# ASYMPTOTIC BEHAVIOUR OF SELF-CONTRACTED PLANAR CURVES AND GRADIENT ORBITS OF CONVEX FUNCTIONS

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Abstract. We hereby introduce and study the notion of self-contracted curves, which encompasses orbits of gradient systems of convex and quasiconvex functions. Our main result shows that bounded self-contracted planar curves have a finite length. We also give an example of a convex function defined in the plane whose gradient orbits spiral infinitely many times around the unique minimizer of the function.

Résumé. Nous introduisons et étudions la notion de courbes auto-contractées, qui englobe les orbites des systèmes de gradient des fonctions convexes et quasi-convexes. Notre résultat principal montre que toute courbe plane auto-contractée bornée est de longueur finie. Nous donnons aussi un exemple de fonction convexe définie sur le plan dont les orbites de gradient spiralent une infinité de fois autour de l'unique minimum de la fonction.

## Contents



## 1. Introduction

This work is mainly devoted to the study of the length of bounded trajectories of the gradient flow of convex (or quasiconvex) functions in the plane. The motivation for this study comes from a well-known result due to S. Łojasiewicz (see [15]), asserting that if $f : \mathbb{R}^n \to \mathbb{R}$ is a real-analytic function and $\bar{x} \in f^{-1}(0)$ is a critical point of $f$, then there exist two constants

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$\rho \in [1/2, 1)$ and $C > 0$ such that

$$||\nabla f(x)|| \geq C\, |f(x)|^{\rho} \tag{1.1}$$

for all $x$ belonging to a neighborhood $U$ of $\bar{x}$. An immediate by-product is the finite length of the orbits of the gradient flow of $f$ lying in $U$. The proof is straightforward using (1.1) and is illustrated below: let $\gamma : [0, +\infty) \to U$ be a gradient trajectory of $f$, that is, $\dot{\gamma}(t) = -\nabla f(\gamma(t))$. Then,

$$-\left(\frac{1}{1-\rho}\right) \frac{d}{dt}\left[f(\gamma(t))^{1-\rho}\right] = -\langle \dot{\gamma}(t), \nabla f(\gamma(t)\rangle\, f(\gamma(t))^{-\rho} = ||\dot{\gamma}(t)||^2\, f(\gamma(t))^{-\rho} \geq C\, ||\dot{\gamma}(t)||,$$

yielding

$$\text{length}(\gamma) = \int_0^{+\infty} ||\dot{\gamma}(t)|| dt < +\infty. \tag{1.2}$$

The aforementioned inequality (1.1) has been extended by K. Kurdyka in [13] for $C^1$ functions belonging to an arbitrary o-minimal structure (we refer to [11] for the relevant definition), in a way that allows us again to deduce the finiteness of the lengths of the gradient orbits in this more general context. In [3] and [4], a further extension has been realized to encompass (nonsmooth) functions and orbits of the corresponding subgradient systems.

It should be noted that in the above cases the functions enjoy an important structural property (o-minimality) and that, for general functions, there is no hope to prove a result like (1.2). A classical example of J. Palis and W. De Melo ([16, page 14]) asserts that the bounded trajectories of the gradient flow of an arbitrary $C^\infty$ function need not converge (in particular, they are of infinite length). In the aforementioned example the critical set of the function is not reduced to a singleton: in Section 7.1, we provide another example of a smooth function having a unique critical point towards which all corresponding orbits converge, but again are of infinite length.

The case when $f$ is a convex coercive function is particularly interesting in view of its potential impact in numerical optimization (see [1], [3], [5], for example). But convex functions are far from being analytic and they do not satisfy neither the Łojasiewicz inequality nor its generalized form established by Kurdyka, unless a growth condition is assumed (see [5, Sections 4.2–4.3] for a sufficient condition and a counter-example). Nevertheless, their rigid structure makes it natural to think that the orbits of their gradient flow are of finite length. It is rather surprising that the answer of this question is not yet known in the literature except in some particular cases.

Let us mention that in the framework of Hilbert spaces, this has been stated as an open problem by H. Brézis [6, Open problems, p. 167]. In infinite dimension, R. Bruck [7] proved that the (sub)gradient orbits of convex coercive functions are converging towards a global minimizer of $f$ but this convergence holds only with respect to the weak topology. Indeed, B. Baillon [2] constructed a counterexample of a lower semicontinuous convex function $f$ in a Hilbert space whose gradient orbits do not converge for the norm topology. A straightforward consequence is that these orbits have infinite length. Concurrently, there are some cases where a convex coercive function $f : H \to \mathbb{R}$ is known to have (sub)gradient orbits of finite length. This is true when the set of minimizers of $f$ has nonempty interior in the Hilbert space $H$ (see H. Brézis [6]), or whenever $f$ satisfies a growth condition. For a detailed discussion and the proofs of these facts, we refer to [5, Section 3-4].

The aforementioned results do not cover the simplest case of a convex smooth function defined in the plane and having a unique minimum. One of the main results of this work is to prove the following:

**Theorem 1.1** (Convex gradient system)**.** *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth convex function which admits a minimum. Then, the trajectories $\gamma$ of the gradient system*

$$\dot{\gamma}(t) = -\nabla f(\gamma(t))$$

*have a finite length.*

The proof of this result does not use the whole convexity of $f$ but, instead, rather relies on the convexity of its level-sets. More precisely, the conclusion of Theorem 1.1 remains also true for the orbits of the gradient flow of a quasiconvex function (see Corollary 6.3).

Actually, both results will follow as consequences of a much more general result (Theorem 1.3) about bounded *self-contracted* planar curves, which allows us to provide a unified framework for this study.

**Definition 1.2** (Self-contracted curve)**.** A curve $\gamma : I \to \mathbb{R}^n$ defined on an interval $I$ of $[0, +\infty)$ is called *self-contracted*, if for every $t_1 \le t_2 \le t_3$, with $t_i \in I$, we have

$$\operatorname{dist}(\gamma(t_1), \gamma(t_3)) \ge \operatorname{dist}(\gamma(t_2), \gamma(t_3)). \tag{1.3}$$

In other words, for every $[a, b] \subset I$, the map $t \in [a, b] \mapsto \operatorname{dist}(\gamma(t), \gamma(b))$ is nonincreasing.

We prove the following.

**Theorem 1.3** (Main result)**.** *Every bounded continuous self-contracted planar curve $\gamma$ is of finite length. More precisely,*

$$\operatorname{length}(\gamma) \le 42\, D(\gamma)$$

*where $D(\gamma)$ is the distance between the endpoints of $\gamma$.*

Let us finally mention that, even if gradient orbits of convex functions have finite length in the plane, they do not enjoy all the properties of the gradient orbits of real-analytic functions. Indeed, on the one hand, the so-called Thom conjecture for the gradient orbits of real-analytic functions holds true: if $x_\infty$ denotes the limit of the orbit $\gamma(t)$, then the secants $(\gamma(t) - x_\infty)/||\gamma(t) - x_\infty||$ converge towards a fixed direction of the unit sphere (see K. Kurdyka, T. Mostowski and A. Parusinski [14]). On the other hand, as we show in Section 7.2, an analogous result fails in the convex case. Indeed, the orbits of a convex gradient flow may turn around their limit infinitely many times.

Our techniques only work in the two-dimensional case. We do not know whether Theorem 1.1 and Theorem 1.3 hold in greater dimension.

The article is organized as follows. In Section 2, we present basic properties of self-contracted curves. In Section 3, we decompose each polygonal approximation of a bounded self-contracted curve in an annulus centered at its endpoint into horizontal and vertical segments. We establish upper bounds on the total length of the vertical segments in Section 3 and on the total length of the horizontal segments in Section 4. The proof of the main result is presented in Section 5. In Section 6, we show that the orbits of various dynamical systems are self-contracted curves. Two (counter)-examples are presented in Section 7.

*Notations.* Throughout the manuscript, we shall deal with the finite-dimensional Euclidean space $\mathbb{R}^n$ equipped with the canonical scalar product $\langle \cdot, \cdot \rangle$. We denote by $\| \cdot \|$ (respectively, $\operatorname{dist}(\cdot, \cdot)$) the corresponding norm (respectively, distance). Therefore, the distance between two points $x$ and $y$ of $\mathbb{R}^2$ will be denoted by $\|x - y\|$, $\operatorname{dist}(x, y)$ or sometimes $|xy|$. We also denote by $\operatorname{dist}(x, S)$ the distance of a given point $x \in \mathbb{R}^n$ to a set $S \subset \mathbb{R}^n$, by $B(x, r)$ the closed ball with center $x \in \mathbb{R}^n$ and radius $r > 0$ and by $S(x, r)$ its boundary, that is, the sphere of the same center and the same radius. For $0 < r < R$, we denote by

$$U(r,R) := \{x \in \mathbb{R}^n \mid r < \|x\| \leq R\} \tag{1.4}$$

the annulus centered at the origin $O$ with outer radius $R$ and inner radius $r$ and by $\Delta R = R - r$ its width. Let

$$[p,q] := \{p + t(q-p) \mid t \in [0,1]\}$$

be the closed segment with endpoints $p, q \in \mathbb{R}^n$. A subset $S$ of $\mathbb{R}^n$ is called convex, if $[p,q] \subset S$ for every $p, q \in S$.

## 2. Self-contracted curves

Throughout this paper, we shall deal with curves $\gamma : I \to \mathbb{R}^n$, defined on an interval $I$ of $\mathbb{R}$. We recall that the length of a curve $\gamma : I \to \mathbb{R}^n$ is defined as

$$\operatorname{length}(\gamma) = \sup \left\{ \sum_{i=1}^{k} \operatorname{dist}(\gamma(t_i), \gamma(t_{i+1})) \right\}$$

where the supremum is taken over all the finite subdivisions $\{t_i\}_{i=1}^{k+1}$ of $I$.

We shall need the following definition.

**Definition 2.1** (Convergence of a curve)**.** A curve $\gamma : I \to \mathbb{R}^n$ is said to converge to a point $x_0 \in \mathbb{R}^n$ if $\gamma(t)$ converges to $x_0$ when $t$ goes to $t_+ := \sup I$.
A curve $\gamma : I \to \mathbb{R}^n$ is said to be *bounded*, if its image $\gamma(I)$ is a bounded subset of $\mathbb{R}^n$.

We start with an elementary property of self-contracted curves.

**Proposition 2.2** (Existence of left/right limits)**.** *Let $\gamma : I \mapsto \mathbb{R}^n$ be a bounded self-contracted curve and $(a,b) \subset I$. Then, $\gamma$ has a limit in $\mathbb{R}^n$ when $t \in (a,b)$ tends to an endpoint of $(a,b)$. In particular, every self-contracted curve can be extended by continuity to the endpoints of $I$ (possibly equal to $\pm\infty$).*

*Proof.* Since $\gamma$ lies in a compact set of $\mathbb{R}^n$, there exists an increasing sequence $\{t_i\}$ in $(a,b)$ with $t_i \to b$ such that $\gamma(t_i)$ converges to some point of $\mathbb{R}^n$, noted $\gamma(b)^+$. Fix any $i, j \in \mathbb{N}^*$ and let $t_i < t < t_{i+j}$. By (1.3), we have

$$\operatorname{dist}(\gamma(t), \gamma(t_{i+j})) \leq \operatorname{dist}(\gamma(t_i), \gamma(t_{i+j})).$$

Letting $j$ go to infinity, we derive

$$\operatorname{dist}(\gamma(t), \gamma(b)^+) \leq \operatorname{dist}(\gamma(t_i), \gamma(b)^+)$$

which gives $\gamma(t) \to \gamma(b)^+$.

Further, using the triangle inequality and the inequality (1.3), we have

$$\operatorname{dist}(\gamma(t_1), \gamma(t_2)) \leq \operatorname{dist}(\gamma(t_1), \gamma(t_3)) + \operatorname{dist}(\gamma(t_3), \gamma(t_2)) \leq 2 \operatorname{dist}(\gamma(t_1), \gamma(t_3)).$$

Using this inequality, we can show, as previously, that $\gamma(t)$ converges as $(a,b) \ni t \to a$.

The last part of the assertion is straightforward. □

The following result is a straightforward consequence of Proposition 2.2.

**Corollary 2.3** (Convergence of bounded self-contracted curves)**.** *Every bounded self-contracted curve $\gamma : (0, +\infty) \to \mathbb{R}^n$ converges to some point $x_0 \in \mathbb{R}^2$ as $t \to +\infty$. Moreover, the function $t \mapsto \operatorname{dist}(x_0, \gamma(t))$ is nonincreasing.*

In the sequel, we shall assume that every self-contracted curve $\gamma : I \mapsto \mathbb{R}^n$ is (defined and) continuous at the endpoints of $I$.

**Remark 2.4** (Basic properties)**.**
(i) Inequality (1.3) shows that the image of a segment $(a,b)$ by a self-contracted curve $\gamma$ lies in a ball of radius $\rho := \mathrm{dist}(\gamma(a),\gamma(b))$.

(ii) A self-contracted curve might not be (left/right) continuous. A simple example is provided by the following planar self-contracted curve:

$$\gamma(t) = \begin{cases} (t,1) & \text{if } t \in (-\infty,0) \\ (0,0) & \text{if } t = 0 \\ (t,-1) & \text{if } t \in (0,+\infty) \end{cases}$$

(iii) If $t \in (a,b) \mapsto \gamma(t)$ is a self-contracted curve, then the curve $t \in (a,b) \mapsto \gamma(a+b-t)$ is not necessarily self-contracted.

(iv) Corollary 2.3 reveals that the trajectories of a general gradient system

$$\dot\gamma(t) = -\nabla f(\gamma(t)), \qquad \gamma(0) = x_0 \in \mathbb{R}^n$$

might fail to be self-contracted curves. Indeed in [16, page 14] an example of a $C^\infty$ function $f : \mathbb{R}^2 \to \mathbb{R}$ is given, for which all trajectories of its gradient system are bounded but fail to converge.

(v) In Section 6, we show that whenever $f$ is (quasi)convex, the gradient trajectories are self-contracted curves. Thus, bounded self-contracted curves might fail to converge for the strong topology in a Hilbert space (see Baillon's example in [2]).

From now on, we restrict ourselves to the two-dimensional case, and study the asymptotic behaviour of self-contracted planar curves.

## 3. Horizontal and Vertical directions

In this section, we introduce a binary-type division of planar segments into horizontal and vertical ones. We shall apply this decomposition for segments issued from polygonal line approximations of a bounded self-contracted curve. In this section, we derive an upper bound on the total length of the vertical segments, while in the next section we shall do the same for the total length of the horizontal ones. Combining both results we shall thus obtain an upper bound estimation on the total length of a bounded self-contracted curve, establishing Theorem 1.3.

Fix $0 < r < R$ and let $U(r,R)$ be the annulus defined in (1.4). Let $\sigma$ be a segment of $U(r,R)$, not reduced to a point. Denote by $p$ and $q$ the endpoints of $\sigma$ and by $m$ its midpoint. Switching $p$ and $q$ is necessary, we can assume that $q$ is closer to the origin $O$ than $p$, that is $\mathrm{dist}(O,q) \le \mathrm{dist}(O,p)$. Let $\theta := \widehat{Omq}$ be the angle between the vectors $\overrightarrow{mO}$ and $\overrightarrow{mq}$, *cf.* Fig. 1. Note that $\theta \in [-\frac{\pi}{2},\frac{\pi}{2}]$ (by convention, inverse-clockwise angles are positive).

**Lemma 3.1** (Segment length estimate)**.** *Let $\sigma$ be a segment of $U(r,R)$ with endpoints $p$ and $q$ such that $\theta \neq \pm\frac{\pi}{2}$. Then*

$$\mathrm{length}(\sigma) \le \frac{2}{\cos\theta}\,|\mathrm{dist}(O,p) - \mathrm{dist}(O,q)|.$$

*Proof.* Let $\bar p$ be the orthogonal projection of $p$ to the line $Om$. Using elementary trigonometry in the right-angled triangle $p\bar p m$, we derive

$$\mathrm{dist}(m,\bar p) = \frac{1}{2}\cos\theta \cdot \mathrm{length}(\sigma).$$

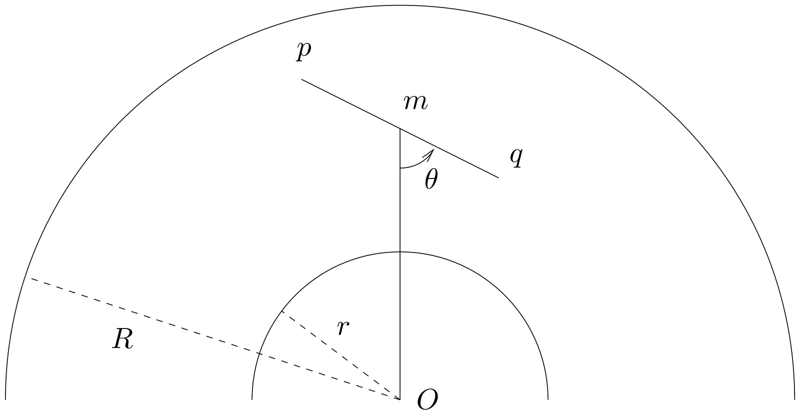


FIGURE 1

Hence,

$$\operatorname{dist}(O,p) - \operatorname{dist}(O,m) \geq \frac{1}{2}\cos\theta \cdot \operatorname{length}(\sigma).$$

Since $\operatorname{dist}(O,q) \leq \operatorname{dist}(O,m)$, the conclusion follows. ☐

Let $\lambda \in (0,1)$ and $\alpha \in (0, \frac{\pi}{2})$ be such that $\sin\alpha < \lambda < 1$, and set $r := \lambda R$. Denote by

$$A := U(\lambda R, R)$$

the corresponding annulus of (1.4), with width equal to $\Delta R = (1-\lambda)R$. We now introduce a crucial definition in the proof of our main result.

**Definition 3.2** (Classification of the segments)**.** Let $\alpha \in (0, \frac{\pi}{2})$, $\lambda \in (0,1)$ and $A$ as above. A nontrivial segment $\sigma$ of $A$ is said to be

- *vertical*, if $\theta$ lies in $(-\frac{\pi}{2}+\alpha, \frac{\pi}{2}-\alpha)$ ;
- *horizontal, pointing in the positive direction,* if $\theta$ lies in $[-\frac{\pi}{2}, -\frac{\pi}{2}+\alpha]$ ;
- *horizontal, pointing in the negative direction,* if $\theta$ lies in $[\frac{\pi}{2}-\alpha, \frac{\pi}{2}]$.

For instance, the segment $[p,q]$ in Fig. 1 points in the negative direction.

**Definition 3.3** (Polygonal approximation)**.** Let $\gamma : I \to \mathbb{R}^2$ be a continuous self-contracted planar curve converging to the origin $O$. We consider polygonal approximations $\{\sigma_i\}_{i=1}^k$ of $\gamma$ in the annulus $A$, as follows: Let $t_1 < t_2 < \cdots < t_{k+1}$ be a sequence of points of $I$ with $\gamma(t_i) \neq \gamma(t_{i+1})$ such that the restriction of $\gamma$ to $[t_1, t_{k+1}]$ lies in $A$. Refining the subdivision if necessary, we can further assume that for every $i \in \{1,\ldots,k\}$ the segment $\sigma_i$ with endpoints $p_i = \gamma(t_i)$ and $q_i = p_{i+1} = \gamma(t_{i+1})$ lies in $A$ and that the length of $\sigma_i$ is within any desired precision $\eta > 0$ of the length of $\gamma_{|[t_i,t_{i+1}]}$ (this precision $\eta > 0$ will be defined at the beginning of Section 4 and will only depend on $\alpha$, $\lambda$ and $R$). Since the function $t \mapsto \operatorname{dist}(O,\gamma(t))$ is nonincreasing (*cf.* Corollary 2.3), we can further assume that, if $\sigma_i := [p_i, q_i]$ is vertical, then $q_i$ is the closest point of $\sigma_i$ to the origin. We denote by $m_i := \frac{1}{2}(p_i + q_i)$ the midpoint of $\sigma_i$.

**Remark 3.4.** It is worth noticing that the polygonal approximation of a self-contracted curve introduced above is no more a self-contracted curve in general. Nevertheless, one still has

$$\operatorname{dist}(p_i, p_l) \geq \operatorname{dist}(p_j, p_l) \quad \text{when } 1 \leq i \leq j \leq l \leq k+1,$$

which is the property we will use.

The total length of the vertical segments satisfies the following upper bound.

**Lemma 3.5** (Total vertical length upper bound)**.** *If $\{\sigma_i\}_{i=1}^k$ is a polygonal approximation of $\gamma$ in the annulus $A$, cf. Definition 3.3, then*

$$\sum_{i\in\mathcal{V}} \text{length}(\sigma_i) \le \frac{2}{\sin\alpha}\,\Delta R$$

*where the sum is taken over all indices $i \in \mathcal{V} \subset \{1,\dots,k\}$ corresponding to the vertical segments.*

*Proof.* Let $\theta_i$ denote the angle between $\overrightarrow{m_iO}$ and $\overrightarrow{m_iq_i}$. Since $i\in\mathcal{V}$, it follows that $|\theta_i| < \frac{\pi}{2}-\alpha$, whence $(\cos\theta_i)^{-1} < (\sin\alpha)^{-1}$. From Lemma 3.1 (segment length estimation), we obtain

$$\sum_{i\in\mathcal{V}} \text{length}(\sigma_i) < \frac{2}{\sin\alpha}\sum_{i\in\mathcal{V}} \text{dist}(O,p_i) - \text{dist}(O,q_i).$$

Since $\text{dist}(O,p_i) \ge \text{dist}(O,q_i)$, for all $i\in\{1,\dots,k\}$ we deduce

$$\sum_{i\in\mathcal{V}} \text{dist}(O,p_i) - \text{dist}(O,q_i) \le \sum_{i=1}^{k} \text{dist}(O,p_i) - \text{dist}(O,q_i).$$

Now, since $q_i = p_{i+1}$, the right-hand term is equal to $\text{dist}(O,p_1) - \text{dist}(O,q_k)$, which is less or equal to the width $\Delta R$ of $A$. The proof is complete. □

## 4. Length estimate for horizontal directions

In this section, we keep the notations and the definitions from the previous section. In particular,

$$\alpha \in (0,\frac{\pi}{2}), \qquad \sin\alpha < \lambda < 1, \qquad A := U(\lambda R, R) \tag{4.1}$$

and $\{\sigma_i\}_{i=1}^k$ is a polygonal approximation of $\gamma$ in the annulus $A$, *cf.* Definition 3.3.

We establish an upper bound on the total length of the horizontal segments issued from the polygonal approximation of $\gamma$.

Let $x\in A$. The distance from the origin to the half-line $L_x$ passing through $x$ and making an angle $\alpha > 0$ with $\overrightarrow{xO}$ is equal to $\sin\alpha\cdot\text{dist}(O,x)$. Thus, the half-line $L_x$ intersects the circle $S(0,\lambda R)$ of radius $\lambda R$ centered at the origin at two points. These two points are noted $\pi(x)$ and $\pi'(x)$, with $\pi(x)$ closer to $x$ than $\pi'(x)$. Furthermore, the half-line $L_x$ intersects $A$ along two segments $\Delta_x$ and $\Delta'_x$, where the endpoints of $\Delta_x$ agree with $x$ and $\pi(x)$, and one of the endpoints of $\Delta'_x$ agrees with $\pi'(x)$. Note that $\min_{x\in A} \parallel \pi(x) - \pi'(x) \parallel > \delta > 0$. The half-line $L_x$ extends to a line which bounds a (closed) half-plane $H_x$ containing the origin of the plane. Denote by $\overline{H_x^c}$ the (closed) half-plane with the same boundary as $H_x$, not containing the origin (see Fig. 2 for an illustration of these notations). The mappings $x\mapsto\pi(x)$ and $x\mapsto\pi'(x)$ from $A$ to $S(0,\lambda R)$ are clearly continuous, thus also uniformly continuous. Therefore, there exists $\eta>0$ such that for every pair of points $x,y\in A$ which are $\eta$-close from each other (*i.e.* $\text{dist}(x,y)<\eta$), we have

$$\text{dist}(x,\pi(y)) < \text{dist}(x,\pi'(y)), \quad \text{dist}(\pi(x),\pi(y)) < \text{dist}(\pi(x),\pi'(x)),$$

$$\text{and} \quad \text{dist}(\pi(x),\pi(y)) < \text{dist}(\pi(x),\pi'(y)). \tag{4.2}$$

We shall further need the following technical lemmas.

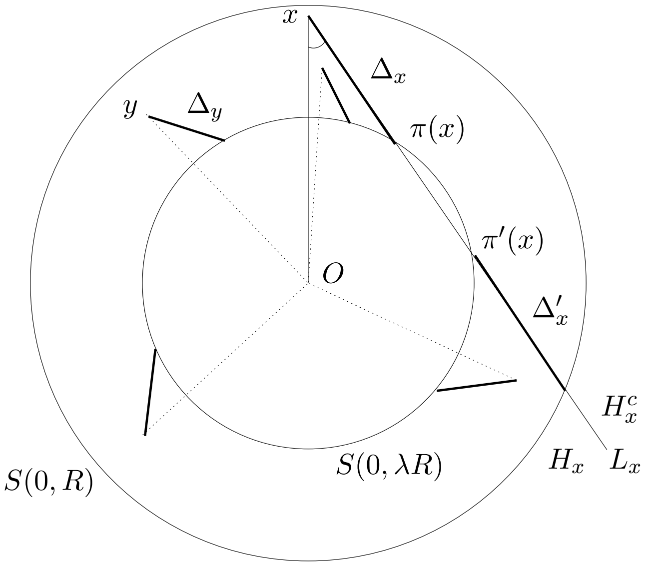


FIGURE 2

**Lemma 4.1** (Essential disjointness of $\Delta_x$, $\Delta_y$)**.** *Let $x$ and $y$ be two distinct points of $A$ such that* $\operatorname{dist}(O,x) > \operatorname{dist}(O,y)$*. If $y \in H_x$, then the segment $\Delta_y$ lies in $H_x$. Furthermore, $\Delta_y$ does not intersect $\Delta_x$, except possibly at $y$.*

*Proof.* Suppose first that $y$ lies in $\Delta_x$. One easily checks that the angle $\widehat{Oz\pi(x)}$ increases when the point $z$ moves from $x$ to $\pi(x)$ along $\Delta_x$. In particular, the angle $\widehat{Oy\pi(x)}$ is greater than $\alpha$. Therefore, the segment $\Delta_y$ lies in $H_x$ and meets $\Delta_x$ only at $y$.

Suppose now that $y \notin \Delta_x$ and (towards a contradiction) that $\Delta_y$ intersects $\Delta_x$ at $z \neq y$. Let $y'$ denote the intersection point of $\Delta_x$ with the circle of radius $|Oy|$ centered at the origin. Then, the image of $\Delta_y$ by the rotation around the origin taking $y$ to $y'$ does not lie in $H_x$ (the image of $z$ should lie in $H_x^c$). On the other hand, this image agrees with $\Delta_{y'}$ since the rotation sends the ray $Oy$ to $Oy'$. From the previous discussion, we conclude that the image of $\Delta_y$ is contained in $H_x$. Hence a contradiction.

Finally, suppose that $y \notin \Delta_x$, $\Delta_y \cap \Delta_x = \emptyset$ and $\Delta_y$ intersects $\Delta'_x$ at $z$. If $z = \pi'(x)$, then obviously $\Delta_y$ lies in $\overline{H_x^c}$ and $y = \pi'(x)$. Suppose now that $z \neq \pi'(x)$ (thus $z \neq \pi(y)$). Since the angle $\widehat{Oz\pi(y)}$ is positive while the angle $\widehat{Oz\pi'(x)}$ is negative, we obtain that $\widehat{\pi'(x)z\pi(y)}$ is positive which is not possible. The proof is complete. □

**Lemma 4.2** (Injectivity of $\pi$)**.** *Let $\sigma := [p,q]$ be an horizontal segment of $A$, with midpoint $m$, pointing in the positive direction. Assume* $\operatorname{dist}(O,q) \leq \operatorname{dist}(O,p)$*. Then,*

(1) *the restriction of $\pi$ to $\sigma$ is injective;*
(2) *if the length of $\sigma$ is at most $\eta$, then the circular arc $\pi([p,m])$ lies in $\overline{H_m^c}$.*

*Proof.* Let $x, y \in [p,q]$ with $\operatorname{dist}(p,x) < \operatorname{dist}(p,y)$. Since the horizontal segment $\sigma$ points in the positive direction, the angle $\widehat{xOy}$ is positive and $y$ lies in $H_x$. From Lemma 4.1 (Essential disjointness of $\Delta_x$, $\Delta_y$), $\pi(x)$ and $\pi(y)$ are distinct (the case $y = \pi(x)$ is impossible since it would yield that the angle $\widehat{\pi(x)Ox} = \widehat{yOx}$ is positive, a contradiction). Hence the first part of the lemma follows.

Let $x \in (m, p]$. From above, the midpoint $m$ of $\sigma$ lies in $H_x$ and the segments $\Delta_x$ and $\Delta_m$ do not intersect each other from Lemma 4.1. By definition of $\eta$, in view of (4.2) the points $x$ and $\pi(x)$ are closer to $\pi(m)$ than to $\pi'(m)$. Thus, the segment $\Delta_x$, which does not intersect $\Delta_m$, does not intersect $\Delta'_m$ either. That is, $\Delta_x$ lies either in $H_m$ or in $\overline{H^c_m}$. Since the horizontal segment $\sigma$ is pointing towards the positive direction, the point $x$ belongs to $H^c_m$. Therefore, the same holds true for the other endpoint $\pi(x)$ of $\Delta_x$. It follows that the circular arc $\pi([p, m])$ with endpoints $\pi(p)$ and $\pi(m)$ is contained in $H^c_m$. □

**Lemma 4.3** (Length estimate for horizontal segments)**.** *Let* $\sigma := [p, q]$ *be an horizontal segment of* $A$ *with midpoint* $m$*, pointing in the positive direction. Assume* $\operatorname{dist}(O, q) \leq \operatorname{dist}(O, p)$*. Then,*

$$\operatorname{length}(\sigma) \leq \frac{2}{\lambda} \operatorname{length}(\pi([p, m])).$$

*Proof.* The line passing through $p$ and the origin $O$ together with the circle of radius $|Op|$ centered at the origin define a decomposition of the circle of radius $|pm|$ centered at $p$ into four arcs. One of these arcs, denoted by $C$, contains the point $m$. Let $m'$ be the endpoint of $C$ lying in the circle of radius $|Op|$ centered at the origin, *cf.* Fig. 3 below. By construction,

$$\operatorname{length}(\sigma) = 2\,|pm| = 2\,|pm'|. \tag{4.3}$$

Since $m'$ is at the same distance from the origin as $p$, there exists a rotation $\rho$ centered at the origin which takes $p$ to $m'$. This rotation sends the ray $[O, p]$ to $[O, m']$ and preserves distances and angles. Therefore, it also sends $\Delta_p$ to $\Delta_{m'}$. In particular, the rotation $\rho$ maps $\pi(p)$ to $\pi(m')$. From Thales' formula, we derive

$$\frac{|\pi(p)\pi(m')|}{|pm'|} = \frac{|O\pi(p)|}{|Op|}.$$

Hence,

$$|\pi(p)\pi(m')| \geq \lambda\,|pm'|. \tag{4.4}$$

Since the endpoints of the segment $[\pi(p), \pi(m')]$ lie in the arc $\pi([pm'])$, we have

$$|\pi(p)\pi(m')| \leq \operatorname{length}(\pi([p, m'])). \tag{4.5}$$

When a point $x$, starting at $m'$, moves along $C$, the angle $\widehat{Oxp}$ increases from less than $\frac{\pi}{2}$ to $\pi$. Thus, there exists a unique point $m''$ of $C$ where the angle $\widehat{Om''p}$ is equal to $\frac{\pi}{2} + \alpha$. By definition of an horizontal segment pointing in the positive direction, the angle $\widehat{Omp}$ lies between $\frac{\pi}{2}$ and $\frac{\pi}{2} + \alpha$. Therefore, the point $m$ lies in $C$ between $m'$ and $m''$, *cf.* Fig. 3.

The angles $\widehat{Om''p}$ and $\widehat{Om''\pi(m'')}$ are equal to $\frac{\pi}{2} + \alpha$ and $\alpha$. Therefore, the ray $pm''$ makes a right angle at $m''$ with the line $D''$ passing through $m''$ and $\pi(m'')$. Thus, the line $D''$ is tangent to $C$ at $m''$. This implies that the points $O$, $m$ and $m''$ lie in the same half-plane delimited by the line $D'$ passing through $m'$ and parallel to $D''$. Since the ray $Om'$ makes an angle less than $\alpha$ with $D'$ at $m'$, the angle between $D'$ and $L_{m'}$ (the half-line passing through $m'$ and making an angle $\alpha > 0$ with $\overrightarrow{m'O}$) is positive, *cf.* Fig. 3. Therefore, the points $O$, $m$ and $m''$ lie in $H_{m'}$.

By applying Lemma 4.1 (Essential disjointness of $\Delta_x$, $\Delta_y$), successively for $x = m'$ and $y = m$, and for $x = p$ and $y = m'$, we obtain that $\pi([p, m'])$ is contained in $\pi([p, m])$ from the injectivity of the restriction of $\pi$ to the segments $[p, m']$ and $[p, m]$, *cf.* Lemma 4.2. Hence,

$$\operatorname{length}(\pi([p, m'])) \leq \operatorname{length}(\pi([p, m])). \tag{4.6}$$

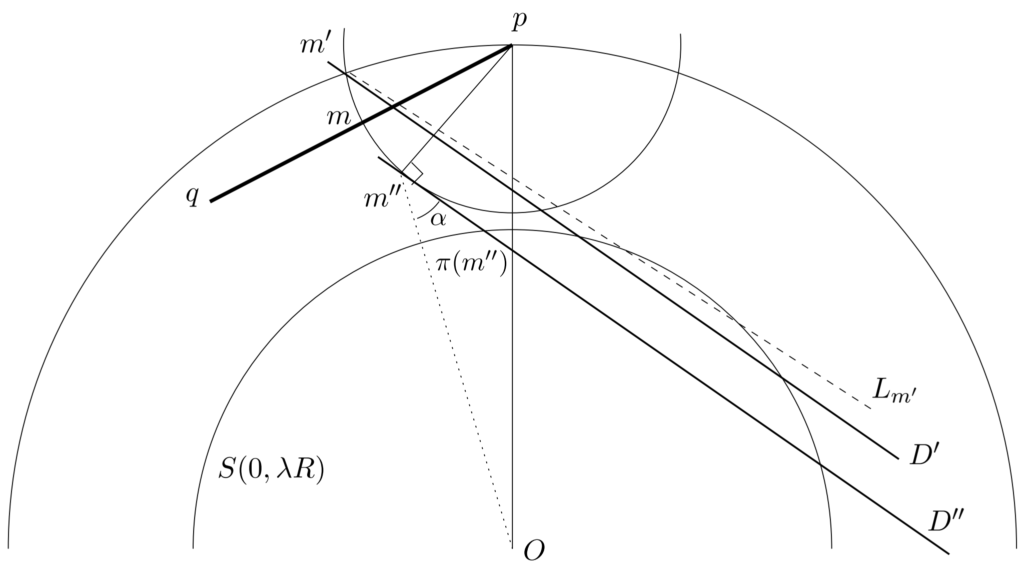


Figure 3

Putting together the inequalities (4.3), (4.4), (4.5) and (4.6), we obtain the desired bound. □

Let us now consider a polygonal decomposition in $A$

$$\sigma_i := [p_i, q_i], \quad i \in \{1, \dots, k\}$$

of a bounded self-contracted curve $\gamma$ converging to $O$, *cf.* Definition 3.3.

**Lemma 4.4** (Disjointness of $\pi(p_i, m_i)$ and $\pi(p_j, m_j)$)**.** *Let $\sigma_i$ and $\sigma_j$ be two distinct horizontal segments of a polygonal approximation of $\gamma$ in $A$, cf. Definition 3.3, both pointing in the positive direction. Then, the images by $\pi$ of $[p_i, m_i]$ and $[p_j, m_j]$ are disjoint.*

*Proof.* Switching the indices $i$ and $j$ if necessary, we can assume that $i < j$.

From Lemma 4.2, the arc $\pi([p_i, m_i])$ is contained in $\overline{H^c_{m_i}}$. To prove the desired result, it is enough to show that $\pi([p_j, m_j])$ lies in the complement of $\overline{H^c_{m_i}}$ (*i.e.* the interior of $H_{m_i}$).

From the definition of a self-contracted curve, the points $p_j$ and $q_j$ are closer to $q_i$ than to $p_i$ (see Remark 3.4). Thus, $p_j$ and $q_j$, and so their barycenter $m_j$, lie in the half-plane delimited by the perpendicular bisector of $\sigma_i$. (Notice that this half-plane also contains the origin $O$, in view of Corollary 2.3.) The half-line of this bisector with endpoint $m_i$ which makes an acute angle with the ray $m_iO$ is noted $D_{m_i}$. Since the horizontal segment $\sigma_i$ points in the positive direction, its half-bisector $D_{m_i}$ makes an angle less or equal to $\alpha$ with $m_iO$. Thus, $L_{m_i}$ lies in the half-plane delimited by the perpendicular bisector of $\sigma_i$ not containing the origin.

Now, since the function $t \mapsto \mathrm{dist}(O, \gamma(t))$ is nonincreasing, the points $p_j$ and $q_j$, and so their barycenter $m_j$, belong to the disk of radius $|Oq_i| < |Om_i|$ centered at the origin. Therefore, the points $p_j$ and $m_j$ lie in $H_{m_i}$, and $\mathrm{dist}(O, m_i) > \max\{\mathrm{dist}(O, p_j), \mathrm{dist}(O, m_j)\}$. From Lemma 4.1 (Essential disjointness of $\Delta_x$, $\Delta_y$), the segments $\Delta_{p_j}$ and $\Delta_{m_j}$ lie in $H_{m_i}$ and do not intersect its boundary. Therefore, their endpoints $\pi(p_j)$ and $\pi(m_j)$ also lie in the interior $H_{m_i}$. □

The total length of the horizontal segments satisfies the following upper bound.

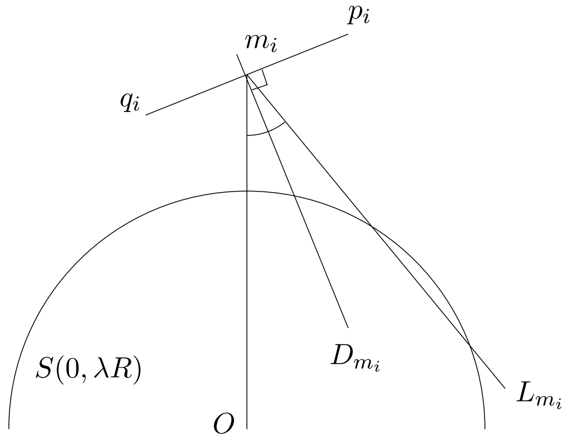


FIGURE 4

**Lemma 4.5** (Total horizontal length upper bound)**.** *If $\{\sigma_i\}_{i=1}^k$ is a polygonal approximation of $\gamma$ in the annulus $A$, cf. Definition 3.3, then*

$$\sum_{i \in \mathcal{H}} \text{length}(\sigma_i) \leq \frac{8\pi}{1-\lambda} \Delta R$$

*where $\mathcal{H}$ is the set of indices corresponding to the horizontal segments.*

*Proof.* From Lemma 4.3 (Length estimate for horizontal segments), the sum of the lengths of the horizontal segments $\sigma_i$ pointing in the positive direction satisfies

$$\sum_{i \in \mathcal{H}_+} \text{length}(\sigma_i) \leq \frac{2}{\lambda} \sum_{i \in \mathcal{H}_+} \text{length}(\pi([p_i, m_i])),$$

where $\mathcal{H}_+$ is the set of indices corresponding to the horizontal segments pointing in the positive direction. Since the arcs $\pi([p_i, m_i])$ of the circle $S(0, \lambda R)$ are disjoint, *cf.* Lemma 4.4, we have

$$\sum_{i \in \mathcal{H}_+} \text{length}(\pi([p_i, m_i])) \leq 2\pi\lambda R.$$

Analogous arguments lead to a similar bound for the sum of the lengths of the horizontal segments $\sigma_i$ pointing in the negative direction. Recalling that $\Delta R = (1-\lambda)R$ the result follows. □

## 5. Proof of the main result

In order to prove our main theorem (*cf.* Theorem 1.3), we shall first need the following result.

**Proposition 5.1** (Length estimate in the annulus $A$)**.** *Every continuous self-contracted planar curve $\gamma$ converging to the origin $O$ satisfies*

$$\text{length}(\gamma \cap A) \leq 42\, \Delta R.$$

*Proof.* Consider a decomposition of $\gamma$ into segments $\sigma_i$ as in Definition 3.3 (refining a subdivision does not decrease the sum). From Lemma 3.5 (Total vertical length upper bound) and Lemma 4.5 (Total horizontal length upper bound), the length $L$ of the polygonal line $\sigma_1 \sigma_2 \cdots \sigma_k$

satisfies

$$\begin{aligned} L &= \sum_{i\in\mathcal{H}} \operatorname{length}(\sigma_i) + \sum_{i\in\mathcal{V}} \operatorname{length}(\sigma_i) \\ &\leq \left( \frac{8\pi}{1-\lambda} + \frac{2}{\sin\alpha} \right) \Delta R. \end{aligned}$$

By taking the supremum of $L$ over all such decompositions respecting the annulus $A$, we derive the same upper bound for the length of $\gamma \cap A$. Finally, by letting $\sin\alpha$ go to $\lambda$, *cf.* (4.1) , we obtain

$$\operatorname{length}(\gamma \cap A) \leq \left( \frac{8\pi}{1-\lambda} + \frac{2}{\lambda} \right) \Delta R.$$

Hence the result since $\min_{0<\lambda<1} \frac{8\pi}{1-\lambda} + \frac{2}{\lambda} \leq 42.$ □

Now, we can derive our main result.

*Proof of Theorem 1.3.* From Corollary 2.3, the bounded continuous self-contracted curve $\gamma$ converges to a point. Using a translation if necessary, we can assume that this point agrees with the origin $O$ of $\mathbb{R}^2$.

Let $t_- = \inf I$. Denote by $\gamma(t_-)$ the limit of $\gamma(t)$ when $t$ goes to $t_-$, (*cf.* Proposition 2.2 (Existence of left/right limits)). Set $R_0 = \operatorname{dist}(O, \gamma(t_-))$. For $i \in \mathbb{N}$, let $A_i$ be the planar annulus centered at the origin with outer radius $R_i$ and inner radius $R_{i+1}$, where $R_{i+1} = \lambda R_i$, with $\lambda \in (0,1)$ given in (4.1). From Proposition 5.1 (Length estimate in the annulus), we have

$$\operatorname{length}(\gamma \cap A_i) \leq 42\, \Delta R_i \tag{5.1}$$

where $\Delta R_i$ is the width of $A_i$. Since $\lambda < 1$, the sequence $R_i$ goes to zero and the sum of the width of the disjoint annulus $A_i$ is equal to $R_0$. Thus, taking the sum of the above inequalities (5.1) for $i \in \mathbb{N}$ we obtain the desired bound

$$\operatorname{length}(\gamma) \leq 42 \operatorname{dist}(O, \gamma(t_-)).$$

The proof is complete. □

## 6. Gradient and subgradient systems, and convex foliations

In this section, we apply Theorem 1.3 to derive length estimates, first for orbits of dynamical systems of gradient or subgradient type, then for orbits orthogonal to a convex foliation. The key fact is to observe that in some interesting particular cases (for instance, $f$ convex or quasiconvex) these curves are self-contracted. Recall however that this is not the case for gradient dynamical systems defined by a general $C^\infty$ function, as already observed in Remark 2.4 (iv).

6.1. **Gradient dynamical system – quasiconvex case.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $C^k$ function ($k \geq 1$), $x_0 \in \mathbb{R}^n$ and consider the Gradient Dynamical System

$$\begin{cases} \dot\gamma(t) = -\nabla f(\gamma(t)),\ t > 0 \\ \gamma(0) = x_0 \in \mathbb{R}^n. \end{cases} \tag{6.1}$$

It follows directly from the standard theory of Ordinary Differential Equations (see [12], for example) that the system (6.1) admits a solution (trajectory) $\gamma : I \longmapsto \mathbb{R}^n$, where $I \subset [0, +\infty)$, which is a curve of class $C^{k-1}$. Note that the case $k = 1$ corresponds to mere continuity of $\gamma$, while for $k > 1$ (or more generally, if $f$ is assumed $C^{1,1}$, that is, $\nabla f$ is Lipschitz continuous), the trajectory $\gamma$ is unique. In the sequel, we shall always consider maximal solutions of (6.1), that is, for which $I = [0, T)$, where $T > 0$ is the maximal time such that $\gamma$ is defined in $[0, T)$. We shall refer to them as orbits of the *gradient flow* of $f$.

We will also need the following definition.

**Definition 6.1** (Convex, quasiconvex and coercive functions)**.** A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *convex* (respectively, *quasiconvex*) if its epigraph

$$\operatorname{epi} f := \{(x, y) \in \mathbb{R}^{n+1} \mid f(x) \le y\}$$

is a convex subset of $\mathbb{R}^n \times \mathbb{R}$ (respectively, if for every $y \in \mathbb{R}$ the sublevel set $\{x \in \mathbb{R}^n \mid f(x) \le y\}$ is a convex subset of $\mathbb{R}^n$). A function $f$ is called *coercive* (or *proper*), if its level sets are bounded, or equivalently, if

$$\lim_{\|x\| \to +\infty} f(x) = +\infty. \tag{6.2}$$

It is straightforward to see that whenever $f$ is coercive the corresponding flow orbits are bounded curves and therefore $I = [0, +\infty)$ (the trajectories are defined for all $t \ge 0$). Notice in particular that the function

$$t \longmapsto f(\gamma(t)), \quad t \in [0, +\infty)$$

is a natural Lyapunov function for the orbits of the flow, *i.e.* it is nonincreasing along the trajectories. Moreover, unless $\gamma$ meets a critical point (*i.e.* $\nabla f(\gamma(t_*)) = 0$ for some $t_* \in [0, +\infty)$), the function defined in (6.1) is decreasing and $\gamma$ is injective.

Let us finally recall (*e.g.* [9, Theorem 2.1]) that a (differentiable) function $f : \mathbb{R}^n \to \mathbb{R}$ is quasiconvex if and only if for every $x, y \in \mathbb{R}^n$ the following holds:

$$\langle \nabla f(x), y - x \rangle > 0 \Rightarrow f(y) \ge f(x). \tag{6.3}$$

We are now ready to establish the following result.

**Proposition 6.2** (Quasiconvex orbits are self-contracted curves)**.** *The orbits of the gradient flow of a quasiconvex $C^{1,1}$ function are self-contracted curves.*

*Proof.* Let $\gamma : I \longmapsto \mathbb{R}^n$ be an orbit of the gradient flow of $f$. Let $0 \le t \le t_1$ be in $I$ and consider the function

$$g(t) = \frac{1}{2} ||\gamma(t) - \gamma(t_1)||^2, \quad t \in I.$$

In view of (6.1), we easily deduce that

$$g'(t) = \langle \nabla f(\gamma(t)), \gamma(t_1) - \gamma(t) \rangle.$$

If $g'(t) > 0$ for some $t \in [0, t_1)$, then the quasiconvexity of $f$ would imply that $f(\gamma(t_1)) \ge f(\gamma(t))$ (see (6.3)), which in view of (6.1) would yield that $\gamma(t') = \gamma(t_1)$ for all $t' \in [t, t_1]$ and $\nabla f(\gamma(t)) = 0$, a contradiction. Thus, $g$ is nonincreasing in the interval $[0, t_1]$, which proves the assertion. □

The following corollary is a straightforward consequence of the previous proposition and Theorem 1.3 (Main result).

**Corollary 6.3** (Orbits of a gradient quasiconvex flow)**.** *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a coercive $C^{1,1}$ quasiconvex function. Then, for every $x_0 \in \mathbb{R}^2$, the orbit of the gradient flow* (6.1) *converges and has finite length.*

6.2. **Subgradient dynamical systems - convex case.** A convex function is a particular case of a quasiconvex function. Therefore, Corollary 6.3 implies that the orbits of the gradient flow of $C^{1,1}$ convex functions are of finite length. It is well-known ([6]) that in the case of a (nonsmooth) convex function $f : \mathbb{R}^n \to \mathbb{R}$ (or more generally, for a semiconvex function [10]), the gradient system (6.1) can be generalized to the following differential inclusion, called Subgradient Dynamical System

$$\begin{cases} \dot\gamma(t) \in -\partial f(\gamma(t)) & \text{a.e. } t \in [0, +\infty), \\ \gamma(0) = x_0 \in \mathbb{R}^n, \end{cases} \tag{6.4}$$

where $\partial f$ is the set of the subgradients (subdifferential) of $f$. If $f : \mathbb{R}^n \to \mathbb{R}$ is convex, then this latter set is defined as

$$\partial f(x) = \{p \in \mathbb{R}^n \mid f(y) \geq f(x) + \langle p, y - x\rangle,\ \forall y \in \mathbb{R}^n\} \quad \text{for all } x \in \mathbb{R}^n .$$

The above formula defines always a nonempty convex compact subset of $\mathbb{R}^n$, and reduces to $\{\nabla f(x)\}$ whenever $f$ is differentiable at $x$, *cf.* [8]. It is also known that (6.4) has a unique absolutely continuous solution $\gamma : [0, +\infty) \to \mathbb{R}^n$, that is, the derivative $\dot\gamma(t) = \frac{d}{dt}\gamma(t)$ exists almost everywhere and for every $0 \leq t_1 \leq t_2$,

$$\gamma(t_2) = \gamma(t_1) + \int_{t_1}^{t_2} \dot\gamma(t)\, dt .$$

The analogous of Proposition 6.2 holds true.

**Proposition 6.4** (Subgradient convex flow orbits are self-contracted curves)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex continuous function. Then, for every $x_0 \in \mathbb{R}^n$, the trajectory $\gamma$ of the subgradient system* (6.4) *is a self-contracted curve.*

*Proof.* We give a sketch of proof for the reader convenience (we refer to [6] for details). It is easy to prove that $t \in [0, +\infty) \mapsto f(\gamma(t))$ is convex and that for almost all $t \geq 0$ we have

$$\frac{d}{dt} f(\gamma(t)) = -||\dot\gamma(t)||^2 \leq 0.$$

Therefore $t \mapsto f(\gamma(t))$ is nonincreasing and $\gamma(t) \in \{f \leq f(x_0)\}$ is bounded. Moreover, for all $t_1 > 0$ and for almost all $t \in [0,\ t_1]$

$$\frac{1}{2}\frac{d}{dt}||\gamma(t) - \gamma(t_1)||^2 = \langle\dot\gamma(t), \gamma(t) - \gamma(t_1)\rangle \leq f(\gamma(t_1)) - f(\gamma(t)) \leq 0.$$

This implies that $t \in [0, t_1) \mapsto ||\gamma(t) - \gamma(t_1)||^2$ is nonincreasing yielding that $\gamma$ is a self-contracted curve. □

When $n = 2$, we have the following generalization of Theorem 1.1.

**Corollary 6.5** (Orbits of a subgradient convex flow)**.** *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a convex continuous function which admits a minimum. Then, for every $x_0 \in \mathbb{R}^2$, the orbit of the gradient flow* (6.4) *converges and has finite length.*

6.3. **Trajectories orthogonal to a convex foliation.** In this section we consider orbits that are “orthogonal” to a *convex foliation*. Let us introduce the definition of the latter. (For any subset $C \subset \mathbb{R}^2$, $\operatorname{int} C$ denotes the interior of $C$ and $\partial C$ its boundary.)

Let $\{C_\alpha\}_{\alpha\in[0,A]}$ (where $A > 0$) be a family of subsets of $\mathbb{R}^2$ such that

(i) For all $\alpha \in [0, A]$, $C_\alpha$ is convex compact
(ii) If $\alpha > \alpha'$, then $C_\alpha \subset \operatorname{int} C_{\alpha'}$, (6.5)
(iii) For every $x \in C_0 \setminus \operatorname{int} C_A$, there exists a unique $\alpha \in [0, A]$ such that $x \in \partial C_\alpha$.

We shall refer to the above as a foliation made up of convex surfaces. We shall now define a notion of "orthogonality" for an orbit $\gamma$ with respect to this foliation. To this end, let $T \in (0, +\infty]$ and $\gamma : [0, T) \to \mathbb{R}^2$ be an absolutely continuous curve. We say that the curve $\gamma$ is "orthogonal" to the foliation defined in (6.5) if the following conditions hold:

$$\begin{aligned}&\text{(i) for every } t \in [0,T), \text{ there exists } \alpha \in [0,A] \text{ such that } \gamma(t) \in \partial C_\alpha,\\ &\text{(ii) for almost all } t \in (0,T), \text{ if } \gamma(t) \in \partial C_\alpha, \text{ then for all } x \in C_\alpha,\ \langle \dot{\gamma}(t), x - \gamma(t)\rangle \geq 0,\\ &\text{(iii) if } t' > t \text{ and } \gamma(t) \in C_\alpha, \text{ then } \gamma(t') \in C_\alpha.\end{aligned} \tag{6.6}$$

Condition (ii) in (6.6) is a nonsmooth generalization of orthogonality: if $\partial C_\alpha$ is smooth at $\gamma(t)$ and $\gamma$ is differentiable at $t$ then $\dot{\gamma}(t)$ is orthogonal to the tangent space of $\partial C_\alpha$ at $\gamma(t)$. Further, condition (iii) guarantees that the curve $\gamma(t)$ enters into each convex set of the foliation. In this context, one has the following result.

**Proposition 6.6** (Orbits orthogonal to a convex foliation)**.** *The curve $\gamma$ is a bounded self-contracted curve, thus, of bounded length.*

*Proof.* The curve $\gamma$ is clearly bounded. Let $0 \leq t_1 < T$. Then, for almost all $t \in [0, t_1]$, we have

$$\frac{1}{2}\frac{d}{dt}||\gamma(t) - \gamma(t_1)||^2 = \langle \dot{\gamma}(t), \gamma(t) - \gamma(t_1)\rangle. \tag{6.7}$$

By (6.6) (i), we have $\gamma(t) \in \partial C_\alpha$ for some $\alpha$. By (6.6) (iii) and since $t_1 \geq t$, we also have $\gamma(t_1) \in C_\alpha$. Therefore, (6.6) (ii) implies that the right-hand side of (6.7) is nonpositive. It follows that $t \in [0, t_1] \mapsto ||\gamma(t) - \gamma(t_1)||^2$ is nonincreasing and $\gamma$ is self-contracted. Applying Theorem 1.3 finishes the proof. □

**Remark 6.7.** (i) The sublevel sets of a continuous quasiconvex function need not define a convex foliation. Indeed, consider the quasiconvex function $f : [-2, 2] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x, & \text{if } -2 \leq x \leq 0,\\ 0, & \text{if } \ 0 \leq x \leq 1,\\ x-1, & \text{if } \ 1 \leq x \leq 2.\end{cases}$$

Then the sublevel sets of $f$ do not define a foliation on $[-2, 2] \subset \mathbb{R}$ since property (iii) of (6.5) fails at the level set $[f = 0]$. This drawback appears whenever the level sets of such functions have "flat" parts outside the set of their global minimizers. Actually, it follows from [9, Theorem 3.1] that the sublevel sets of a continuous quasiconvex coercive function $f$ define a convex foliation if and only if the function is *semi-strictly quasiconvex.* (We refer to [9] for the exact definition and basic properties of semi-strictly quasiconvex functions.)

(ii) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a coercive $C^{1,1}$ quasiconvex function and $\gamma : [0, +\infty) \to \mathbb{R}^2$ be the solution of (6.1). Let $x_\infty$ be the limit of $\gamma(t)$ as $t \to +\infty$ and assume that $f$ has no critical point in $\{f(x_\infty) < f \leq f(x_0)\}$. Then, it is not difficult to see that $\{f \leq \alpha\}_{\alpha \in [f(x_\infty), f(x_0)]}$ is a family of $C^1$ convex compact subsets which satisfies (6.5) (in fact, $f$ is semi-strictly quasiconvex in $[f(x_\infty), f(x_0)]$) and $\gamma$ satisfies (6.6).

(iii) Despite the first remark, Proposition 6.6 can be used to obtain the result of Corollary 6.3 without the extra assumption made in the second remark. The reason is that the trajectory of the gradient flow will not pass through the flat parts of $f$ anyway (if it does, then it stops there). We leave the technical details to the reader.

## 7. Two counter-examples

7.1. **Absence of Convexity.** The second conclusion of Corollary 6.3 (about the finite length of the orbits) fails if $f$ is not quasiconvex, even when the function is $C^\infty$ and has a unique critical point at its global minimum. Let us give an explicit example below:

Define a function $f : \mathbb{R}^2 \to \mathbb{R}$ in polar coordinates as

$$f(r, \theta) = e^{-1/r}(1 + r + \sin(\tfrac{1}{r} + \theta))$$

with $f(O) = 0$. The graph of $f$ in the plane $\theta = 0$ looks like the graph of Fig. 5.

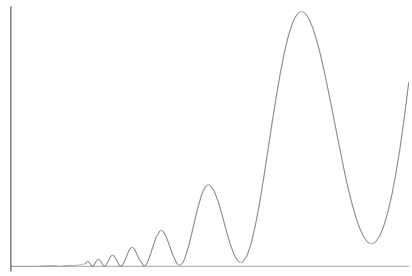

Figure 5

One can check that $f$ is smooth, positive away from $O$, with no critical point except at the origin. The gradient trajectory of $f$ issued from the point $(r, \theta) = ((\frac{3\pi}{2})^{-1}, 0)$ remains close to the spiral given by

$$\begin{cases} r &= \left(\frac{3\pi}{2} + t\right)^{-1} \\ \theta &= -t \end{cases}$$

where $t$ runs over $[0, \infty)$. Therefore, it converges to the origin and its length is infinite.

7.2. **Thom conjecture fails for convex functions.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a convex continuous function which admits a minimum. Then, Corollary 6.5 guarantees that the orbits of the gradient flow of $f$ have finite length (thus, *a fortiori*, converge to a minimizer of $f$). However, it may happen that each orbit turns around its limit infinitely many times. In particular the corresponding statement of the Thom conjecture fails in the convex case.

We construct below a counter-example using a technique due to D. Torralba [18] which allows us to build a convex function with prescribed level-sets given by a sequence of nested convex sets. Let us recall his result.

For any convex set $C \subset \mathbb{R}^n$, the support function of $C$ is defined as $\delta_C(x^*) = \sup_{x \in C}\langle x, x^*\rangle$ for all $x^* \in \mathbb{R}^n$. Let $\{C_k\}_{k\in\mathbb{N}}$ be a decreasing sequence of convex compact subsets of $\mathbb{R}^2$ such that $C_{k+1} \subset \operatorname{int} C_k$. Set

$$K_k = \max_{||x^*||=1} \frac{\delta_{C_{k-1}}(x^*) - \delta_{C_k}(x^*)}{\delta_{C_k}(x^*) - \delta_{C_{k+1}}(x^*)}.$$

Then Torralba's theorem [18] asserts that for every real sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ satisfying

$$0 < K_k(\lambda_k - \lambda_{k+1}) \le \lambda_{k-1} - \lambda_k \quad \text{for every } k \ge 1, \tag{7.1}$$

there exists a continuous convex function $f$ such that for every $k \in \mathbb{N}$, $\{f \le \lambda_k\} = C_k$. Moreover, $\lambda_k$ converges to $\min f$ and, for any $k \ge 0$ and $\lambda \in [\lambda_{k+1}, \lambda_k]$, we have

$$\{f \le \lambda\} = \left(\frac{\lambda - \lambda_{k+1}}{\lambda_k - \lambda_{k+1}}\right) C_k + \left(\frac{\lambda_k - \lambda}{\lambda_k - \lambda_{k+1}}\right) C_{k+1} \tag{7.2}$$

(*i.e.,* the level-sets of $f$ are convex interpolations of the two nearest prescribed level-sets).

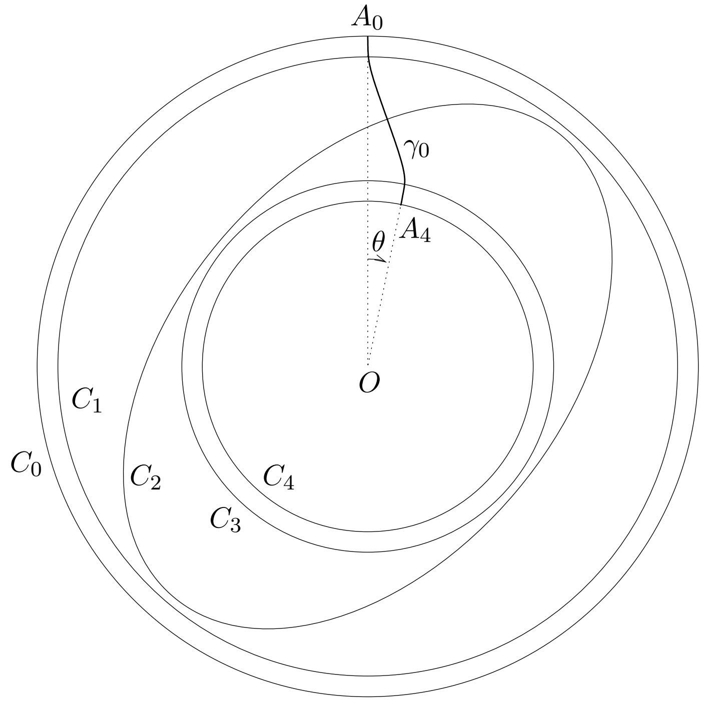


FIGURE 6

*Step 1. Constructing a first piece of trajectory.* Consider the finite decreasing sequence of convex sets $C_0 = B(O,1)$, $C_1 = B(O,0.9)$, $C_2 = E$, $C_3 = B(O,0.6)$ and $C_4 = B(O,1/2)$ where $E$ is a compact set bounded by an ellipse (see Fig. 6). It is easy to find a sequence $\{\lambda_k\}$ which satisfies (7.1): set $K = \max\{K_1, K_2, K_3, K_4\} + 1 > 1$ (since $C' \subset \operatorname{int} C$ implies $\delta_C > \delta_{C'}$), take $\lambda_0 = 1$, $\lambda_1 = 1/2$ and

$$\lambda_k - \lambda_{k+1} = \frac{1}{K^k}(\lambda_0 - \lambda_1). \tag{7.3}$$

We then obtain a positive function $f_0 : C_0 \to \mathbb{R}$ with $\operatorname{argmin} f_0 := \{f_0 = \min f_0\} = C_4$. Consider the subgradient trajectory $\gamma_0$ starting from the point $A_0$ of $C_0$ (see Fig. 6). It reaches $A_4 \in \partial C_4$. From (7.2) this trajectory is radial (pointing towards the origin) between $\partial C_0 = \{f_0 = \lambda_0\}$ and $\partial C_1 = \{f_0 = \lambda_1\}$ and between $\partial C_3 = \{f_0 = \lambda_3\}$ and $\partial C_4 = \{f_0 = \lambda_4\}$. Due to the presence of the ellipse $C_2$, the trajectory deflects with an angle $\theta := \widehat{A_0 O A_4} > 0$ in the clockwise direction.

*Step 2. Construction of the function from the previous step.* Consider the transformation $\mathcal{T} = r \circ h$, where $h$ is the homothety of center $O$ and coefficient $1/2$ and $r$ is the rotation of center $O$ and angle $\theta$. We define, for all $k \in \mathbb{N}$ and $\bar{k} \in \{0,1,2,3\}$

$$C_k = \mathcal{T}^{[k/4]}(C_{\bar{k}}) \quad \text{where } [k/4] \text{ is the integer part of } k/4 \text{ and } \bar{k} = k \text{ (modulo 4)}$$

(see Fig. 7 for the first steps of the construction).

The sequence of convex sets $\{C_k\}$ satisfies the assumptions of Torralba's theorem and we can define a sequence $\{\lambda_k\}$ as in (7.3) which satisfies (7.1) (note that $\{K_k\}$ is 4–periodic since, for all convex set $C \subset \mathbb{R}^2$ and $x^* \in \mathbb{R}^2$, $\delta_{\mathcal{T}(C)}(x^*) = \frac{1}{2}\delta_C(r^{-1}(x^*)))$. We obtain a convex continuous function $f : C_0 \to \mathbb{R}^+$ with $\operatorname{argmin} f = \{O\}$. The trajectory starting from the top of $C_0$ spirals

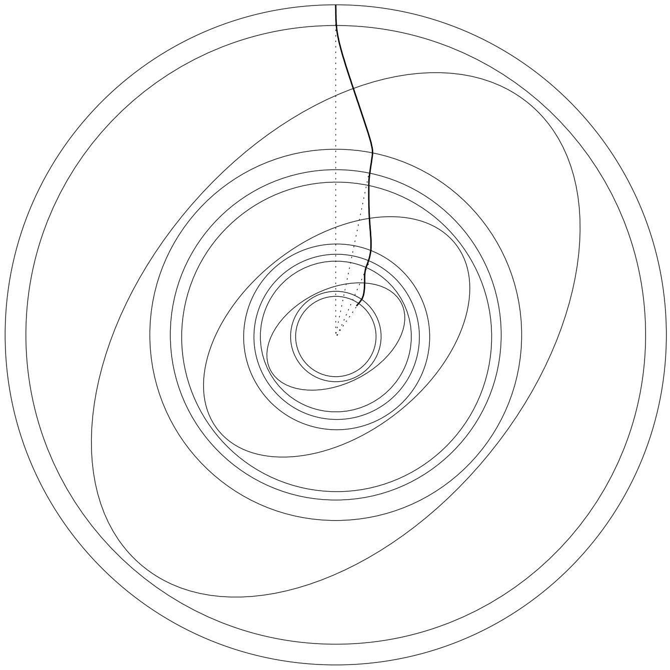

FIGURE 7

around the origin and converges to $O$ (see Fig. 7 where the beginning of the trajectory is drawn with a deflection of $3\theta$).

*Step 3. Smoothing $f$.* Actually, the function $f$ built above is $C^\infty$ except at the origin and at the boundaries $\partial C_k$. It is possible to smooth out $f$ in order to obtain a function which is $C^\infty$ everywhere except at the origin and $C^m$ at the origin (for any fixed $m \geq 1$). The smoothing procedure is quite involved from a technical point of view and is omitted. We refer the interested reader to [5, Section 4.3] where such a smoothing is realized (in a different context). This procedure does not modify significantly neither the function nor its gradient trajectories (*i.e.* they remain a spiral). This concludes the construction.

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