$$

Since $t \mapsto t^{-1}\sqrt{1+t^2}$ is decreasing for $t > 0$ and $\mu < 1/2$, we have $\mu^{-1}\sqrt{1+\mu^2} > \sqrt{5}$, therefore (4.4) is satisfied. □

We shall now enhance in the above construction to deduce that the cone $C(\tau, \alpha)$ avoids a thin (infinite) cylinder

$$\mathrm{Cyl}\,(r_0) = \{(x, y, z) \in \mathbb{R}^3;\, x^2 + y^2 = r_0^2,\, z \in \mathbb{R}\}$$

containing the z-axis. Indeed, taking $r_0 << r$ the above cylinder is very close to the z-axis, therefore we obtain (almost) the same result as before. This is formulated in the next lemma.

Lemma 4.6. *There exists an integer $N \geq 2$ such that whenever $r = N r_0$ and $\alpha = \arccos 1/\sqrt{5}$, we have:*

$$C(\alpha, \tau) \cap \mathrm{Cyl}\,(r_0) = \emptyset, \quad \textit{for all } \tau \geq 0.$$

Proof. We consider again the curve γ given by (4.2). Thanks to the symmetry, it is enough to check the assertion for $\tau = 0$. Therefore, for $\sigma(\theta, z) = (r_0\cos\theta, r_0 \sin\theta, z)$, it is enough to verify

$$\langle \gamma'(0), \sigma(\theta, z) - \gamma(0)\rangle \leq \cos\alpha \, \left\|\gamma'(0)\right\| \, \|\sigma(\theta, z) - \gamma(0)\| \qquad \forall\theta \in [0, 2\pi],\, \forall z \geq 0,$$

where $\gamma(0) = (r, 0, 0)$ and $\gamma'(0) = (0, r, \mu r)$ .The above condition reads

$$r r_0 \sin\theta + \mu r z \leq \cos\alpha \sqrt{r^2(1+\mu^2)} \sqrt{\left(r_0\cos\theta - r\right)^2 + r_0^2 \sin^2\theta + z^2} \quad \forall\theta \in [0, 2\pi],\, \forall z \geq 0.$$

Dividing by $r r_0$, setting $w = z/r_0$, and since $\cos\alpha = 1/\sqrt{5}$, we deduce

$$\sin\theta + \mu w \leq \frac{1}{\sqrt{5}} \sqrt{1+\mu^2} \sqrt{\left(\frac{r}{r_0} - 1\right)^2 + 2\frac{r}{r_0}(1 - \cos\theta) + w^2}.$$

Setting $r = Nr_0$ we obtain the condition

$$\frac{1}{\sqrt{5}} \geq \frac{1}{\sqrt{1+\mu^2}} \sup_{\theta\in[0,2\pi], w\in\mathbb{R}} \left\{ \frac{\sin\theta + \mu|w|}{\sqrt{(N-1)^2 + 2N(1-\cos\theta) + w^2}} \right\}.$$

But for any $\theta \in [0, 2\pi]$, $u = |w| \geq 0$

$$\frac{\sin\theta + \mu u}{\sqrt{(N-1)^2 + 2N(1-\cos\theta) + u^2}} \leq \frac{1+\mu u}{\sqrt{(N-1)^2 + u^2}}$$

and

$$\sup_{u\geq 0} \left\{ \frac{1+\mu u}{\sqrt{(N-1)^2 + u^2}} \right\} = \frac{1}{N-1} \sqrt{1+\mu^2(N-1)^2} \longrightarrow \mu \quad \text{as} \quad N \to +\infty.$$

Since $\mu\left(1+\mu^2\right)^{-1/2} < \left(\sqrt{5}\right)^{-1}$, we can choose N large enough such that

$$\frac{\sqrt{1+\mu^2(N-1)^2}}{(N-1)\sqrt{1+\mu^2}} < \frac{1}{\sqrt{5}}.$$

Therefore, for this choice of N, we get $C(0,\alpha) \cap \mathrm{Cyl}\,(r_0) = \emptyset$. □

Let γ be given by (4.2). We shall now include a further restriction. We shall show that the cone $C(\tau, \alpha)$ associated to γ also avoids radial segments S of the form:

$$S = \{(x, 0, 0);\ 0 \leq x \leq r\}.$$

This is the aim of the following lemma.

Lemma 4.7. *If* $\lambda = \frac{1}{\sqrt{5}}$ *and* $\alpha = \arccos(\lambda)$, *then* $C(\tau, \alpha) \cap S = \emptyset$ *for all* $\tau \geq 0$.

Proof. It is enough to verify

$$\langle \gamma'(\tau), (x,0,0) - \gamma(\tau)\rangle \leq \cos\alpha\, \|\dot\gamma(\tau)\|\, \|(x,0,0) - \gamma(\tau)\|, \qquad \text{for all } 0 \leq x \leq r,$$

where

$$\gamma(\tau) = (r\,\cos\tau,\, r\,\sin\tau,\, \mu r\tau) \quad \text{and} \quad \dot\gamma(\tau) = (-r\sin\tau,\, r\cos\tau,\, \mu r).$$

Setting $\lambda = \cos\alpha$ and simplifying by r, we obtain for all $0 \leq x \leq r$

$$-\,x\sin\tau - \mu^2 r\tau \;\leq\; \lambda\,\sqrt{1+\mu^2}\,\sqrt{(x - r\cos\tau)^2 + r^2\sin^2\tau + \mu^2 r^2\tau^2}, \tag{4.5}$$

Notice that μ satisfies $\sin\tau + \mu^2\tau > 0$ for every $\tau \geq 0$. Therefore,

$$-x\sin\tau - \mu^2 r\tau \leq -x(\sin\tau + \mu^2\tau) \leq 0,$$

so (4.5) is clearly satisfied. □

4.2. **Arbitrary long eels inside a bounded cylinder.** We are now ready to construct arbitrarily long λ-eels lying inside the following bounded cylinder:

$$\mathrm{Cyl}\,(r, [a, a+2\pi\mu r]) := \{(x,y,z) \in \mathbb{R}^3;\ x^2 + y^2 = r^2,\ a \leq z \leq a + 2\pi\mu r\}. \tag{4.6}$$

Indeed we have the following result.

Proposition 4.8. *Let* $\lambda \geq 1/\sqrt{5}$ *and let* $\mathrm{Cyl}\,(r, [a, a+2\pi\mu r])$ *be the bounded cylinder defined in (4.6). Then there exists a* λ*-eel*

$$\gamma : I \longmapsto \mathrm{Cyl}\,(r, [a, a+2\pi\mu r])$$

whose length is greater than 1. *Moreover, the initial point of* γ *lies in the upper part of the cylinder* ($z = a + 2\pi\mu r$) *while the last point lies at the bottom* ($z = a$).

Proof. Without loss of generality, we assume $a = 0$. Below, N is a fixed integer given by Lemma 4.6. Let us fix an odd integer n such that $2\pi\mu r n > 1$. Then for $1 \le k \le n$, we define internal cylinders

$$C_k := \mathrm{Cyl}\,(\frac{r}{N^{n-k}}, [0, 2\pi\mu r]) = \{(x, y, z) \in \mathbb{R}^3;\ x^2 + y^2 = \Big(\frac{r}{N^{n-k}}\Big)^2,\ 0 \le z \le 2\pi\mu r\}.$$

For $k = 2\ell + 1 \le n$ (odd) we define a downward spiral curve $\gamma_k^{\downarrow}$ as follows:

$$\gamma_k^{\downarrow}(t) = \frac{r}{N^{n-k}}\big(\cos(t),\, \sin(t),\, \mu(2\pi N^{n-k} - t)\big), \qquad \text{for } 0 \le t \le 2\pi N^{n-k}.$$

while for $k = 2\ell \le n$ (even) we define an upward spiral curve $\gamma_k^{\uparrow}$ as follows:

$$\gamma_k^{\uparrow}(t) = \frac{r}{N^{n-k}}\big(\cos(t),\, \sin(t),\, \mu t\big), \qquad \text{for } 0 \le t \le 2\pi N^{n-k}.$$

Notice that if k odd,

$$\gamma_k^{\downarrow}(0) = (\frac{r}{N^{n-k}}, 0, 2\pi\mu r) \quad \text{and} \quad \gamma_k^{\downarrow}(2\pi N^{n-k}) = (\frac{r}{N^{n-k}}, 0, 0),$$

while for k even

$$\gamma_k^{\uparrow}(0) = (\frac{r}{N^{n-k}}, 0, 0) \quad \text{and} \quad \gamma_k^{\uparrow}(2\pi N^{n-k}) = (\frac{r}{N^{n-k}}, 0, 2\pi\mu r).$$

Each spiral γ_k lies on the surface of the cylinder C_k and makes N^{n-k} loops to reach the upper part of the cylinder starting from the bottom and going upwards if k is even (respectively, to reach the bottom, starting from the upper part and going downward, if k is odd). We finally define parametrized segments e_k^+ joining the end point of $\gamma_k^{\downarrow}$ to the initial point of $\gamma_k^{\uparrow}$ (for $k = 2\ell + 1$), and respectively e_k^- joining the end point of $\gamma_k^{\uparrow}$ to the initial point of $\gamma_{k+1}^{\downarrow}$ (for $k = 2\ell$), that is:

$$e_k^+(t) = \Big(\frac{r}{N^{n-k}}(1 + t(N-1)), 0, 2\pi\mu r\Big) \ \text{ and } \ e_k^-(t) = \Big(\frac{r}{N^{n-k}}(1 + t(N-1)), 0, 0\Big), \quad t \in [0, 1].$$

The curve γ will now be defined concatenating the above curves: we start with $k = 1$ and the downward spiral $\gamma_1^{\downarrow}$ and we concatenate with the segment e_1^-. We continue with the upward spiral $\gamma_1^{\uparrow}$ and the segment e_1^+ and concatenate with $\gamma_2^{\downarrow}$ ($k = 2$), then the segment e_2^- and so on, up to the final downward spiral $\gamma_n^{\downarrow}$. The resulting curve is clearly continuous. Applying Lemma 4.6 and Lemma 4.7 we deduce that γ is a λ-eel. The length of γ is clearly greater than 1 since we cross n times the cylinder of length $2\pi\mu r$ (and we have taken $n \ge 2$ such that $2\pi\mu r n > 1$). ∎

Remark 4.9. Proposition 4.8 ensures, by rescaling, that we can construct arbitrarily long λ-eels inside arbitrarily small cylinders. The λ-eel γ inside the cylinder $\mathrm{Cyl}\,(r, [a, b])$, where $b = a + 2\pi\mu r$, is obtained by concatenating pieces of three different types:

- Type 1: a spiral going downward: $\gamma_i^{\downarrow}(t) = (\rho\cos(t), \rho\sin(t), b - \rho\mu t)$,
- Type 2: a spiral going upward: $\gamma_i^{\uparrow}(t) = (\rho\cos(t), \rho\sin(t), a + \rho\mu t)$,
- Type 3: a segment parametrized by $e_i^-(t) = (t, 0, a)$ or by $e_i^+(t) = (t, 0, b)$.

Remark 4.10. It is possible to modify slightly the above construction to get a λ-ell (with $\lambda = 1/\sqrt{5}$) $\gamma : (-\infty, 0] \to \mathrm{Cyl}\,(r, [a, a + 2\pi\mu r])$, with infinite length, and such that $\lim_{t\to-\infty} \gamma(t)$ does not exist. Since the curve constructed γ above depends on the parameter n, let us denote it γ_n. We can assume, without loss of generality, by choosing a suitable parametrization, that γ is defined on $[-n, 0]$ and that for each n, the restriction of γ_{n+1} to $[-n, 0]$ coincides with γ_n.

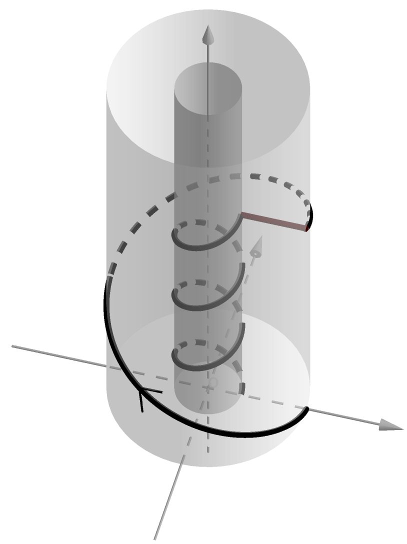

FIGURE 3. A block of the construction

Now we define γ on $(-\infty, 0]$, satisfying, for each n, $\gamma_{|[-n,0]} = \gamma_n$. Since each γ_n is a λ-ell, it is clear that γ is a λ-ell. Morover, the z-coordinate of $\gamma(t)$ oscillates infinitely many times between a and $a + 2\pi\mu r$. This shows both that γ has infinite length and that $\lim_{t\to-\infty} \gamma(t)$ does not exist.

4.3. **Constructing bounded eels of infinite length in 3D.** To construct a bounded λ-eel with infinite length, we need to glue together curves of length greater than 1 (constructed in the previous subsection) that lie each time in prescribed disjoint bounded cylinders, all taken along the z-axis, of the form $C_n := \mathrm{Cyl}\,(r_n, [a_n, b_n])$ with $a_n > b_{n+1}$ and $r_n \searrow 0^+$. To construct efficiently such a curve, and to establish that it is a λ-eel, we shall need the following result, asserting that a λ-eel lying in a small cylinder does not see a bigger remote cylinder of the same axis.

Lemma 4.11. *Let* $\lambda = 1/\sqrt{5}$, $\alpha = \arccos(\lambda)$, *and let us set*

$$\mathrm{Cyl}\,(R, [a, b]) := \{(x, y, z) \in \mathbb{R}^3;\ x^2 + y^2 \leq R,\ a \leq z \leq b\}.$$

Then there exists $M > 1$ *such that, for every* $r \in (0, R/2)$ *and* $a', b' \in \mathbb{R}$ *such that*

$$a' < b' < a < b \quad \text{and} \quad b' - a' \geq MR$$

the curve $\gamma : [a', b'] \to \mathbb{R}^3$ *with equation* $\gamma(t) = (r \cos t,\ r \sin t,\ \mu r t)$*, satisfies*

$$C(\tau, \alpha) \cap \mathrm{Cyl}\,(R, [a, b]) = \emptyset.$$

Proof. Without loss of generality, we can assume $b' = 0$. The equation of the spiral $\gamma : [0, \infty) \to \mathbb{R}^3$ is of the form

$$\gamma(t) = (r \cos t,\ r \sin t,\ \mu r t)\,, \qquad a' \leq t \leq 0,$$

where $\mu < 1/2$ is given by Lemma 4.4. Set

$$\sigma(\theta, z, u) = (u \cos\theta, u \sin\theta, z)\,, \quad \gamma(0) = (r, 0, 0) \quad \text{and} \quad \gamma'(0) = (0, r, \mu r)\,.$$

It is enough to check that

$\langle\gamma'(0), \sigma(\theta,z,u) - \gamma(0)\rangle \le \cos\alpha \,\|\gamma'(0)\|\, \|\sigma(\theta,z,u)-\gamma(0)\| \qquad \forall\theta\in[0,2\pi],\ \forall z\in[a,b],\ \forall u\in[0,R].$

The above condition reads, for all $\theta\in[0,2\pi]$, for all $z\in[a,b]$ and for all $u\in[0,R]$,

$$ru\sin\theta+\mu rz \le \frac{1}{\sqrt{5}}\sqrt{r^2(1+\mu^2)}\sqrt{(u\cos\theta-r)^2+u^2\sin^2\theta+z^2}.$$

So it is enough to check that for all $z\in[a,b]$ and $u\in[0,R]$ it holds

$$u+\mu z\le\frac{1}{\sqrt{5}}\,z\sqrt{1+\mu^2}.$$

In order to do that, let us fix the value of M. For $\mu<1/2$, we have $\sqrt{1+\mu^2}>\sqrt{5}\mu$. Therefore, we can choose $M>0$ such that $\sqrt{1+\mu^2}>\left(\mu+\frac{1}{M}\right)\sqrt{5}$. Now for all $u\le R$ and $z\ge a$ we have

$$\frac{u+\mu z}{z}\le\frac{R}{a}+\mu\le\frac{1}{M}+\mu<\frac{\sqrt{1+\mu^2}}{\sqrt{5}},$$

This completes the proof of the lemma. □

We are now ready to prove Theorem 4.2, that is, given $\lambda=\frac{1}{\sqrt{5}}$, we construct a continuous curve $\gamma:[0,+\infty]\to\mathbb{R}^3$ of infinite length, lying in the unit ball, with nonzero right derivative at each point and satisfying the λ-cone property (λ-eel).

Proof of Theorem 4.2. We claim that we can construct a sequence of disjoint bounded cylinders

$$C_n=\mathrm{Cyl}\,(r_n,[a_n,a_n+2\pi\mu r_n]),\quad n\ge1$$

along the z-axis, such that $a_n\in[0,1)$, $r_{n+1}\le r_n/2$, $\ell_n:=a_n-(a_{n+1}+2\pi\mu r_{n+1})>0$ and ℓ_n/r_n is sufficiently big to ensure that the cylinder C_n is not seen by any λ-eel lying in a (smaller) cylinder C_m for $m>n$ (*c.f.* Lemma 4.11). More precisely, we define $a_0=0$, and, for $n\ge1$,

$$a_n=2^{-n}\quad\text{and}\quad r_n=\frac{1}{2^{n+1}(\pi\mu+M)},$$

where $M>0$ is given by Lemma 4.11. Let us check that the conditions of Lemma 4.11 are fulfilled for the cylinders C_n (big remote cylinder) and C_{n+1} (small cylinder):

$$\ell_n=a_n-(a_{n+1}+2\pi\mu r_{n+1})=\frac{1}{2^{n+1}}-\frac{2\pi\mu}{2^{n+2}(\pi\mu+M)}=\frac{M}{2^{n+1}(\pi\mu+M)}\ge Mr_n.$$

Now the construction is as follows. For each n, let γ_n be the λ-eel given by Proposition 4.8, of length greater than 1 lying inside the cylinder C_n, entering this cylinder from the upper part ($z=a_n+2\pi\mu r_n$) and having its endpoint at the bottom ($z=a_n$). Let $\tilde{e}_n$ be the oriented segment going from the endpoint of the curve γ_n (bottom of the cylinder C_n) to the starting point of γ_{n+1} (upper part of the cylinder C_{n+1}). We now define $\gamma:[0,+\infty)\to\mathbb{R}^3$ by concatenation of the following curves : γ_1, $\tilde{e}_1$, γ_2, $\tilde{e}_2$, and so on. It is clear that γ is continuous and has right derivative at each point. Morever, γ is contained in the unit ball of $\mathbb{R}^3$ and its length $\ell(\gamma)$ is greater than $\ell(\gamma_1)+\ell(\gamma_2)+\cdots+\ell(\gamma_n)\ge n$ for every n, therefore it is infinite. Observe that $\gamma(t)$ has limit 0 as $t\to+\infty$. It remains to prove that γ is a λ-eel, that is, it satisfies the λ-cone condition. Notice that each curve γ_n, $\tilde{e}_n$ is individually a λ-eel (that is, it satisfies the λ-cone property with respect to itself). Provided M is sufficiently big, the segment $\tilde{e}_n$ is almost parallel to the z-axis and it is oriented to the opposite direction of the previous curves γ_1, $\tilde{e}_1,\cdots$, γ_n. Therefore, if the λ-cone $C(t,\alpha)$, given in (4.1), has its origin onto a segment $\tilde{e}_n$, then it does not meet the union of the ranges of γ_1, $\tilde{e}_1, \cdots$, γ_n. It remains to treat the case where $C(t,\alpha)$ has its origin to a curve of the form γ_n. These curves are constructed (for each n) by concatenating pieces of the form $\gamma_i^{\downarrow}$ (of type 1), $\gamma_i^{\uparrow}$ (of type 2) and e_i^+ or e_i^- (of type 3) (*c.f.* Remark 4.9). If the λ-cone lies on a piece of type 1 or of type 3 of γ_n, then it is oriented to the opposite directions of all of the previous pieces γ_1, $\tilde{e}_1, \cdots$, γ_{n-1}, $\tilde{e}_{n-1}$ of γ, therefore it does not meet the union of their ranges. If now the cone $C(t,\alpha)$ has its origin on an upward piece $\gamma_i^{\uparrow}$ (type 2) of the curve γ_n, then the result follows from Lemma 4.11. The proof is complete. □

5. Curves with the λ-cone property in 2 dimensions

It is remarkable that there is no analogue of the construction in Theorem 4.2 in dimension 2. Indeed, we shall show that for any value of the parameter $\lambda \in [-1,1)$, any bounded planar λ-eel (that is, continuous curve with right derivative at each point that satisfies the λ-cone property) is rectifiable and has finite length. We shall need the following lemmas. (Recall $\alpha = \arccos(\lambda)$.)

Lemma 5.1. *Let $\gamma : I \to \mathbb{R}^2$ be a planar λ-eel and $t_1 < t_2 < t_3$ in I. Then*

$$\gamma(t_3) \notin [\gamma(t_1), \gamma(t_2)].$$

Proof. Set $A = \gamma(t_1)$, $B = \gamma(t_2)$, $C = \gamma(t_3)$ and assume towards a contradiction that $C \in [A,B]$. Choosing adequate coordinates in $\mathbb{R}^2$ we may assume that $A = (0,0)$, $B = (1,0)$ and $C = (c,0)$ with $c \in (0,1)$. In the sequel, we shall write $\gamma = (\gamma_1, \gamma_2)$ in these coordinates.

Before we proceed, notice that we may assume

$$\gamma(t) \notin (A,C) \quad \text{for all } t \in (t_1, t_2]. \tag{5.1}$$

Indeed, set $N_1 = \{t \in [t_1,t_2) : \gamma(t) \in [A,C]\} = \{t \in [t_1,t_2) : \gamma_1(t) \in [0,c], \gamma_2(t) = 0\}$ and $\alpha_1 := \sup\{\gamma_1(t) : t \in N_1\}$. Then $\alpha_1 < c$ (since γ is continuous and injective) and consequently, there exists $t_1 \le \tilde{t}_1 < t_2$ with $\gamma(\tilde{t}_1) = (\alpha_1, 0) = \tilde{A}$. In this case we can replace A by $\tilde{A}$ and t_1 by $\tilde{t}_1$ and get (5.1).

We set $\tilde{t}_2 = \inf\{t \in [t_1,t_2] : \gamma_1(t) \ge c, \gamma_2(t) = 0\}$. There is no loss of generality to assume $t_2 = \tilde{t}_2$, since we can always replace B by $\tilde{B} = (\gamma_1(\tilde{t}_2), 0)$ (notice that $\gamma_1(\tilde{t}_2) > c$ by injectivity).

Therefore for all $t \in (t_1,t_2)$ we have $\gamma(t) \notin (A,B)$. Setting $\Gamma_{AB} = \{\gamma(t) : t \in [t_1,t_2]\}$ we deduce that $\Gamma_{AB} \cup (B,A]$ is a Jordan curve which separates $\mathbb{R}^2$ in two regions, exactly one of them being bounded. Call $\mathcal{R}$ this bounded region, set $H^+ = \{x = (x_1,x_2) : x_2 > 0\}$, $H^- = \{x = (x_1,x_2) : x_2 < 0\}$ and let $\varepsilon > 0$ be such that $B(C,\varepsilon) \cap \Gamma_{AB} = \emptyset$. Then at least one of the sets $B(C,\varepsilon) \cap H^+$ and $B(C,\varepsilon) \cap H^-$ has nonempty intersection with $\mathcal{R}$. Assume, with no loss of generality, that

$$B(C,\varepsilon) \cap H^- \cap \mathcal{R} \neq \emptyset.$$

Then for every $x \in H^- \cap \operatorname{int} \mathcal{R}$ and every direction $d = (d_1,d_2) \in \mathbb{S}^1$ (the unit sphere of $\mathbb{R}^2$) with $d_2 \le 0$ it holds $\ell_{x,d} \cap \Gamma_{AB} \neq \emptyset$, where $\ell_{x,d} := \{x + \mu d : \mu \ge 0\}$ is the half-line emanating from x with direction d. In particular, shrinking $\varepsilon > 0$ if necessary, and recalling notation (1.5) we deduce that

$$C_x(d,\alpha) \cap \Gamma_{AB} \neq \emptyset, \quad \text{for all } x \in B(C,\varepsilon) \cap H^- \cap \mathcal{R} \text{ and all } d = (d_1,d_2) \text{ with } d_2 \le 0. \tag{5.2}$$

Let $\tau_3 \in (t_2,t_3)$ be such that for all $t \in (\tau_3,t_3]$ we have $\gamma(t) \in B(C,\varepsilon)$ (such τ_3 exists by continuity). Then it follows by (5.2) and the λ-eel property that $\gamma_2'(t) > 0$, and consequently, $\gamma_2(t) < 0$ (since $\gamma_2(t_3) = 0$). Let further $\tau \in [t_2,t_3]$ be such that

$$\gamma_2(\tau) = \min_{t \in [t_2,t_3]} \gamma_2(t) \; (< 0\,).$$

Then since $\gamma_2(t_2) = 0$, there exists $\tilde{t} \in [t_2, \tau]$ with ($\gamma(t) \in \mathcal{R}$ and) $\gamma_2'(t) < 0$ which together with (5.2) contradicts the λ-eel property. □

For the next statement, recall notation (4.1) and (1.5).

Lemma 5.2. *Under the assumptions of the previous lemma we have:*

$$C(\gamma'(t), \alpha) \cap K(t) = \{0\}, \quad \text{for all } t \in I.$$

Proof. Fix $t \in I$ and assume with no loss of generality (by translation) that $\gamma(t) = 0$. Then $K(t) = \overline{\text{cone}}\,(\Gamma(t) - \gamma(t)) = \overline{\text{cone}}\,\Gamma(t)$, where $\Gamma(t) = \{\gamma(\tau) : t \in [0, t]\}$. Let assume that there exists $x \in C(\gamma'(t), \alpha) \cap K(t)$, $x \neq 0$. Then by Caratheodory theorem, there exist $x_i = \gamma(\tau_i)$, $i \in \{1, 2, 3\}$ with $\tau_1 \leq \tau_2 \leq \tau_3 < t$ and $x \in \text{conv}\{x_1, x_2, x_3\}$ (convex envelope). Set $\ell_1 := \{x_1 + \mu(x - x_1) : \mu \geq 0\}$ and $\ell_2 = \{x_2 + \mu(x - x_2) : \mu \geq 0\}$. If $\ell_1 \cap \Gamma(t) = \ell_2 \cap \Gamma(t) = \emptyset$, then for μ_1, μ_2 sufficiently big, the point x_3 should belong to the triangle defined by the points $\ell_1(\mu_1) := x_1 + \mu_1(x - x_1)$, x and $\ell_2(\mu_2) = x_2 + \mu_2(x - x_2)$. Then by connectedness of $\gamma([\tau_1, \tau_3])$, we deduce that for some $s < \tau_3 < t$ it holds $\gamma(s) \in \ell_1 \cup \ell_2$. We deduce that x is a convex combination of two points of $\Gamma(t)$, that is, $x \in [\gamma(s_1), \gamma(s_2)]$ for some $s_1 < s_2 < t$. Set $\Gamma_{12} := \{\gamma(\tau\} : \tau \in [s_1, s_2]\}$. Since γ is a λ-eel, we have $C(t, \alpha) \cap \Gamma_{12} = \emptyset$. Then $[\gamma(s_2), \gamma(s_1)] \cup \Gamma_{12}$ is a Jordan curve and $\gamma(t) = 0 \in \mathcal{R}$ where $\mathcal{R}$ is the bounded region delimited by the Jordan curve. This yields that for some $t_1 < t_2 < t$, $\gamma(t) = 0$ is a convex combination of $\gamma(t_1)$ and $\gamma(t_2)$, which contradicts Lemma 5.1. □

In view of Lemma 5.2 and Remark 3.7 we obtain our main result.

Theorem 5.3 (bounded planer eels have finite length)**.** *Let $\gamma : I \to \mathbb{R}^2$ be a bounded λ-eel. Then γ is rectifiable and has finite length.* □

Acknowledgment. The authors wish to thank Ludovic Rifford for several useful discussions.

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Research supported by the grants:
BASAL PFB-03, FONDECYT 1171854, ECOS/CONICYT C14E06, REDES/CONICYT 15040 (Chile) and MTM2014-59179-C2-1-P (MINECO of Spain and ERDF of EU).

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Research supported by the grants:
ECOS/CONICYT C14E06 (France) and REDES/CONICYT-15040 (Chile).

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Research supported by the grant MTM2015-65825-P (MINECO of Spain) and 2018-MAT14 (ETSI Industriales, UNED).