Cyclic hypomonotonicity, cyclic submonotonicity and integration

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Abstract Rockafellar has shown that subdifferentials of convex functions are always cyclically monotone operators. Moreover, maximal cyclically monotone operators are necessarily operators of this type, since one can formally construct a convex function, which turns out to be unique up to a constant, whose subdifferential gives back the operator. This result is a cornerstone in convex analysis and tightly relates convexity and monotonicity. In this paper we establish analogous robust results that relate weak convexity notions to corresponding notions of weak monotonicity, provided one deals with locally Lipschitz functions and locally bounded operators. In particular, subdifferentials of locally Lipschitz functions that are d-hypomonotone (respectively, d-submonotone) also enjoy an additional cyclic strengthening of this notion and, in fact, are maximal under this new property. Moreover, every maximal cyclically hypomonotone (respectively, maximal cyclically submonotone) operator is always the Clarke subdifferential of some d-weakly convex (respectively, d-approximately convex) locally Lipschitz function, unique up to a constant, which, in finite dimensions, is a lower-C^2 function (respectively, a lower-C^1 function).

Key words Submonotonicity, hypomonotonicity, cyclicity, integration.

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1 Introduction

Spingarn introduced the notion of submonotonicity (Definition 2(i)) and showed that this notion characterizes the Clarke subdifferentials of lower C^1 functions ([17, Theorem 3.9]). We recall that a locally Lipschitz function $f : U \to \mathbb{R}$ is said to be *lower-C^k* (where U is a nonempty open subset of $\mathbb{R}^n$ and $k = 1, 2, \ldots$) if for every $x_0 \in U$, there exist a neighborhood V of x_0, a compact set S and a jointly continuous function $F : V \times S \to \mathbb{R}$, such that for all $x \in V$ we have

$$f(x) = \max_{s \in S} F(x, s) \tag{1}$$

and the derivatives of F of order k with respect to x exist and are jointly continuous ([16, Definition 10.29] e.g.). These functions enjoy interesting stability properties for optimization, see [7], [12], [16] for example. In [6] submonotonicity has been extended in infinite dimensions. (We call here this notion d-submonotonicity to stress out the directional character of this extension, see Definition 2(ii).) It has subsequently been established that d-submonotonicity characterizes subdifferentials of d-approximately convex functions ([4]).

If X is a Banach space, a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is called *d-approximately convex* ([4, Definition 8]), if for every $x_0 \in X$, $e \in S_X$ and $\varepsilon > 0$ there exists $\delta = \delta(x_0, e, \varepsilon) > 0$ such that for all $(x, y) \in U(x_0, e, \delta)$ and $t \in (0, 1)$

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon t(1-t)\|x - y\|, \tag{2}$$

where

$$U(x_0, e, \delta) := \left\{ (x_1, x_2) \in X \times X : x_1 \neq x_2,\, x_i \in B(x_0, \delta) \;\&\; \|\frac{x_1 - x_2}{\|x_1 - x_2\|} - e\| < \delta \right\} \tag{3}$$

(S_X is the unit sphere of X and $B(x_0, \delta)$ is the open ball of center $x_0 \in X$ and radius $\delta > 0$.)

If, in the above definition, δ does not depend on e, case in which condition (3) simply reads $x, y \in B(x_0, \delta)$, then f is called *approximately convex*. (This latter notion has been introduced in [11] for another purpose.) In finite dimensions, thanks to the compactness of the unit ball, d-approximate convexity and approximate convexity coincide. If, in addition, the functions are locally Lipschitz, then the aforementioned Spingarn's result yields that both notions coincide also with the notion of a lower-C^1 function.

Concurrently to the notion of submonotonicity, Rockafellar introduces the (stronger) notion of hypomonotonicity (Definition 3(i)), and shows that this notion characterizes in finite dimensions the Clarke subdifferentials of lower-C^2 functions ([15]).
Lower-C^2 functions have several known equivalent descriptions:

- they are locally decomposable to a difference of a convex continuous and convex quadratic function ([15], [7])

- they coincide with the class of locally Lipschitz weakly convex functions ([8], [18]).

We recall from [18, page 232] that a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is *weakly convex*, if for every $x_0 \in X$, there exists $\delta > 0$ and $\rho > 0$ such that for all $x, y \in B(x_0, \delta)$ and $t \in (0,1)$

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \rho t(1-t)\|x-y\|^2. \tag{4}$$

In accordance with the notion of approximate convexity, let us call the function f *d-weakly convex* if for every $x_0 \in X$ and $e \in S_X$, there exists $\delta, \rho > 0$ such that (4) holds provided $(x, y) \in U(x_0, e, \delta)$ and $t \in (0,1)$.

Cyclicity properties of subdifferentials

Rockafellar has shown that subdifferentials of convex functions are not only monotone, but also cyclically monotone. This latter property is not just an artificial strengthening of monotonicity, but a very important notion. The following powerful characterization ([14]) justifies this assertion:

(M) A (locally bounded) multivalued operator is maximal cyclically monotone if, and only if, it is the subdifferential of a (locally Lipschitz) convex function, which is unique up to a constant.

In this paper we show that robust results in the spirit of (M) can be achieved under an appropriate definition of cyclicity over the notion of d-hypomonotonicity (respectively, d-submonotonicity). The class of locally Lipschitz d-weakly convex (respectively, d-approximately convex) functions will therefore play the role of the class of (locally Lipschitz) convex functions in the above characterization. If U is a nonempty open subset of X and $T : X \rightrightarrows X^*$ with $U \subset \text{dom}(T)$, we establish that:

(H) T is maximal cyclically hypomonotone if, and only if, there exists a locally Lipschitz d-weakly convex function $f : U \to \mathbb{R}$ such that $T = \partial f$.

(S) T is maximal cyclically submonotone if, and only if, there exists a locally Lipschitz d-approximately convex function $f : U \to \mathbb{R}$ such that $T = \partial f$.

In both cases, f is unique up to a constant in every connected component of U.

Let us mention that a notion of radial cyclic submonotonicity had been introduced by Janin in [9]. In this work it has been established that every bounded operator defined on a convex compact subset of $\mathbb{R}^n$ is the Clarke subdifferential of lower-C^1 function, provided it is maximal under the property of being submonotone and radially cyclically submonotone. A careful elaboration of Janin's proof shows that his result holds also for locally bounded operators defined on open connected subsets of $\mathbb{R}^n$. On the other hand, the proof is heavily based on compactness arguments valid only in finite dimensions. We stress out this important difference: in infinite dimensions compactness arguments have to be replaced by more sophisticated techniques. Moreover, a robust (and self-contained) definition of cyclic submonotonicity is needed. This definition appears for the first time in [5], where characterization (S) has implicitly been established (it follows by combining [5, Theorem C] and [4, Theorem 13], but the reader should beware of differences in terminology). Both definition of cyclic hypomonotonicity and characterization (H) appear in this work for the first time.

In Section 2 we recall the definitions of hypomonotonicity and submonotonicity (being used for operators in $\mathbb{R}^n$) and of d-hypomonotonicity and d-submonotonicity that correspond to their appropriate (i.e. directional) extensions in infinite dimensions. We also introduce the notion of cyclic hypomonotonicity (which is new) and of cyclic submonotonicity (which is used in [5]). These definitions relay on the concept of δ-subdivision $\{x_i\}_{i=1}^n$ of a given closed polygonal path $[w_h]_{h=1}^m$ (Definition 4), notion introduced in [5, Definition 5]. This consists on a variational δ-partition of $[w_h]_{h=1}^m$, with respect to its directions $e_h := (w_{h+1} - w_h)/\|w_{h+1} - w_h\|$.

In Section 3 we establish characterizations (H) and (S). In contrast to (M), in this work we assume that $U \subset \mathrm{dom}(T)$, and therefore that T is locally bounded (see Proposition 8). The question whether (H) and (S) remain valid in the general case remains open after the current work.

2 Weak monotonicity and cyclicity

Let $T : X \rightrightarrows X^*$ be a multivalued operator with domain $\mathrm{dom}(T) = \{x \in X : T(x) \neq \emptyset\}$. The operator T is called monotone, if for all $x_1, x_2 \in X$, $x_1^* \in T(x_1)$, $x_2^* \in T(x_2)$ we have

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq 0. \tag{5}$$

The operator T is called cyclically monotone, if for every $\{x_i\}_{i=1}^n \subset X$, $\{x_i^*\}_{i=1}^n \subset X^*$, with $x_n = x_1$ and $x_i^* \in T(x_i)$, $i \in \mathbb{N}_{n-1} := \{1, \ldots, n-1\}$ we have

$$\sum_{i=1}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle \leq 0. \tag{6}$$

We say in particular that a cyclically monotone operator T is maximal cyclically monotone, if there is no other cyclically monotone operator S whose graph strictly contains the graph of T. The following result is well known (see [14], [3])

Theorem 1 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper (i.e. not identically equal to $+\infty$) lower semicontinuous function and $\partial f : X \rightrightarrows X^*$ its Clarke-Rockafellar subdifferential. The following are equivalent:*

(i) *f is convex*

(ii) *∂f is monotone*

(iii) *∂f is cyclically monotone*

(iv) *∂f is maximal cyclically monotone*

In the sequel, we shall survey definitions of d-hypomonotonicity and d-submonotonicity, where "d-" stands for directional, precision superfluous in finite dimensions. We also state the corresponding notions of cyclicity. These definitions are justified by results analogous to Theorem 1 (given in this section) and to (M) (established in Section 3).

2.1 D-submonotone and d-hypomonotone operators

Let us start by the notions of submonotonicity and d-submonotonicity.

Definition 2 *The operator T is called*
(i) submonotone, *if for every $x_0 \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for all $(x_1, x_2) \in B(x_0, \delta)$ and all $x_i^* \in T(x_i)$, $i \in \{1, 2\}$ we have*

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -\varepsilon \|x_1 - x_2\|. \tag{7}$$

(ii) d-submonotone, *if for every $x_0 \in X$, $e \in S_X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that* (7) *holds provided $(x_1, x_2) \in U(x_0, e, \delta)$ and $x_i^* \in T(x_i)$, $i \in \{1, 2\}$.*
($U(x_0, e, \delta)$ *is defined in* (3)).

Remark 1. Submonotonicity had been called "strict submonotonicity" in [17, page 79]. Spingarn employed the term "submonotonicity" for the more restrictive notion where (7) holds only if $x_2 = x_0$. To distinguish between these two notions we propose for the latter the term "semi-submonotonicity" (respectively "d-semi-submonotonicity"). An easy example of semi-submonotone operator which is not submonotone can be found in [6, Example 1.3]. Finally let us quote the following interesting characterization of semi-submonotonicity ([17, Proposition 2.4]):

"*The Clarke subdifferential ∂f of a locally Lipschitz function on $\mathbb{R}^n$ is semi-submonotone if, and only if, the function f is regular and* (*Mifflin*) *semi-smooth*".

Remark 2. In finite dimensions the notions of submonotonicity and d-submonotonicity coincide. (This follows from a standard argument evoking the compactness of the unit ball of $\mathbb{R}^n$.) However the notion of "d-submonotonicity" is more appropriate in infinite dimensions, "submonotonicity" thus corresponding to a restrictive uniform notion (see [6], [5] for details).

We now recall the stronger notion of hypomonotonicity (for finite dimensions) and d-hypomonotonicity (for infinite dimensions).

Definition 3 *The operator T is called*
(i) hypomonotone, *if for every $x_0 \in X$, there exists $\delta > 0$ and $\rho > 0$ such that for all $(x_1, x_2) \in B(x_0, \delta)$ and all $x_i^* \in T(x_i)$, $i \in \{1, 2\}$ we have*

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -\rho \|x_1 - x_2\|^2. \tag{8}$$

(ii) d-hypomonotone, *if for every $x_0 \in X$ and $e \in S_X$ there exists $\delta > 0$ and $\rho > 0$ such that* (8) *holds provided $(x_1, x_2) \in U(x_0, e, \delta)$ and $x_i^* \in T(x_i)$, $i \in \{1, 2\}$.*

Remark 3. The definition of hypomonotonicity appears for the first time in [15, Section 4] under the name "strict hypomonotonicity" (see also [13, page 5234] and references therein). The definition of d-hypomonotonicity is new and appears here for the first time. Similarly to Remark 2, in finite dimensions there is no distinction among these notions, while the latter is to be considered in infinite dimensions.

2.2 Cyclicity and notion of path subdivision

Cyclic monotonicity (see (6)) has a relatively simple definition. On the contrary, defining robust cyclicity notions for d-hypomonotonicity and d-submonotonicity is more complicated and uses the notion of path δ-subdivision.

Given a finite family of points $\{w_h\}_{h=1}^m$ where $m > 1$ and $w_1 = w_m$ we call "closed polygonal path" (or simply "path") and we denote by $[w_h]_{h=1}^m$, the union of the consecutive segments $[w_h, w_{h+1}]$ for $h = 1, \dots m-1$. Let us now recall from [4, Definition 5] the notion of path δ-subdivision.

Definition 4 *Given $\delta > 0$ and a closed polygonal path $[w_h]_{h=1}^m$ $(m > 1)$, we say that $\{x_i\}_{i=1}^n$ is a δ-*subdivision *of $[w_h]_{h=1}^m$ $(\{x_i\}_{i=1}^n \in \mathcal{D}([w_h]_{h=1}^m, \delta))$, if*

(i) $\{x_i\}_{i=1}^n \subseteq B_\delta([w_h]_{h=1}^m)$ *and* $x_1 = x_n$

(ii) $\|x_{i+1} - x_i\| < \delta$, *for* $i \in \mathbb{N}_{n-1}$

(iii) *there exists a finite sequence $\{k_h\}_{h=1}^m$ with $1 = k_1 < k_2 < \dots < k_m := n$ such that for $1 \le h \le m-1$ we have:*

$$k_h \le i < k_{h+1} \implies \|\frac{x_{i+1} - x_i}{\|x_{i+1} - x_i\|} - e_h\| < \delta,$$

where

$$e_h := \frac{w_{h+1} - w_h}{\|w_{h+1} - x_h\|}. \tag{9}$$

We are ready to state the definitions of cyclic submonotonicity ([5, Definition 6]) and of cyclic hypomonotonicity (that appears here for the first time).

Definition 5 *The operator T is called*

(i) cyclically submonotone, *if for every path $[w_h]_{h=1}^m$ and $\varepsilon > 0$, there exists $\delta > 0$, such that for all $\{x_i\}_{i=1}^n \in \mathcal{D}([w_h]_{h=1}^m, \delta)$ and all $x_i^* \in T(x_i)$*

$$\sum_{i=1}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle \le \varepsilon \sum_{i=1}^{n-1} \|x_{i+1} - x_i\|; \tag{10}$$

(ii) cyclically hypomonotone, *if for every path $[w_h]_{h=1}^m$, there exist $\delta, \rho > 0$ such that for all $\{x_i\}_{i=1}^n \in \mathcal{D}([w_h]_{h=1}^m, \delta)$ and all $x_i^* \in T(x_i)$*

$$\sum_{i=1}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle \le \rho \sum_{i=1}^{n-1} \|x_{i+1} - x_i\|^2. \tag{11}$$

If U is a nonempty open subset of X, we say that T is cyclically submonotone (respectively, cyclically hypomonotone) on U if in the above definitions both path and path subdivisions are taken in U. Furthermore, we say that T is maximal cyclically submonotone (on U), if it is cyclically submonotone (on U) and if its graph (restricted to U) cannot be strictly contained in the graph of any other cyclically submonotone operator.

The notions of maximal submonotonicity, maximal cyclic hypomonotonicity and maximal hypomonotonicity can be defined analogously.

It is clear that cyclic monotonicity implies cyclic hypomonotonicity, which in turn implies cyclic submonotonicity. The following proposition shows that cyclic hypomonotonicity (respectively, cyclic submonotonicity) implies d-hypomonotonicity (respectively, d-submonotonicity).

Proposition 6 (i) *Every cyclically hypomonotone operator is d-hypomonotone.*
(ii) *Every cyclically submonotone operator is d-submonotone.*

Proof We shall only prove (i). Assertion (ii) follows analogously. So, let us suppose that T is cyclically hypomonotone and let $x_0 \in X$ and $e \in S_X$. Then set $m = 3$, $w_1 = x_0 = w_3$ and $w_2 = x_0 + e$ and take $\delta, \rho > 0$ as in Definition 5 for the path $[w_h]_{h=1}^3$. Then for every $(x_1, x_2) \in U(x_0, e, \delta/2)$ and for $x_3 = x_1$, it is easily seen that $\{x_i\}_{i=1}^3 \in \mathcal{D}([w_h]_{h=1}^m, \delta)$. Thus relation (11) holds true yielding (8). □

A careful reader might wonder what would have happened if one had tried to define a notion of "cyclic monotonicity" in the spirit of Definition 5, i.e. by using path subdivisions. The following result shows that this would have lead to an equivalent reformulation of the classical definition of cyclic monotonicity in (6).

Proposition 7 *Let $T : X \rightrightarrows X^*$ be a multivalued operator. Then the following assertions are equivalent:*
(i) *T is cyclically monotone*
(ii) *for every $[w_h]_{h=1}^m$ there exists $\delta > 0$ such that for all $\{x_i\}_{i=1}^n \in \mathcal{D}([w_h]_{h=1}^m, \delta)$ and all $x_i^* \in T(x_i)$*

$$\sum_{i=1}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle \leq 0. \tag{12}$$

Proof Implication (i)$\Longrightarrow$(ii) is obvious. Thus, let us suppose that T satisfies (ii) and let us show that it is cyclically monotone.
We first prove that T is monotone. To this end, let $u_1, u_2 \in \mathrm{dom}(T)$, $u_1^* \in T(u_1)$ and $u_2^* \in T(u_2)$. Consider the path $[w_h]_{h=1}^3$, where $w_1 = w_3 = u_1$ and $w_2 = u_2$ and let $\delta > 0$ given by (ii). We may clearly suppose that $\delta < \|u_1 - u_2\|$. Let $n \in \mathbb{N}$ be such that $n > \delta^{-1}(\|u_1 - u_2\|)$. For $i = 1, \ldots, n+1$ we set

$$x_i := u_1 + \frac{(i-1)}{n}(u_2 - u_1)$$

and

$$x_{n+1} := u_2 + \frac{(i-1)}{n}(u_1 - u_2).$$

Then $\{x_i\}_{i=1}^{2n+1} \in \mathcal{D}([w_h]_{h=1}^m, \delta)$, hence (12) gives

$$\sum_{i=1}^{2n} \langle x_i^*, x_{i+1} - x_i \rangle \leq 0. \tag{13}$$

Note that

$$x_{i+1} - x_i = \begin{cases} n^{-1}(u_2 - u_1), & \text{if } i \in \mathbb{N}_n \\ n^{-1}(u_1 - u_2), & \text{if } i \in \mathbb{N}_{2n} \setminus \mathbb{N}_n \end{cases}$$

whence it follows easily

$$\sum_{i=2}^{n} \langle x_i^*, x_{i+1} - x_i \rangle = - \sum_{i=n+2}^{2n} \langle x_i^*, x_{i+1} - x_i \rangle.$$

Since $x_1 = u_1$ and $x_{n+1} = u_2$ (13) resumes to

$$n^{-1}(\langle u_1^*, u_2 - u_1 \rangle + \langle u_2^*, u_1 - u_2 \rangle) \leq 0$$

for all $u_1^* \in T(u_1)$ and $u_2^* \in T(u_2)$, which shows that T is monotone.

Let us now show that T is cyclically monotone. To this end, let $\{u_h\}_{h=1}^m$ be any finite sequence in $\mathrm{dom}(T)$ with $u_1 = u_m$ and let $u_i^* \in T(u_i)$. Let $\delta > 0$ be the one given by (ii) corresponding to the path $[u_h]_{h=1}^m$ and consider a δ-partition $\{x_i\}_{i=1}^n$ (i.e. $\|x_{i+1} - x_i\| \leq \delta$) with the property that for every $h \in \mathbb{N}_m$ there exists $1 \leq i_1 < i_2 \leq n$ such that $\{x_j\}_{j=i_1}^{i_2}$ is a partition of $[u_h, u_{h+1}]$.

Since

$$u_{h+1} - u_h = \sum_{j=i_1}^{i_2 - 1} (x_{j+1} - x_j),$$

applying the monotonicity of T successively to the points x_j, x_{j+1} (for $j = i_1, \ldots, i_2 - 1$) we easily get

$$\langle u_h^*, u_{h+1} - u_h \rangle \leq \sum_{j=i_1}^{i_2 - 1} \langle x_j^*, x_{j+1} - x_j \rangle$$

and consequently

$$\sum_{h=1}^{m-1} \langle u_h^*, u_{h+1} - u_h \rangle \leq \sum_{i=1}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle.$$

Since $\{x_i\}_{i=1}^n \in \mathcal{D}([w_h]_{h=1}^m, \delta)$, the result follows from (11). $\square$

We finally recall from [6, Theorem 2.4] the following result.

Proposition 8 *Every d-submonotone (a fortiori, monotone or d-hypomonotone) operator is locally bounded on* $\mathrm{intdom}(T)$.

Let us summarize the notions that concern a multivalued operator $T : X \rightrightarrows X^*$ in the following diagram

$$\boxed{\begin{array}{ccc} \text{cyclic monotonicity} & \Longrightarrow & \text{monotonicity} \\ \Downarrow & & \Downarrow \\ \text{cyclic hypomonotonicity} & \Longrightarrow & \text{d-hypomonotonicity} \\ \Downarrow & & \Downarrow \\ \text{cyclic submonotonicity} & \Longrightarrow & \text{d-submonotonicity} \end{array}}$$

2.3 Characterizations via weak monotonicity notions

The forthcoming results are analogous to Theorem 1 and concern the notions of weak monotonicity (Section 2.1) and their cyclic versions (Section 2.2). Throughout this section $f : U \to \mathbb{R}$ denotes a locally Lipschitz function defined on a nonempty open subset U of X, and $\partial f : X \rightrightarrows X^*$ denotes its Clarke subdifferential.

Lemma 9 (i) *The function $f : U \to \mathbb{R}$ is d-weakly convex if, and only if, for every $x_0 \in U$ and $e \in S_X$ there exists $\delta, \rho > 0$ such that for all $(x, y) \in U(x_0, e, \delta)$ and all $x^* \in \partial f(x)$*

$$f(y) - f(x) \geq \langle x^*, y - x \rangle - \rho \|x - y\|^2. \tag{14}$$

(ii) *The function $f : U \to \mathbb{R}$ is d-approximately convex if, and only if, for every $x_0 \in U$, $\varepsilon > 0$ and $e \in S_X$ there exists $\delta > 0$ such that for all $(x, y) \in U(x_0, e, \delta)$ and all $x^* \in \partial f(x)$*

$$f(y) - f(x) \geq \langle x^*, y - x \rangle - \varepsilon \|x - y\|. \tag{15}$$

Proof (i). This result is known in finite dimensions (see [18, Proposition 4.8] or [2, Theorem 5.1]). Let us include a simple proof for the general case.

Suppose that f is d-weakly convex, that is, for every $x_0 \in U$ and $e \in S_X$ there exists $\delta, \rho > 0$ such that for every $(x, y) \in U(x_0, e, \delta)$ and $t \in (0, 1)$ formula (4) holds. Setting $s = 1 - t$ we infer that

$$f(x + s(y - x)) - f(x) \leq s\,[f(y) - f(y)] + \rho s(1 - s)\|x - y\|^2,$$

which yields

$$\frac{f(x + s(y - x)) - f(x)}{s} \leq f(y) - f(y) + \rho(1 - s)\|x - y\|^2. \tag{16}$$

Since f is d-weakly convex, it is also d-approximately convex, hence regular ([6, Theorem 4.1]). In particular, the limit in the left part of (16) as $s \searrow 0^+$ exists and yields the Clarke directional derivative of f. Thus letting $s \searrow 0^+$ in (16) we obtain (14).

For the inverse implication, we suppose that for every $x_0 \in U$ and $e \in S_X$ there exists $\delta, \rho > 0$ such that (14) holds for all $(x, y) \in U(x_0, e, \delta) \cup U(x_0, -e, \delta)$ and all $x^* \in \partial f(x)$. To show that f is weakly convex, let any $t \in (0, 1)$, set $x_t := tx + (1 - t)y$ and note that $(x_t, y) \in U(x_0, e, \delta)$ and $(x_t, x) \in U(x_0, -e, \delta)$. Thus for any $x_t^* \in \partial f(x_t)$ (14) yields:

$$f(y) \geq f(x_t) + \langle x_t^*, y - x_t \rangle - \rho \|x_t - y\|^2; \tag{17}$$

and

$$f(x) \geq f(x_t) + \langle x_t^*, x - x_t \rangle - \rho \|x_t - y\|^2. \tag{18}$$

Since $y - x_t = t(y - x)$ and $x - x_t = (1 - t)(x - y)$, multiplying (17) by $(1 - t)$ and (18) by t and adding the corresponding inequalities we infer that (14) holds (for $\rho' = 2\rho$), hence f is d-weakly convex. This completes the proof of (i).

(ii). This assertion follows analogously. $\square$

We also need to recall the notion of a w*-cusco mapping ([1] e.g.).

- A multivalued mapping $T : X \rightrightarrows X^*$ is called *w*-cusco* on the open set U, if it is w*-upper semicontinuous with nonempty w*-compact convex values on U. A w*-cusco mapping on U that does not strictly contain any other w*-cusco mapping with domain in U is called a *minimal w*-cusco* on U.

The following result has been established in [5, Proposition 9].

Proposition 10 *Let T be cyclically submonotone on U. The following statements are equivalent:*

(i) *T is w*-cusco on U.*

(ii) *T is minimal w*-cusco on U.*

(iii) *T is maximal cyclically submonotone on U.*

We are ready to state the main results of this section.

Theorem 11 *The following are equivalent:*

(i) *f is d-weakly convex*

(ii) *∂f is d-hypomonotone*

(iii) *∂f cyclically hypomonotone*

(iv) *∂f maximal cyclically hypomonotone*

Proof One obviously has (iv)$\Longrightarrow$(iii), while Proposition 6 shows that (iii)$\Longrightarrow$(ii).

Let us prove (ii)$\Longrightarrow$(i). Suppose that ∂f is d-hypomonotone and let $x_0 \in X$ and $e \in S_X$. Then there exist $\delta, \rho > 0$ such that relation (8) holds. Then for every $(x, y) \in U(x_0, e, \delta)$ and $t \in (0, 1)$ we set $x_t = tx + (1-t)y$. Applying Lebourg's Mean Value theorem ([10]) on the segments $[x, x_t]$ and $[y, x_t]$ we obtain points $z_1 \in [x, x_t[$ and $z_2 \in [y, x_t[$ such that for some $z_1^* \in \partial f(z_1)$ and $z_2^* \in \partial f(z_2)$ we have

$$\langle z_1^*, x_t - x\rangle = f(x_t) - f(x)$$

and

$$\langle z_2^*, x_t - y\rangle = f(x_t) - f(y).$$

Since $x_t - x = (1-t)(y - x)$ and $x_t - y = t(x - y)$, multiplying the first relation above by t and the second by $(1-t)$ and adding the resulting equalities we obtain

$$tf(x) + (1-t)f(y) - f(x_t) = t(1-t)\langle z_1^* - z_2^*, x - y\rangle.$$

Since $(x, y) \in U(x_0, e, \delta)$ we have

$$\frac{x - y}{\|x - y\|} = \frac{z_1 - z_2}{\|z_1 - z_2\|} \in B(e, \delta).$$

Thus using (8) we infer

$$tf(x) + (1-t)f(y) \geq f(x_t) - \rho t(1-t)\|z_1 - z_2\|\|x - y\|.$$

Since $\|x-y\| \geq \|z_1 - z_2\|$, the above inequality implies (4). This shows that f is d-weakly convex.

Let us finally prove that (i)$\Longrightarrow$(iv). We suppose that f is a locally Lipschitz d-weakly convex function.

Step 1. ∂f is d-hypomonotone.
Let $x_0 \in X$ and $e \in S_X$. Then there exist $\delta, \rho > 0$ such that (14) holds. It follows that for all $(x, y) \in U(x_0, e, \delta)$, $x^* \in \partial f(x)$ and $y^* \in \partial f(y)$ we have

$$\langle x^*, y - x\rangle \leq f(y) - f(x) + \frac{\rho}{2}\|x-y\|^2$$

and

$$\langle y^*, x - y\rangle \leq f(x) - f(y) + \frac{\rho}{2}\|x-y\|^2.$$

Adding the above inequalities we obtain (8).

Step 2. ∂f is cyclically hypomonotone.
For any path $[w_h]_{h=1}^m$ $(m > 1)$ we set $C = \cup_{h\in\mathbb{N}_{m-1}}[w_h, w_{h+1}]$ and

$$e_h = \frac{w_{h+1} - w_h}{\|w_{h+1} - w_h\|}$$

for $h \in \mathbb{N}_{m-1}$. Since ∂f is d-hypomonotone, for every $x \in C$ there exists $\rho(x) > 0$ and $\delta(x) > 0$ such that

$$\langle x_1^* - x_2^*, x_2 - x_1\rangle \geq -\rho(x)\|x_1 - x_2\|^2 \tag{19}$$

whenever $x_1, x_2 \in \bigcup\limits_{h\in\mathbb{N}_{m-1}} U(x, e_h, \delta(x))$, $x_1^* \in T(x_1)$ and $x_2^* \in T(x_2)$. Let $\{B(x_i, \delta(x_i))\}_{i=1}^k$ be a finite subcovering of the open covering $B(x, \delta(x))_{x\in C}$ of the compact set C. Set $\rho = \max\{\rho(x_i) : i \in \mathbb{N}_k\}$ and $\delta > 0$ be a Lebesgue number of the open subcovering $\{B(x_i, \delta(x_i))\}_{i=1}^k$, that is

$$\forall u \in C, \exists i \in \mathbb{N}_k : B(u, \delta) \subseteq B(x_i, \delta(x_i)). \tag{20}$$

Let any δ-subdivision $\{x_i\}_{i=1}^n \in \mathcal{D}([w_h]_{h=1}^m, \delta)$. Since f is locally Lipschitz, applying Lebourg's Mean Value theorem ([10]) on the segment $[x_i, x_{i+1}]$ (for $i \in \mathbb{N}_{n-1}$), we infer the existence of $z_i \in \,]x_i, x_{i+1}[$ and $z_i^* \in T(z_i)$ such that

$$f(x_{i+1}) - f(x_i) = \langle z_i^*, x_{i+1} - x_i\rangle. \tag{21}$$

Adding the above equalities, we get

$$\sum_{i=1}^{n-1}\langle z_i^*, x_{i+1} - x_i\rangle = 0.$$

Consequently,

$$\sum_{i=1}^{n-1}\langle x_i^*, x_{i+1} - x_i\rangle = \sum_{i=1}^{n-1}\langle x_i^* - z_i^*, x_{i+1} - x_i\rangle. \tag{22}$$

Evoking (19) and (20) we obtain for every $i \in \mathbb{N}_{n-1}$ that

$$\langle x_i^* - z_i^*, \frac{x_{i+1} - x_i}{\|x_{i+1} - x_i\|} \rangle = \langle x_i^* - z_i^*, \frac{z_i - x_i}{\|z_i - x_i\|} \rangle < \rho \|z_i - x_i\|.$$

Since $\|z_i - x_i\| \leq \|x_{i+1} - x_i\|$ the above formula yields

$$\langle x_i^* - z_i^*, x_{i+1} - x_i \rangle < \rho \|x_{i+1} - x_i\|^2.$$

Adding the above equalities for $i = 1, 2, \ldots, n-1$, and using (22) we obtain

$$\sum_{i=1}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle \leq \rho \sum_{i=1}^{n-1} \|x_{i+1} - x_i\|^2.$$

Step 3. ∂f is maximal cyclically hypomonotone.
Since ∂f is the Clarke subdifferential of a locally Lipschitz function, it is a w*-cusco mapping. Being also cyclically submonotone, it follows from Proposition 10 that ∂f is maximal cyclically submonotone. This implies in particular that ∂f is maximal cyclically hypomonotone. This finishes the proof. $\square$

The analogous result for d-submonotone operators and d-approximately convex functions is essentially known in the literature (under a different terminology).

Theorem 12 *The following are equivalent:*

(i) f *is d-approximately convex*

(ii) ∂f *is d-submonotone*

(iii) ∂f *cyclically submonotone*

(iv) ∂f *maximal cyclically submonotone.*

Proof The equivalence (ii)$\Longleftrightarrow$(iii)$\Longleftrightarrow$(iv) has been established in [5, Theorem A], while (i)$\Longleftrightarrow$(ii) can be found in [4, Theorem 13]. $\square$

We summarize in the following diagram, where $f : U \to \mathbb{R}$ denotes a locally Lipschitz function and $\partial f : U \rightrightarrows X^*$ its Clarke subdifferential.

$f : U \to \mathbb{R}$		$\partial f : U \rightrightarrows X^*$		$\partial f : U \rightrightarrows X^*$
convex	$\Longleftrightarrow$	monotone	$\Longleftrightarrow$	maximal cyclically monotone
$\Downarrow$		$\Downarrow$		$\Downarrow$
d-weakly convex ($\equiv$ lower-C^2 if $X = \mathbb{R}^n$)	$\Longleftrightarrow$	d-hypomonotone ($\equiv$ hypomonotone if $X = \mathbb{R}^n$)	$\Longleftrightarrow$	maximal cyclically hypomonotone
$\Downarrow$		$\Downarrow$		$\Downarrow$
d-approximately convex ($\equiv$ lower-C^1 if $X = \mathbb{R}^n$)	$\Longleftrightarrow$	d-submonotone ($\equiv$ submonotone if $X = \mathbb{R}^n$)	$\Longleftrightarrow$	maximal cyclically submonotone

.

As mentioned in the introduction, in finite dimensions the characterization of lower-C^2 functions (via hypomonotonicity) and of lower-C^1 functions (via submonotonicity) have been established in [15] and [9] respectively. In [9] we also find the notion of radial cyclic submonotonicity.

- An operator T is called *radially cyclically submonotone*, if for every $[w_h]_{h=1}^m$ and $\varepsilon > 0$, there exists $\delta > 0$, such that for all δ-partitions $\{x_i\}_{i=1}^n$ of $[w_h]_{h=1}^m$ and all $x_i^* \in T(x_i)$ one has:

$$\sum_{i=1}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle \leq \varepsilon \sum_{i=1}^{n-1} \|x_{i+1} - x_i\|. \tag{23}$$

Similarly, one can define the notion of radial cyclic hypomonotonicity.

- An operator T is called *radially cyclically hypomonotone*, if for every $[w_h]_{h=1}^m$ there exist $\rho, \delta > 0$, such that for all δ-partitions $\{x_i\}_{i=1}^n$ of $[w_h]_{h=1}^m$ and all $x_i^* \in T(x_i)$ one has:

$$\sum_{i=1}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle \leq \rho \sum_{i=1}^{n-1} \|x_{i+1} - x_i\|^2. \tag{24}$$

In both cases, by the term "δ-partition" of the path $[w_h]_{h=1}^m$ we mean a finite collection of points $\{x_i\}_{i=1}^n$ in $\bigcup_{h \in \mathbb{N}_{m-1}} [w_h, w_{h+1}]$, where $x_1 = x_n = w_1$, that satisfy $\|x_{i+1} - x_i\| < \delta$

(for $i \in \mathbb{N}_{n-1}$) and

$$\sum_{i=1}^{n-1} \|x_{i+1} - x_i\| = \sum_{h=1}^{n-1} \|w_{h+1} - w_h\|. \tag{25}$$

Remark 4 In view of (25) it is easily seen that (23) can be replaced by the following simpler condition

$$\sum_{i=1}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle \leq \varepsilon.$$

Janin called an operator T "cyclically submonotone", if it is both submonotone and radially cyclically submonotone, and established the corresponding finite dimensional characterization for lower-C^1 functions. In the forthcoming Remark 5 we will see that in finite dimensions Janin's definition is equivalent to Definition 5(i) (and consequently, his result is a particular case of Theorem 12).

Proposition 13 *Let T be a multivalued operator in $\mathbb{R}^n$.*

(i) *T is cyclically submonotone if, and only if, T is d-submonotone and radially cyclically submonotone.*

(ii) *T is cyclically hypomonotone if, and only if, T is d-hypomonotone and radially cyclically hypomonotone.*

Proof (i) The "only if" part being obvious, let us suppose that T is d-submonotone and radially cyclically submonotone and let us show that it satisfies Definition 5(i). To this end, let $[w_h]_{h=1}^m$ be a closed polygonal path and let $\varepsilon > 0$. Set $P = [w_h]_{h=1}^m$ and $F = \{e_h : h \in \mathbb{N}_{m-1}\}$ (the set of directions of the path, given by (9)). Since F is finite and P is compact, using the definition of d-submonotonicity of T and a standard argument we infer that there exists $\delta > 0$ with the following property: for every $x \in P$ and $e \in F$, and for all $(x_1, x_2) \in U(x, e, \delta)$ and $x_i^* \in T(x_i)$ $(i = 1, 2)$

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -\frac{\varepsilon}{2} \|x_1 - x_2\|. \tag{26}$$

Let $\{x_i\}_{i=1}^n \in \mathcal{D}([w_h]_{h=1}^m, \delta)$ and let us show that (10) is satisfied.
Indeed, since T is radially cyclically submonotone, there exists $\delta_1 < \delta$ such that for every δ_1-partition $\{y_j\}_{j=1}^k$ of $\{x_i\}_{i=1}^n$ and for every $y_i^* \in T(y_i)$ one has

$$\sum_{j=1}^{k-1} \langle y_j^*, y_{j+1} - y_j \rangle \leq \frac{\varepsilon}{2} \sum_{j=1}^{k-1} \|y_{j+1} - y_j\|,$$

which in view of (25) becomes

$$\sum_{j=1}^{k-1} \langle y_j^*, y_{j+1} - y_j \rangle \leq \frac{\varepsilon}{2} \sum_{i=1}^{n-1} \|x_{i+1} - x_i\|. \tag{27}$$

Let now $1 = j_1 < j_2 < \ldots < j_{n-1} = k-1$ be such that for every $i \in \mathbb{N}_{n-1}$ the subfamily $\{y_{j_i}, \ldots y_{j_{i+1}}\}$ is a δ_1-partition of the segment $[x_i, x_{i+1}]$ (in particular, $y_{j_i} = x_i$ and $y_{j_{i+1}} = x_{i+1}$). Since for every $j \in \{j_i, \ldots j_{i+1}\}$ we have $\|y_j - x_i\| < \delta$, it follows from (26) that

$$\langle x_i^*, \frac{y_j - x_i}{\|y_j - x_i\|}\rangle \leq \langle y_j^*, \frac{y_j - x_i}{\|y_j - x_i\|}\rangle + \frac{\varepsilon}{2},$$

which yields

$$\langle x_i^*, x_{i+1} - x_i\rangle \frac{\|y_{j+1} - y_j\|}{\|x_{i+1} - x_i\|} \leq \langle y_j^*, y_{j+1} - y_j\rangle + \frac{\varepsilon}{2}\|y_{j+1} - y_j\|.$$

Summing the above inequality for $j = j_i, \ldots, j_{i+1}$ we obtain

$$\langle x_i^*, x_{i+1} - x_i\rangle \leq \sum_{j=j_i}^{j_{i+1}} \langle y_j^*, y_{j+1} - y_j\rangle + \frac{\varepsilon}{2}\|x_{i+1} - x_i\|.$$

Summing again for $i = 1, \ldots, n-1$ we get

$$\sum_{i=1}^{n-1} \langle x_i^*, x_{i+1} - x_i\rangle \leq \sum_{j=1}^{k-1} \langle y_j^*, y_{j+1} - y_j\rangle + \frac{\varepsilon}{2} \sum_{i=1}^{n-1} \|x_{i+1} - x_i\|. \tag{28}$$

Adding (27) and (28) we obtain (10). The proof is complete.

(ii). This assertion follows analogously and the proof is omitted. $\square$

Remark 5 Cyclic submonotonicity differs from radial cyclic submonotonicity from the fact that it uses the notion of path subdivision (instead of path partition), imposing thus a robust assumption on the behavior of the operator around the chosen path. Let us mention that in view of Remark 2, Proposition 13(i) shows in particular that both Janin's definition of cyclic submonotonicity and his characterization of lower-C^1 functions are particular cases of Definition 5(i) and Theorem 12 respectively.

3 Integration of multivalued operators

A classical result of convex analysis (see (M)) ensures that every maximal cyclically monotone operator T is the subdifferential of some lower semicontinuous convex function f_T. Given $x_0 \in \mathrm{dom}(T)$, a function f_T with the aforementioned property is constructed in the following way:

$$f_T(x) := \sup \left\{ \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i\rangle \;+\; \langle x_n^*, x - x_n\rangle \right\} \tag{29}$$

where the supremum is taken for all $n \geq 1$, all $x_1, x_2, ..., x_n$ in $\mathrm{dom}\,(T)$ and all $x_0^* \in T(x_0), x_1^* \in T(x_1), \ldots, x_n^* \in T(x_n)$. In turns out that the function f_T is proper, lower semicontinuous and convex (as supremum of affine functions), that $T = \partial f_T$, and that functions satisfying this property are equal to f_T up to an additive constant. Moreover, if T is locally bounded, then f_T will be locally Lipschitz.

The above result of Rockafellar does not depend on the dimension of the space, since it is based on global arguments and makes no use of topological notions (see [14] for details). We now give the analogous results for maximal cyclic hypomonotone (respectively, maximal cyclically submonotone) operators. Let us point out the following important difference: In contrast to the above classical result, the forthcoming integration results provide subdifferential representations for the operator T on $\mathrm{intdom}(T)$ (and not on the whole space), using thus implicitly the local boundedness of T (see Proposition 8). We conjecture that the results remain true in the general case.

Theorem 14 *Let $T : X \rightrightarrows X^*$ be a multivalued operator and U a nonempty open and connected subset of* $\mathrm{dom}(T)$.

(H) *T is maximal cyclically hypomonotone on U if, and only if, $T = \partial f$ for some locally Lipschitz d-weakly convex function $f : U \to \mathbb{R}$ unique up to an additive constant.*

(S) *T is maximal cyclically submonotone on U if, and only if, $T = \partial f$ for some locally Lipschitz d-approximately convex function $f : U \to \mathbb{R}$, unique up to an additive constant.*

Proof Theorem 11 (respectively, Theorem 12) corresponds to the "if" part of (H) (respectively, (S)), while the "only if" part of (S) follows from [5, Theorem C].

Let us give an easy argument for establishing the "only if" part of (H). To this end, let T be maximal cyclically hypomonotone on U. Then T is also cyclically submonotone. Let S be any maximal cyclically submonotone extension of S on U. Using (S) we conclude that $S = \partial f$, for some locally Lipschitz d-approximately convex function $f : U \to \mathbb{R}$. It follows from Proposition 10 that S is minimal w*-cusco. In particular, S is a minimal element of the family of w*-cusco operators containing T. Since T is maximal cyclically hypomonotone, it can be shown (in a way analogous to [5, Proposition 8]) that T is w*-cusco. This shows that $T = S = \partial f$ and Theorem 11 guarantees that f is d-weakly convex. Uniqueness of f follows from the uniqueness of f in (S) (see [5, Theorem B]). $\square$

References

[1] Borwein, J.M., Minimal cuscos and subgradients of Lipschitz functions, in: *Fixed Point Theory and its Applications*, (J.-B. Baillon & M. Théra eds.), Pitman Res. Notes in Math. Series, No. 252, Longman, Essex, (1991), 57-82.

[2] Clarke, F., Stern, R. & Wolenski, P., Proximal smoothness and the Lower-C^2 property, *J. Convex Anal.* **2** (1995), 117-144.

[3] Correa, R., Joffre, A. & Thibault L., Subdifferential monotonicity as characterization of convex functions, *Numer. Funct. Anal. Optim.* **15** (1994), 531-535.

[4] Daniilidis, A., Georgiev, P., Approximate convexity and submonotonicity, preprint 13p (2001).

[5] DANIILIDIS, A., GEORGIEV, P. & PENOT, J.-P., Integration of multivalued operators and cyclic submonotonicity, *Trans. Amer. Math. Soc.* **355** (2003), 177-195 (Published electronically on September 6, 2002)

[6] GEORGIEV, P., Submonotone Mappings in Banach Spaces and Applications, *Set-Valued Analysis* **5** (1997), 1-35.

[7] HIRIART-URRUTY, J.-B., "Generalized differentiability, duality and optimization for problems dealing with differences of convex functions", in: Convexity and Duality in Optimization", *Lecture Notes in Econom. Math. Systems* **256** (1984), 37-70.

[8] JANIN, R., *Sur la dualité et la sensibilité dans les problèmes de programmation mathématique*, Thesis de doctorat, Paris 1974.

[9] JANIN, R., Sur des multiapplications qui sont des gradients généralisés, *C.R. Acad. Sc. Paris* **294** (1982), 117-119.

[10] LEBOURG, G., Generic differentiability of Lipschitzian functions, *Trans. Amer. Math. Soc.* **256** (1979), 125-144.

[11] NGAI, H.V., LUC, D.T. & THERA, M., Approximate convex functions, *Journal of Nonlinear and Convex Analysis* **1** (2000), 155-176.

[12] PENOT, J.-P. & BOUGEARD, M., Approximation and decomposition properties of some classes of locally d.c. functions, *Math. Programming* **41** (1988), 195-227.

[13] POLIQUIN, R., ROCKAFELLAR, R.T. & THIBAULT, L., Local differentiability of Distance functions, *Trans. Math. Math. Soc.* **352** (2000), 5251-5262.

[14] ROCKAFELLAR, R.T., On the maximal monotonicity of subdifferential mappings, *Pacific J. Math.* **33** (1970), 209-216.

[15] ROCKAFELLAR, R.T., 'Favorable Classes of Lipschitz Continuous Functions in Subgradient Optimization' in *Nondifferentiable Optimization* (1982), Nurminski E. (eds), Pergamon Press, New York.

[16] ROCKAFELLAR, R.T. & WETS, J.-B., *Variational Analysis*, Springer, New York (1998).

[17] SPINGARN, J.E., Submonotone subdifferentials of Lipschitz functions, *Trans. Amer. Math. Soc.* **264** (1981), 77-89.

[18] VIAL, J.-P., Strong and weak convexity of sets and functions, *Math. Oper. Res.* **8** (1983), 231-259.