STUDIA MATHEMATICA
Online First version

Asymmetric free spaces and canonical asymmetrizations

by

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Abstract. A construction analogous to that of Godefroy–Kalton for metric spaces allows one to embed isometrically, in a canonical way, every quasi-metric space (X, d) in an asymmetric normed space $\mathcal{F}_a(X, d)$ (its quasi-metric free space, also called asymmetric free space or semi-Lipschitz free space). The quasi-metric free space satisfies a universal property (linearization of semi-Lipschitz functions). The (conic) dual of $\mathcal{F}_a(X, d)$ coincides with the non-linear asymmetric dual of (X, d), that is, the space $\mathrm{SLip}_0(X, d)$ of semi-Lipschitz functions on (X, d), vanishing at a base point. In particular, for the case of a metric space (X, D), the above construction yields its usual free space. On the other hand, every metric space (X, D) naturally inherits a canonical asymmetrization coming from its free space $\mathcal{F}(X)$. This gives rise to a quasi-metric space (X, D_+) and an asymmetric free space $\mathcal{F}_a(X, D_+)$. The symmetrization of the latter is isomorphic to the original free space $\mathcal{F}(X)$. The results of this work are illustrated with explicit examples.

Contents

2020 *Mathematics Subject Classification*: Primary 46B20; Secondary 39B82, 46A22.
Key words and phrases: free space, canonical asymmetrization, semi-Lipschitz functions, quasi-metric space.
Received 27 May 2020; revised 16 November 2020.
Published online 28 May 2021.

DOI: 10.4064/sm200527-24-11

1. Introduction. Arens and Eells [4] showed that every metric space (X, D) can be isometrically embedded as a closed subset of a normed linear space. The closed linear span of the image of X under this embedding is called the *Arens–Eells space* [39, Section 2.2]. The idea of considering isometric embeddings of metric spaces in linear spaces goes back to Kuratowski, Wojdysławski and Klee [29]. It also implicitly appears in classical works of Kantorovich [27, 28], where a new distance in the space of finite measures on X was defined (known as the Kantorovich–Rubinstein distance, [24] e.g.) in a way that entails an isometric embedding of (X, D) into the dual space $C(X)^*$. The authors of [4] also obtained an analogous embedding of a uniform space in a locally convex linear space. The terms *free Banach space* and, respectively, *free locally convex space* have then been conceived [35, 34] to refer to the resulting spaces.

The terminology *Lipschitz free space* (or simply, *free space*) over a metric space (X, D) has been introduced and highly popularized with the seminal work of Godefroy and Kalton [22], where they employed this term to describe a very similar construction to the one of the Arens–Eells space (see Remark 3.9 for a comparison), but with emphasis on the linearization of both the metric space and its natural morphisms (Lipschitz functions between metric spaces). Free spaces, in this new terminology, have rapidly gained the interest of many researchers in functional analysis ([2, 3, 8, 13, 16, 20, 23] e.g.) and the topic has become, arguably, one of its most active trends nowadays.

Let us outline the construction. Given a metric space (X, D) with a distinguished point x_0 (called a *base point*), the free space $\mathcal{F}(X)$ is constructed as follows: we first consider as *pivot* space (*non-linear dual* of X) the Banach space $\mathrm{Lip}_0(X)$ of real-valued Lipschitz functions vanishing at the base point, endowed with the norm

$$\|f\|_{\mathrm{Lip}} = \sup_{\substack{x,y\in X\\ x\neq y}} \frac{|f(x)-f(y)|}{D(x,y)}.$$

Then each $x \in X$ is identified to a Dirac measure δ_x acting linearly on $\mathrm{Lip}_0(X)$ as evaluation. Then the mapping

$$\widehat{\delta} : X \to \mathrm{Lip}_0(X)^*$$

that maps x to δ_x is an isometric embedding. The Lipschitz free space $\mathcal{F}(X)$

over X is defined as the closed linear span of $\widehat{\delta}(X)$ in $\mathrm{Lip}_0(X)^*$. Furthermore, the free space is a predual for $\mathrm{Lip}_0(X)$, meaning that $\mathcal{F}(X)^*$ is isometrically isomorphic to $\mathrm{Lip}_0(X)$ (therefore, the space $\mathrm{Lip}_0(X)$ is at the same time the (linear) dual of $\mathcal{F}(X)$ and the non-linear dual of X). For a survey on the properties and development of Lipschitz free spaces, we refer the reader to [21]. We also refer to Bachir [5] for prior constructions based on evaluations over some algebra of functions acting on X.

In the present work, using the aforementioned embedding, we show that metric spaces can be asymmetrized in a canonical way, giving rise to quasi-metric spaces, that is, spaces equipped with an asymmetric distance (see Definition 2.2). Semi-Lipschitz functions (Definition 2.26) are the natural morphisms for such spaces. Starting from a quasi-metric space (X, d) with a base point $x_0 \in X$, the normed cone structure (Definition 2.16) of the set $\mathrm{SLip}_0(X)$ of real-valued semi-Lipschitz functions on X vanishing at x_0 is used as an *asymmetric pivot* space to obtain a semi-Lipschitz free construction, which is analogous to the Kalton–Godefroy symmetric construction (this latter uses as pivot the Lipschitz functions). This leads the notion of *semi-Lipschitz free space* (or *quasi-metric free space*) $\mathcal{F}_a(X, d)$ for (X, d), where the set $\mathrm{SLip}_0(X)$ is both the non-linear (conic) dual of X and the (linear, conic) dual of $\mathcal{F}_a(X, d)$. We emphasize that $\mathrm{SLip}_0(X)$ is not a linear space in general, therefore we need to enhance the duality of *normed cones*. This being said, the semi-Lipschitz free construction remains compatible with the classical one in the symmetric case. Moreover, it is also compatible with the aforementioned canonical asymmetrization, in the sense that the semi-Lipschitz free space of the canonical asymmetrization of a metric space and the asymmetrization of its free space are often identical (Proposition 3.18) and in any case they have isomorphic symmetrizations (Theorem 3.12).

Quasi-metric spaces and asymmetric norms have recently attracted a lot of interest: they arise naturally when considering non-reversible Finsler manifolds [9, 14, 33] (see also [7, 15]), and have applications in physics [25], as well as in game theory [1, 19]. The properties of spaces with asymmetric norms have been studied by several authors (see [11, 36] and references therein), emphasizing similarities and differences with respect to the theory of (symmetric) normed spaces. Besides its intrinsic interest, and the aforementioned applications, this theory was also stimulated by the study of oriented graphs and by applications in computer science, mainly to the complexity of algorithms.

In this work we endeavor a new insight into the current state-of-the-art, by showing that morphisms of quasi-metric spaces can be linearized in a similar manner to what is done in the symmetric case through an asymmetric free space, and that this asymmetric free theory behaves equally well and is fully compatible with the symmetric theory in a canonical manner. Indeed, there is a canonical way to move from a symmetric to an asymmetric space and vice versa, which in addition is compatible with the embeddings into their free spaces.

The manuscript is organized as follows: In Section 2 we recall basic notions and definitions and we fix our notation. We also give some auxiliary results required for the development of the theory in the asymmetric case, together with results about linear functionals, dual conic norms and continuity on normed cones. We also give the definition of a canonical asymmetrization of a metric space. The main result is established in Section 3, with the definition of the semi-Lipschitz free space $\mathcal{F}_a(X)$ of a quasi-metric space X (Definition 3.4) and its characteristic feature that its dual is exactly the space $\mathrm{SLip}_0(X)$ (Theorem 3.5). The semi-Lipschitz free space $\mathcal{F}_a(X)$ is a bicomplete asymmetric normed space (it is naturally endowed with an asymmetric norm). For this reason, we shall also refer to it as the asymmetric free space of X. In Section 4, through a simple diagram chasing argument, we show that semi-Lipschitz free spaces enjoy a canonical (and useful) linearization property: every semi-Lipschitz map between pointed quasi-metric spaces extends to a linear map between the corresponding semi-Lipschitz free spaces (Corollary 4.4). In Section 5 we give concrete examples of asymmetric free spaces in order to help the reader to get an insight into this new theory. Finally, Section 6 contains open questions and outlines possible further research lines.

2. Notation and preliminaries. Throughout this article we denote by $\mathbb{R}_+$ the set of non-negative real numbers and we use the convention $\inf \emptyset = +\infty$. Given a vector space E, we denote by $\|\cdot\| : E \to \mathbb{R}_+$ a norm on E and by $\|\cdot| : E \to \mathbb{R}_+$ an *asymmetric norm* on E, that is, a function satisfying:

(i) $\forall x, y \in E : \|x+y| \le \|x| + \|y|$;
(ii) $\forall x \in E : x = 0 \Leftrightarrow \|x| = 0$;
(iii) $\forall x \in E,\ \forall r > 0 : \|rx| = r\|x|$.

If we replace the second condition by

$$\text{(ii)}' \qquad x = 0 \Leftrightarrow \begin{cases} \|x| = 0, \\ \|-x| = 0 \end{cases}$$

then we say that $\|\cdot| : E \to \mathbb{R}_+$ is an *asymmetric hemi-norm* on E. The term *asymmetric normed space* refers to pairs $(E, \|\cdot|)$ having either an asymmetric norm or an asymmetric hemi-norm. Notice that an asymmetric (hemi-)norm differs from a norm in that $\|-x| = \|x|$ is not necessarily true.

We may also consider, keeping the same notation, *extended asymmetric norms*, allowing $\|\cdot|$ to take the value $+\infty$. Finally, we denote by u the asymmetric hemi-norm on $\mathbb{R}$ defined by

$$u(x) = \max\{x, 0\} \quad \text{for every } x \in \mathbb{R}. \tag{2.1}$$

REMARK 2.1 (Asymmetrizations in $\mathcal{F}(X)$). There is a natural way to asymmetrize the norm $\|\cdot\|_{\mathcal{F}}$ of the free space $\mathcal{F}(X)$ of a given metric space (X, D), based on the dual space $L := \mathrm{Lip}_0(X)$. Let us denote by $\langle\cdot,\cdot\rangle$ the duality map of the duality pair $(L, \mathcal{F}(X))$. Then the norm $\|\cdot\|_{\mathcal{F}}$ of $\mathcal{F}(X)$ can be represented as follows:

$$\|Q\|_{\mathcal{F}} := \sup_{\substack{\phi\in L\\ \|\phi\|_L\le 1}} \langle\phi, Q\rangle \quad \text{for every } Q \in \mathcal{F}(X). \tag{2.2}$$

Let us recall that a (convex) *cone* in a linear space is a (convex) subset P such that $\lambda x \in P$ for every $x \in P$ and $\lambda \in \mathbb{R}_+$. In this work we shall use the term *cone* to refer to a convex cone. Consider any generating closed cone P of L (i.e., $L = \mathrm{span}(P) = P - P$) that satisfies

$$\forall\phi \in L,\ \exists\phi_1, \phi_2 \in P: \quad \begin{cases} \phi = \phi_1 - \phi_2, \\ \max\{\|\phi_1\|_L, \|\phi_2\|_L\} \le \|\phi\|_L \le \|\phi_1\|_L + \|\phi_2\|_L. \end{cases} \tag{2.3}$$

We set

$$\|Q|_{\mathcal{F}_P} := \sup_{\substack{\phi\in P\\ \|\phi\|_L\le 1}} \langle\phi, Q\rangle \quad \text{for every } Q \in \mathcal{F}(X). \tag{2.4}$$

Notice that for any $Q \in \mathcal{F}(X)$ we have $\max\{\|Q|_{\mathcal{F}_P}, \|{-Q}|_{\mathcal{F}_P}\} \le \|Q\|_{\mathcal{F}}$. Since the supremum in (2.2) is attained at some $\phi \in L$ with $\|\phi\|_L = 1$ (by the Hahn–Banach theorem), using the decomposition (2.3) we deduce

$$\|Q\|_{\mathcal{F}} = \langle\phi, Q\rangle = \langle\phi_1, Q\rangle + \langle\phi_2, -Q\rangle \le \|Q|_{\mathcal{F}_P} + \|{-Q}|_{\mathcal{F}_P}. \tag{2.5}$$

This shows that (ii)$'$ holds and (2.4) defines an asymmetric (hemi-)norm $\|\cdot|_{\mathcal{F}_P}$ on the vector space $\mathcal{F}(X)$.

We shall refer to the asymmetric norm $\|\cdot|_{\mathcal{F}_P}$ defined in (2.4) as the P-*asymmetrization* of the free space $\mathcal{F}(X)$, for which we implicitly assume that (2.3) holds. We shall mainly deal with the case where P is the cone of positive Lipschitz functions, that is,

$$P = L_+ := \{\phi \in L : \phi \ge 0\}.$$

In this case, we denote the arising asymmetric norm by $\|\cdot|_{\mathcal{F}_+}$. Notice that if $\phi\ (= \phi^+ - \phi^-) \in L$ then both its positive part ϕ^+ and its negative part ϕ^- are also in L and they satisfy $|\phi^+(x) - \phi^+(y)| \le |\phi(x) - \phi(y)|$ and $|\phi^-(x) - \phi^-(y)| \le |\phi(x) - \phi(y)|$ for all $x, y \in X$, which leads to (2.3).

More generally, a P-asymmetrization of $\mathcal{F}(X)$ is called *canonical* if P is of the form

$$P := \{\phi \in L : T\phi \ge 0\},$$

where T is a *linear isometry* that identifies L with some Banach lattice in a canonical way.

2.1. Quasi-metric spaces. Let us introduce the notion of a quasi-metric space, which will be the main focus of this work.

Definition 2.2 (Quasi-metric space). A *quasi-metric space* is a pair (X, d), where $X \neq \emptyset$ and

$$d : X \times X \to [0, \infty)$$

is a function, called a *quasi-metric* (or *quasi-distance*), satisfying:

(i) $\forall x, y, z \in X$: $d(x, y) \leq d(x, z) + d(z, y)$ (*triangular inequality*);
(ii) $\forall x, y \in X$: $x = y \Leftrightarrow d(x, y) = 0$.

Note that a quasi-metric does not have the symmetry property $d(x, y) = d(y, x)$ of a distance. If we replace (ii) by

$$\text{(ii)}' \qquad x = y \Leftrightarrow \begin{cases} d(x, y) = 0 \\ d(y, x) = 0 \end{cases}$$

then we say that d is a *quasi-hemi-metric*. In this work we shall also consider *extended quasi-metrics* $\tilde{d} : X \times X \to [0, \infty]$, that is, quasi-metrics that satisfy the same two conditions above, but are also allowed to take the value $+\infty$. If X is a vector space equipped with an (extended) asymmetric (hemi-)norm $\| \cdot |$, then the function

$$d(x, y) := \|y - x| \quad \text{for all } x, y \in X \tag{2.6}$$

is an (extended) quasi-(hemi-)metric on X that satisfies

$$d(x + z, y + z) = d(x, y) \quad \text{and} \quad d(rx, ry) = rd(x, y), \tag{2.7}$$

for all $x, y, z \in X$ and $r \in \mathbb{R}_+$. Furthermore, for all $x, y \in X$ the *reverse quasi-metric* $\bar{d}$ is defined by

$$\bar{d}(x, y) = d(y, x).$$

Throughout this paper, we shall be dealing with both variants of quasi-metric spaces. The term *quasi-metric space* will thus refer to a pair (X, d) where d is either a quasi-distance or a quasi-hemi-distance.

Remark 2.3 (Terminology alert I). The reader should be alerted that the terminology may vary in the literature. Some authors allow a quasi-hemi-metric and a asymmetric hemi-norm to also take negative values. They also use the terms hemi-metric and hemi-norm to refer to what we call quasi-hemi-metric and asymmetric hemi-norm, respectively (see, for instance, [19]). In our work, the qualifier *quasi* refers to the asymmetry of the metric, and *hemi* to the fact that distinct elements x, y in X may have quasi-distance $d(x, y)$ equal to 0.

Two quasi-metric spaces can be completely identified via isometries. (The reader should be alerted that the slightly weaker notion of *almost isometry* also exists, and is more appropriate in relation to Banach–Stone type theorems [9, 14].)

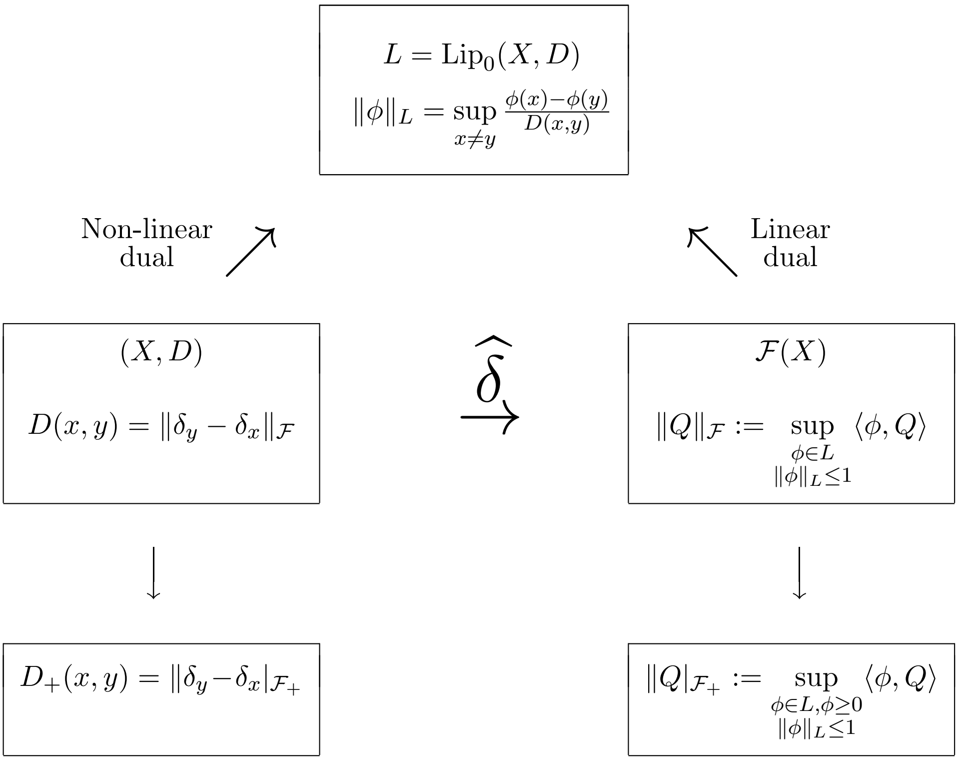
Definition 2.4 (Isometry). A bijective mapping Φ between extended quasi-metric spaces (X, d) and (Y, ρ) is called an *isometry* if for all $x_1, x_2 \in X$,

$$\rho(\Phi(x_1), \Phi(x_2)) = d(x_1, x_2).$$

Definition 2.5 (Canonical asymmetrization of a metric space). Let (X, D) be a metric space with a base point $x_0 \in X$. Every P-asymmetrization of the free space $\mathcal{F}(X)$ (cf. Remark 2.1) induces, via the isometric injection of X into $\mathcal{F}(X)$, an asymmetrization of the distance D, given by

$$D_P(x, y) = \|\delta_y - \delta_x|_{\mathcal{F}_P} = \sup_{\substack{\phi \in P \\ \|\phi\|_L \leq 1}} (\phi(y) - \phi(x)) \quad \text{for all } x, y \in X.$$

The quasi-(hemi-)distance D_P is called the *P-asymmetrization* of (X, D). If $\| \cdot |_{\mathcal{F}_P}$ is a canonical asymmetrization of $\mathcal{F}(X)$, then D_P will be called a *canonical asymmetrization* of D. If $P = L_+$, the canonical asymmetrization will be denoted by D_+. The diagram below depicts the situation.



Let us illustrate the above notion of canonical asymmetrization by means of the following simple example.

Example 2.6 (Canonical asymmetrizations of $\mathbb{R}$). Let us consider $\mathbb{R}$ as a metric space, with its usual distance $D(x,y) = |y-x|$ for all $x, y, \in \mathbb{R}$, and $x_0 = 0$ as a distinguished point. It is well known [20, 39] that the free space $\mathcal{F}(\mathbb{R})$ can be identified with the space of Lebesgue-integrable functions $\mathcal{L}^1(\mathbb{R})$, provided we identify the space $L = (\mathrm{Lip}_0(X,D), \|\cdot\|_L)$ of real-valued Lipschitz functions vanishing at 0 with the Banach space $\mathcal{L}^\infty(\mathbb{R})$ (essentially bounded Lebesgue-measurable functions) via the canonical linear isometry $T\phi = \phi'$ (a.e.), for all $\phi \in L$ (cf. the Rademacher theorem). Then taking either

$$P = L_+ = \{\phi \in L : \phi \geq 0\} \quad \text{or, respectively,} \quad P = \{\phi \in L : \phi' \geq 0\}$$

leads to two different canonical asymmetrizations of $\mathbb{R}$ (via the asymmetrizations $\|\cdot|_{\mathcal{F}_+}$ and respectively $\|\cdot|_{\mathcal{F}_P}$ of its free space). The first asymmetrization is given by the formula

$$D_+(x,y) = \|\delta(y) - \delta(x)|_{\mathcal{F}_+} = \sup_{\substack{\phi \in L_+ \\ \|\phi\|_L \leq 1}} (\phi(y) - \phi(x)).$$

Notice that $D_+(x,y) \leq \max\{|y-x|, |y|\}$. It can be easily seen that if either $y > x > 0$ or $y < x < 0$, then $D_+(x,y) = |y-x|$ (take $\phi_*(t) = |t|$ in L_+ with $\|\phi_*\|_L = 1$). However, $D_+(1,n) = n-1$, while $D_+(n,1) = 1$ for every $n \geq 2$.

The second asymmetrization, thanks to the monotonicity of every ϕ in P, implies that for all $x, y \in X$,

$$\begin{aligned} D_P(x,y) = \|\delta_y - \delta_x|_{\mathcal{F}_P} &= \sup_{\substack{\phi \in L,\, \phi' \geq 0 \\ \|\phi\|_L \leq 1}} (\phi(y) - \phi(x)) \\ &= \max\{y - x,\, 0\} = u(y-x) = d_u(x,y), \end{aligned}$$

where $u(\cdot)$ is the asymmetric hemi-norm given by $u(x) = \max\{x, 0\}$ for all $x \in \mathbb{R}$ and d_u is the corresponding quasi-hemi-distance.

2.2. Symmetrized distance and topologies. Every quasi-metric distance can be symmetrized in the sense of the following definition.

Definition 2.7 (Symmetrized distance). Let (X,d) be a quasi-metric space. Then

$$d^{s_0}(x,y) = \max\{d(x,y), d(y,x)\} \quad \text{and} \quad d^s(x,y) = d(x,y) + d(y,x) \tag{2.8}$$

are two natural symmetrizations of the quasi-distance d, which are equivalent to each other:

$$d^{s_0}(x,y) \leq d^s(x,y) \leq 2d^{s_0}(x,y) \quad \text{for all } x, y \in X.$$

If d is an extended quasi-metric, then so is $\bar{d}$ and consequently the symmetrizations d^s and d^{s_0} give rise to extended metrics. In the case that X is a vector space and d satisfies (2.7), the above symmetrizations preserve the invariance by translations and homothety. Notice further that (2.5) shows that the symmetrization of the P-asymmetrized norm $\|\cdot|_{\mathcal{F}_P}$ of a free space $\mathcal{F}(X)$ is equivalent to $\|\cdot\|_{\mathcal{F}}$ (cf. Remark 2.1). A similar remark applies to the symmetrization of the P-asymmetrization of the distance of a metric space (X, D) (cf. Definition 2.5).

Proposition 2.8 (Asymmetrization vs. symmetrization). *Assume that (X, D_P) is a P-asymmetrization of a metric space (X, D) (cf. Definition 2.5). Then the symmetrizations D_P^s and $D_P^{s_0}$ are bi-Lipschitz equivalent to the initial distance D, and consequently, the Banach spaces $\mathrm{Lip}_0(X, D)$, $\mathrm{Lip}_0(X, D_P^s)$ and $\mathrm{Lip}_0(X, D_P^{s_0})$ are isomorphic.*

Proof. It suffices to prove the result for D_P^s. Take $x, y \in X$. Let $\hat{\phi}$ be a function in $L = \mathrm{Lip}_0(X, D)$ with $\|\hat{\phi}\|_L \leq 1$ such that

$$D(x,y) = \sup_{\substack{\phi\in L\\ \|\phi\|_L\leq 1}} (\phi(y) - \phi(x)) = \hat{\phi}(y) - \hat{\phi}(x).$$

Let $\hat{\phi}_1$ and $\hat{\phi}_2$ be functions in P such that $\hat{\phi} = \hat{\phi}_1 - \hat{\phi}_2$, with the inequality $\max\{\|\hat{\phi}_1\|_L, \|\hat{\phi}_2\|_L\} \leq \|\hat{\phi}\|_L = 1$. Then

$$\begin{aligned} D(x,y) &= (\hat{\phi}_1(y) - \hat{\phi}_1(x)) + (\hat{\phi}_2(x) - \hat{\phi}_2(y)) \\ &\leq \sup_{\substack{\psi\in P\\ \|\psi\|_L\leq 1}} (\psi(y) - \psi(x)) + \sup_{\substack{\psi\in P\\ \|\psi\|_L\leq 1}} (\psi(x) - \psi(y)), \end{aligned}$$

which coincides with $D_P(x,y) + D_P(y,x) = D_P^s(x,y)$. Furthermore, it is clear that

$$D_P^s(x,y) = D_P(x,y) + D_P(y,x) \leq 2D(x,y).$$

Thus, the distances D_P^s and D are equivalent, and $\mathrm{Lip}_0(X, D)$ is linearly isomorphic to $\mathrm{Lip}_0(X, D_P^s)$. ∎

Every (possibly extended) quasi-metric space (X, d) can be endowed with three "natural" topologies:

(i) The *forward topology* $\mathcal{T}(d)$, generated by the family of open *forward*-balls

$$\{B_d(x,r) : x \in X,\ r > 0\},$$

where $B_d(x,r) := \{y \in X : d(x,y) < r\}$ for all $x \in X$ and $r > 0$.

(ii) The *backward topology* $\mathcal{T}(\bar{d})$, generated by the family of *backward*-balls

$$\{B_{\bar{d}}(x,r) : x \in X,\ r > 0\},$$

where $B_{\bar{d}}(x,r) := \{y \in X : d(y,x) < r\}$ for all $x \in X$ and $r > 0$.

(iii) The *symmetric topology* $\mathcal{T}^s$, generated by the family of sets

$$\{B_d(x,r) \cap B_{\hat{d}}(x,r) : x \in X,\ r > 0\}.$$

The symmetric topology being generated by the symmetrized distance d^{s_0} or d^s defined in (2.8) is obviously a metric topology. On the other hand, $\mathcal{T}(d)$ and $\mathcal{T}(\bar{d})$ are not in principle metric topologies. Nevertheless, they are both first countable topologies, since they have local bases consisting of balls of rational radii.

In what follows, unless stated otherwise, the default topology on a quasi-metric space (X, d) will be its forward topology, which is either a T_1-topology (when d is a quasi-metric) or a T_0-topology (when d is a quasi-hemi-metric).

Example 2.9 (The space $(\mathbb{R}, d_u)$). Let us consider $\mathbb{R}$ with its (canonical) asymmetric distance d_u (see Example 2.6). It is easy to check that $\mathcal{T}(d_u)$ has a local basis of the form $\{[x_0, x_0 + \varepsilon) : \varepsilon > 0\}$ for each $x_0 \in \mathbb{R}$, while $\mathcal{T}(\bar{d_u})$ has a local basis consisting of sets of the form $(x_0 - \varepsilon, x_0]$, and $\mathcal{T}(d_u^s)$ is the usual topology of $\mathbb{R}$.

Notice that d_u is issued from the asymmetric hemi-norm $u(x) = \max\{x, 0\}$ for all $x \in \mathbb{R}$; see (2.1) and (2.6). Moreover, the unit ball $\overline{B}(0,1) = \{y \in \mathbb{R} : d_u(0, y) \leq 1\} = (-\infty, 1]$ is not $\mathcal{T}(d_u)$-closed because $(1, \infty)$ is not $\mathcal{T}(d_u)$-open. Notice also that, for every topological space X, a function $f : X \to \mathbb{R}$ is upper semicontinuous if and only if $f : X \to (\mathbb{R}, u)$ is continuous.

The following example reveals that the topology of a quasi-metric space which is T_1 may not be T_2.

Example 2.10. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of distinct elements and consider the space

$$X = \{x_n : n \in \mathbb{N}\} \cup \{\bar{x}, \bar{y}\},$$

where $\bar{x}$ and $\bar{y}$ are different from each other and from any element of the sequence. Then the function d defined on $X \times X$ by $d(\bar{x}, x_n) = d(\bar{y}, x_n) = 1/n$ for every $n \in \mathbb{N}$, and $d(x, y) = 1$ for all other cases where $x \neq y$, is a quasi-metric on X. In this case, the forward topology $\mathcal{T}(d)$ cannot be T_2, since $\{x_n\}_n$ converges to both $\bar{x}$ and $\bar{y}$. Notice that the symmetrized distance d^s is discrete, with $d^s(x, y) > 1$, whenever $x \neq y$.

2.3. Cones and conic norms. In this subsection we shall recall from [38] the notion of an abstract cone. To this end, let us first recall that a *monoid* is a semigroup $(X, +)$ with neutral element 0.

Definition 2.11 (Abstract cone). A *cone* on $\mathbb{R}_+$ is a triple $(C, +, \cdot)$ such that $(C, +)$ is an abelian monoid (with neutral element 0), and $\cdot$ is a mapping from $\mathbb{R}_+ \times X$ to X such that for all $x, y \in C$ and $r, s \in \mathbb{R}_+$:

(i) $r \cdot (s \cdot x) = (rs) \cdot x$;
(ii) $r \cdot (x + y) = (r \cdot x) + (r \cdot y)$ and $(r + s) \cdot x = (r \cdot x) + (s \cdot x)$;
(iii) $1 \cdot x = x$ and $0 \cdot x = 0$.

Note that this definition does not include the existence of additive inverses. However, when such an inverse exists for some $x \in C$, it is unique, and we denote it by $-x$.

A *subcone* of a cone $(C, +, \cdot)$ is a cone $(S, +|_S, \cdot|_S)$ such that S is a subset of C and $+|_S$ and $\cdot|_S$ are, respectively, the restriction of $+$ and $\cdot$ to $S \times S$.

Definition 2.12 (Cancellative cone). A cone $(C, +, \cdot)$ is called *cancellative* if for any $x, y, z \in C$,

$$x + z = y + z \implies x = y.$$

It follows readily that every cone that embeds in a linear space is cancellative. Before we proceed, let us give two examples of abstract cones which are not cancellative.

Example 2.13 (Non-cancellative cone). (i) Consider a cone C and let $\mathcal{S}(C)$ be the set of subcones of C, under the usual operations of subset addition and scalar product. Then $\mathcal{S}(C)$ may not be cancellative. Indeed, for $C = \mathbb{R}^2$, let us consider the following elements of $S(C)$:

$$X = \{(x, 0) : x \in \mathbb{R}\}, \quad Y = \{(0, x) : x \in \mathbb{R}\}, \quad Z = \{(x, x) : x \in \mathbb{R}\}.$$

Then $X + Z = Y + Z$ but $X \neq Y$.

(ii) For a non-empty set X, consider the set $\mathbb{R}_+^X$ of non-negative functions, with the operations $\lambda \odot f = f^\lambda$ (product with external scalar) and $f \oplus g = f \cdot g$ (addition). Then $\mathbb{R}_+^X$ is not cancellative.

Definition 2.14 (Cone morphisms). A *linear mapping* from a cone $(C_1, +, \cdot)$ to a cone $(C_2, +, \cdot)$ is a mapping $f : C_1 \to C_2$ such that $f(\alpha \cdot x + \beta \cdot y) = \alpha \cdot f(x) + \beta \cdot f(y)$ for any $x, y \in C_1$ and any $\alpha, \beta \in \mathbb{R}_+$.

Remark 2.15 (Compatibility of cone morphisms). Let f be a linear mapping between two cones C_1 and C_2. Then if $H_i := \{x \in C_i : -x \in C_i\}$ denotes the linear part of the cone C_i for $i \in \{1, 2\}$, then it is straightforward to see that for every $x \in H_1$, $f(-x) = -f(x)$. In particular, the restriction of f to H_1 yields a linear mapping between the linear spaces H_1 and H_2.

We shall now introduce the notion of a *conic norm*, which will be relevant for our developments.

Definition 2.16 (Conic norm). A *conic norm* on a cone $(C, +, \cdot)$ is a function $\|\cdot|: C \to \mathbb{R}_+$ such that for all $x, y \in C$ and $r > 0$:

(i) $\|x + y| \leq \|x| + \|y|$;
(ii) $\|x| = 0 \Leftrightarrow x = 0$;
(iii) $\|r \cdot x| = r\|x|$.

The pair $(C, \|\cdot|)$ is called a *normed cone*. If we replace condition (ii) by

(ii)′ $x = 0 \Leftrightarrow \forall z \in C : [x + z = 0 \Rightarrow \|x| = \|z| = 0]$, then we say that $\|\cdot|: C \to \mathbb{R}_+$ is a *conic hemi-norm*. A cone equipped with either a conic norm or a conic hemi-norm will be called a *normed cone*. This is in accordance with the term *asymmetric normed space*, which refers to a vector space equipped with either an asymmetric norm or an asymmetric hemi-norm. (The asymmetry is now stemming from the use of a cone, rather than a vector space. Notice however that C is not necessarily a cancellative cone.)

Example 2.17. Consider the pair $(\mathbb{R}^2, \|\cdot|)$, with

$$\|(x_1, x_2)| := u(x_1) + u(x_2),$$

where u is the canonical asymmetric hemi-norm of $\mathbb{R}$ given by $u(x) = \max\{x, 0\}$ for all $x \in \mathbb{R}$ (see also Example 2.6). By restricting $\|\cdot|$ to any cone $C \subseteq \mathbb{R}^2$, we obtain a conic hemi-norm. The case $C = \mathbb{R}^2_-$ corresponds to an example of normed cone with the trivial conic hemi-norm equal to 0 everywhere.

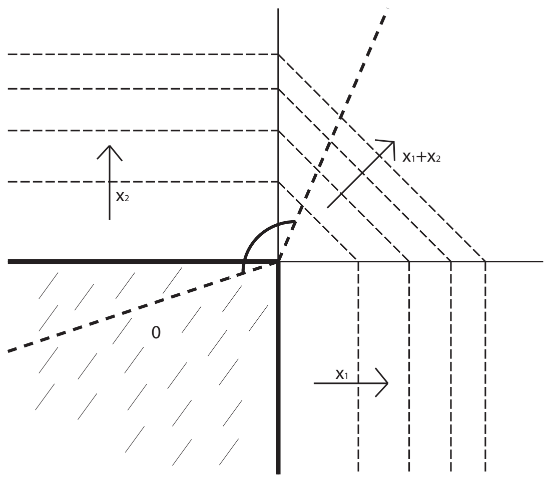


Fig. 1. Illustration of Example 2.17

Remark 2.18 (Terminology alert II). The reader should again be alerted that some authors ([38] e.g.) employ the term *quasi-norm* to refer to what we call "conic hemi-norm". We opt for "conic hemi-norm" because it is more suggestive. At the same time, "quasi-norm" might have a different meaning in the theory of Banach spaces ([2] e.g.). The asymmetric aspect of the conic norm is inherent to the definition of a cone, and therefore does not require the prefix "quasi".

Remark 2.19 (Conic norm vs. asymmetric norm). If the cone happens to be a linear space X, then the conic norm corresponds to an asymmetric norm on X, and instead of the term "normed cone" we use the term *asymmetric* *normed space*, as in [11]. The same applies to the case of conic hemi-norms and asymmetric hemi-norms. Given an asymmetric normed space $(X, \|\cdot|)$, one can define the *reverse* norm of an element $x \in X$ as $\|-x|$, and the (symmetric) norms (symmetrizations of $\|\cdot|$)

$$\|x\|_{s_0} := \max\{\|x|, \|-x|\} \quad \text{and} \quad \|x\|_s := \|x| + \|-x|.$$

It is clear that the above norms are equivalent.

An extended quasi-metric d on a cone $(C, +, \cdot)$ is called *invariant* if it satisfies

$$d(x+z, y+z) = d(x,y) \quad \text{and} \quad d(rx, ry) = rd(x,y), \tag{2.9}$$

which is the case whenever the extended quasi-metric d is induced by a conic norm which is the restriction of an asymmetric norm of a vector space that contains C. An extended quasi-metric d on a cone $(C, +, \cdot)$ is called *subinvariant* if $d(x+z, y+z) \leq d(x,y)$ instead of the first part of (2.9). More generally, the following result, established in [18, Proposition 1], states that given a normed cone $(C, \|\cdot|)$, there is a natural way to generate an extended quasi-metric d_e.

Proposition 2.20 (Extended quasi-metrics generated by conic norms). *Let $\|\cdot|$ be a conic (hemi-)norm on a cone $(C, +, \cdot)$. Then the function d_e defined on $C \times C$ by*

$$d_e(x,y) = \inf_{\substack{z \in C \\ y = x+z}} \|z|,$$

is a subinvariant extended quasi (hemi-)metric on C. If the cone $(C, +, \cdot)$ is cancellative, then d_e is invariant.

For $x \in C$, $r \in \mathbb{R}_+ \setminus \{0\}$ and $\varepsilon > 0$, we have

$$rB_{d_e}(x, \varepsilon) = rx + \{y \in C : \|y| < r\varepsilon\},$$

and the translations are $\mathcal{T}(d_e)$-open.

Remark 2.21. (i) The quasi-metric d_e might take infinite values if C is not a linear space (the infimum may be taken over the empty set).

(ii) If C is a cancellative cone, then the infimum in the above definition becomes superfluous, and if C is a linear space, the definition of d_e coincides with the definition of the quasi-metric given in (2.6).

(iii) The quasi-metric induced by the reverse norm coincides with the one obtained by the reverse quasi-metric. The same is true for the symmetrized metric, which coincides with the metric obtained by the symmetrization of the asymmetric norm.

Using the extended quasi-metric of Proposition 2.20, we define an equivalence between normed cones.

Definition 2.22 (Isomorphisms between normed cones). A bijective mapping $\Phi : X \to Y$ between two normed cones is called an *isometric isomorphism* if it is linear (cf. Definition 2.14) and an isometry between the corresponding extended quasi-metrics, that is,

$$d_e(\Phi x_1, \Phi x_2) = d_e(x_1, x_2) \quad \text{for all } x_1, x_2 \in X.$$

Note that this is equivalent to the relation $\|\Phi x| = \|x|$ for all $x \in X$.

We shall now proceed to define a notion of *completeness* for a quasi-metric space. Even though there are several non-equivalent notions of completeness in quasi-metric spaces (all of them generalizing, in some sense, completeness in metric spaces), we shall focus on the one which is compatible with normed cones and asymmetric normed spaces:

Definition 2.23 (Bicomplete quasi-metric space). A (possibly extended) quasi-metric space (X, d) is called *bicomplete* if the (extended) metric space (X, d^s) is complete, meaning that any d^s-Cauchy sequence in X is d^s-convergent in X. If X is a linear space and d is the quasi-metric induced by an asymmetric norm $\|\cdot|$, we say $(X, \|\cdot|)$ is a *bi-Banach space* whenever X is complete under the symmetrized metric d^s.

Definition 2.24 (Bicompletion of a quasi-metric space). Let (X, d) be an (extended) quasi-metric space. A *bicompletion* of (X, d) is an (extended) quasi-metric space $(\tilde{X}, \tilde{d})$ along with a mapping

$$\iota : (X, d) \to (\tilde{X}, \tilde{d})$$

such that:

(i) ι is an isometric embedding;
(ii) $\iota(X)$ is dense in $\tilde{X}$ for the symmetrized topology;
(iii) $(\tilde{X}, \tilde{d})$ is bicomplete.

An important result regarding bicompleteness of normed cones (and therefore of asymmetric normed spaces) is the existence and uniqueness of the bicompletion (see [32, Theorem 3.13]). This result, once again, generalizes the usual completion of normed linear spaces.

Proposition 2.25 (Uniqueness of bicompletion for cancellative normed cones). *Let $(C, \|\cdot|)$ be a cancellative normed cone. Then there exists a unique (up to an isometric isomorphism) bicompletion of $(C, \|\cdot|)$, which is also a normed cone, and the embedding into the bicompletion is linear. If C is a linear space, then its bicompletion is an asymmetric normed space.*

2.4. Semi-Lipschitz functions and dual cones. Let us now define the class of semi-Lipschitz functions, which reflects naturally the asymmetry in the definition of a quasi-metric space.

Definition 2.26 (Semi-Lipschitz function). Let (X, d) be a quasi-metric space. A function $f : X \to \mathbb{R}$ is said to be *semi-Lipschitz* if there exists $L > 0$ such that for all $x, y \in X$ we have

$$f(x) - f(y) \leq Ld(y, x). \tag{2.10}$$

The class of semi-Lipschitz functions on X is denoted by $\mathrm{SLip}(X)$.

Let us recall that a Lipschitz function f satisfies $|f(x) - f(y)| \leq Ld(x, y)$ for all $x, y \in X$. Therefore, if (X, d) is a metric space, the notions of semi-Lipschitz and Lipschitz function coincide. In a quasi-metric space, f is Lipschitz if and only if both f and $-f$ are semi-Lipschitz. To get easy examples of semi-Lipschitz functions that are not Lipschitz, consider functions of the form $d(x, \cdot)$ on the quasi-metric space of Example 2.6.

Definition 2.27 (Semi-Lipschitz conic norm). Let (X, d) be a quasi-metric space. The *semi-Lipschitz conic* (*hemi-*)*norm* of a function $f : X \to \mathbb{R}$ is defined by

$$\|f|_S := \inf \{L > 0 : (2.10) \text{ holds}\}.$$

The following proposition is easily shown.

Proposition 2.28 (Semi-Lipschitz criterion). *Let (X, d) be a quasi-metric space and $f : X \to \mathbb{R}$.*

(i) *If d is a quasi-metric, then f is semi-Lipschitz if and only if*

$$\|f|_S = \sup_{x \neq y} \frac{\max\{f(x) - f(y), 0\}}{d(y, x)} = \sup_{x \neq y} \frac{f(x) - f(y)}{d(y, x)} < \infty.$$

(ii) *If d is a quasi-hemi-metric, then f is semi-Lipschitz if and only if $\|f|_S < \infty$. In this case,*

$$\|f|_S = \sup_{d(y,x)>0} \frac{\max\{f(x) - f(y), 0\}}{d(y, x)} = \sup_{d(y,x)>0} \frac{f(x) - f(y)}{d(y, x)}.$$

Remark 2.29. Let (X, d) be a quasi-metric space and $f : X \to \mathbb{R}$. If for all $x, y \in X$ we have $f(x) \leq f(y)$ whenever $d(y, x) = 0$ (d-monotonicity), then

$$\sup_{d(y,x)>0} \frac{\max\{f(x) - f(y), 0\}}{d(y, x)} = \sup_{d(y,x)>0} \frac{f(x) - f(y)}{d(y, x)}. \tag{2.11}$$

It follows readily from Definition 2.26 that every semi-Lipschitz function is d-monotonic, and therefore it satisfies (2.11).

Examples 2.30. (i) If $f : X \to \mathbb{R}$ is not semi-Lipschitz or d-monotonic, then the equality (2.11) is not necessarily true. For example, let $X = \{a, b\}$ with $a, b \in \mathbb{R}$, let $d : X \times X \to [0, \infty)$ be the quasi-hemi-metric given by $d(a, b) = 1$ and $d(b, a) = 0$, and let $f : X \to \mathbb{R}$ be defined as $f(a) = 1$ and

$f(b) = 0$. Then f is not semi-Lipschitz,

$$\sup_{d(y,x)>0} \frac{f(x)-f(y)}{d(y,x)} = -1 \quad \text{and} \quad \sup_{d(y,x)>0} \frac{\max\{f(x)-f(y),0\}}{d(y,x)} = 0.$$

(ii) The equality (2.11) could be true without f being semi-Lipschitz. For instance, let $X = \{a, b, c\}$ with $a, b, c \in \mathbb{R}$, let $d : X \times X \to [0, \infty)$ be the quasi-hemi-metric given by

$$d(x,y) = \begin{cases} 1 & \text{if } x = a,\ y = b, \\ 1 & \text{if } x = b,\ y = c, \\ 2 & \text{if } x = a,\ y = c, \\ 0 & \text{otherwise,} \end{cases}$$

and let $f : X \to \mathbb{R}$ be defined as $f(a) = 2$, $f(b) = 1$ and $f(c) = 1$. Then f is not semi-Lipschitz, since $f(a) - f(b) = 1$ and $d(b, a) = 0$. However,

$$\sup_{d(y,x)>0} \frac{f(x)-f(y)}{d(y,x)} = 0 \quad \text{and} \quad \sup_{d(y,x)>0} \frac{\max\{f(x)-f(y),0\}}{d(y,x)} = 0.$$

Remark 2.31 (Terminology alert III). The above definition of a semi-Lipschitz function, introduced in [14], differs from the one that is usually considered in the literature and is based on an inequality of the form

$$f(x) - f(y) \le L d(x, y). \tag{2.12}$$

A function $f : (X, d) \to \mathbb{R}$ is semi-Lipschitz according to Definition 2.26 if and only if it is semi-Lipschitz on $(X, \bar{d})$ according to (2.12). This is also equivalent to $-f$ being semi-Lipschitz on (X, d) according to (2.12). Therefore, the difference between these two definitions of a semi-Lipschitz function is equivalent to either a change of orientation of the quasi-metric (replace d by $\bar{d}$) or of the sign of the values of f (replace f by $-f$). With this in mind, let us now justify our choice for Definition 2.26:

(i) If $(X, \|\cdot|)$ is a normed cone, the norm $\|\cdot|$ may not be semi-Lipschitz according to (2.12), while $-\|\cdot|$ is always semi-Lipschitz according to (2.12).
(ii) In general, if (X, d) is a quasi-metric space, the functions of the form $d(x_0, \cdot)$ that characterize forward convergence (i.e. $x_n \to x_0$ in the forward topology if and only if $d(x_0, x_n) \to 0$) may not be semi-Lipschitz according to (2.12), while $-d(x_0, \cdot)$ and $d(\cdot, x_0)$ will be so.

Therefore, to avoid/circumvent the above inconveniences, we shall opt for Definition 2.26. This definition, in particular, is compatible with the natural definition of a semi-Lipschitz function from a quasi-metric space (X, d) to an asymmetric normed space or a normed cone $(Y, \|\cdot|)$: indeed, denoting by $d^{\|\cdot|}$ the distance associated to the asymmetric norm $\|\cdot|$, it is natural to demand

$$d^{\|\cdot|}(f(x),f(y)) = \|f(y)-f(x)| \leq Ld(x,y),$$

which coincides with our definition when taking $(Y,\|\cdot|) = (\mathbb{R},u)$, with u given by $u(x) = \max\{x,0\}$ for all $x \in \mathbb{R}$. In fact, the quasi-metric space $(\mathbb{R},u)$ is involved in the definition of the dual of both an asymmetric normed space and a normed cone, and consequently it is of great importance in this theory. Furthermore, as we shall see in Proposition 2.35, a real-valued linear functional on a normed cone will belong to the *dual cone* (see Definition 2.37) if and only if it is semi-Lipschitz according to Definition 2.26.

Definition 2.32 (Asymmetric pivot space). Let (X,d) be a quasi-metric space and $x_0 \in X$ be a base point. We define the *asymmetric non-linear dual* (*pivot space*)

$$\mathrm{SLip}_0(X,d) := \{f \in \mathrm{SLip}(X) : f(x_0) = 0\}.$$

In case there is no ambiguity regarding the quasi-metric considered, we simply write $\mathrm{SLip}_0(X)$.

Remark 2.33. (i) It is easy to see that $(\mathrm{SLip}_0(X),\|\cdot|_S)$ is a cancellative normed cone.

(ii) Any semi-Lipschitz function on a quasi-metric space (X,d) is Lipschitz on the (symmetrized) metric space (X,D), where D is either d^s or d^{s_0}. Therefore, both cones of semi-Lipschitz functions $\mathrm{SLip}(X,d)$ and $\mathrm{SLip}(X,\bar{d})$ are contained in the linear space $\mathrm{Lip}(X,D)$ of Lipschitz functions on (X,D).

Let $(\mathbb{R},u)$ be the asymmetric normed space evoked in Example 2.6. Then the asymmetric norm u generates the *upper topology* on $\mathbb{R}$, which is the topology that characterizes upper semicontinuity in the following way: a function from a topological space $f : (X,\tau) \to (\mathbb{R},u)$ is continuous for the forward topology of $(\mathbb{R},u)$ if and only if f is upper semicontinuous for the usual norm on $\mathbb{R}$ (which is the symmetrization of u).

Example 2.34. Let (X,d) be a quasi-metric space with a base point x_0. Then for each $x \in X$, the function $f(\cdot) = d(x,\cdot) - d(x,x_0)$ belongs to $\mathrm{SLip}_0(X,d)$ and satisfies $\|f|_S = 1$. Indeed, it follows directly from the triangular inequality that f is semi-Lipschitz with $\|f|_S \leq 1$. We obviously have $f(x_0) = 0$. Taking $z \in X$ with $z \neq x$ we deduce $f(z)-f(x) = d(x,z)$, that is, $\|f|_S = 1$.

The previous example becomes relevant in order to define duality for normed cones and asymmetric normed spaces. The following proposition gives some insight into this duality. The proof has no essential difficulty and is included for the reader's convenience.

Proposition 2.35 (Linear functionals over a normed cone). *Let* $(C, \|\cdot|)$ *be a normed cone and* $\varphi : C \to \mathbb{R}$ *a linear functional. Then the following are equivalent:*

(i) φ *is upper semicontinuous* (*for short, usc*);
(ii) φ *belongs to* $\mathrm{SLip}_0(C, d_e)$, *where* d_e *is the* (*extended*) *quasi-metric induced by the conic norm* $\|\cdot|$ (*cf. Proposition* 2.20);
(iii) *there exists* $M \geq 0$ *such that* $\varphi(x) \leq M\|x|$ *for all* $x \in C$.

Proof. Let us show that (i) implies (iii). Assume that the linear functional φ is usc. Then there exists $\alpha > 0$ such that $\varphi(B(0, \alpha)) \subseteq (-\infty, 1)$. Set $M = 2/\alpha$. Then for every $x \in C$ with $\|x| \neq 0$, we have $\tilde{x} = \frac{\alpha x}{2\|x|} \in B(0, \alpha)$, hence $\varphi(\tilde{x}) < 1$ and $\varphi(x) < M\|x|$. If $x \in C$ with $\|x| = 0$, then for every $r > 0$ we have $\|rx| = 0$ and $\varphi(rx) < 1$, which implies $\varphi(x) < 1/r$ and necessarily $\varphi(x) \leq 0$.

Let us now show that (iii) implies (ii). We need to establish the inequality $\varphi(x) - \varphi(y) \leq L d_e(y, x)$, $\forall x, y \in C$, for some $L \geq 0$. If $d_e(y, x) = \infty$, the inequality becomes trivial. If not, then $x \in y + C$, so we can write $x = y + z$, and then $\varphi(x) - \varphi(y) = \varphi(z) \leq M\|z|$. By taking the infimum of all z such that $x = y + z$, we get $\varphi(x) - \varphi(y) \leq M d_e(y, x)$, that is, φ is semi-Lipschitz.

Let us finally assume (ii) and recall that the forward topology on $(C, \|\cdot|)$ is first countable. Then take $\{x_n\}_n \subseteq C$ such that $d_e(x, x_n) \to 0$. Since φ is semi-Lipschitz, we have $\varphi(x_n) - \varphi(x) \leq L d_e(x, x_n)$ for some $L \geq 0$, which yields $\varphi(x) \geq \limsup \varphi(x_n)$. ∎

Remark 2.36. Each one of the above statements is also equivalent to φ being lower semicontinuous (for short, lsc) for the reverse extended quasi-metric $\bar{d}_e$.

Indeed, assume there exists $M \geq 0$ such that $\varphi(x) \leq M\|x|$ for all $x \in C$, and consider a sequence $\{z_n\}_n$ and z in C such that $\bar{d}_e(z, z_n) \to 0$. Then $d_e(z_n, z) \to 0$, which yields the existence of a sequence $\{y_n\}_n \subset C$ such that $y_n + z_n = z$ and $\|y_n| \to 0$. Since φ is linear, $\varphi(z) = \varphi(z_n) + \varphi(y_n) \leq \varphi(z_n) + M\|y_n|$, which shows that φ is lsc for $\bar{d}_e$.

On the other hand, if φ is lsc for $\bar{d}_e$, an analogous argument to Proposition 2.35 ((i)$\Rightarrow$(iii)) leads to the same conclusion, that is, the existence of $M \geq 0$ such that $\varphi(x) \leq M\|x|$ for all $x \in C$.

Definition 2.37 (Dual normed cone). Let $(C, \|\cdot|)$ be a normed cone. We define the *dual cone* of C as

$$C^* := \{\varphi : C \to \mathbb{R} : \varphi \text{ usc, linear}\} = \{\varphi \in \mathrm{SLip}_0(C) : \varphi \text{ linear}\}.$$

For any $\varphi \in C^*$, the *dual conic norm* is defined by

$$\|\varphi|^* := \sup_{\|x| \leq 1} \max\{\varphi(x), 0\} = \sup_{\|x| \leq 1} \varphi(x).$$

It is easy to check that $\|\cdot|^*$ is a conic norm on C^* (obviously $\|\varphi|^* \geq 0$, since $\varphi(0) = 0$). Moreover, if $(C, \|\cdot|)$ is a normed cone with conic hemi-norm, then $\|\cdot|^*$ is a conic hemi-norm on C^*.

The proof of the following result is reasonably simple.

Proposition 2.38. *Let $(C, \|\cdot|)$ be a normed cone, and $\varphi \in C^*$. Then*

$$\|\varphi|^* = \inf\{M > 0 : \varphi(x) \leq M\|x| \text{ for all } x \in C\}.$$

As in the case of normed spaces, there is a direct relation between the semi-Lipschitz constant and the dual norm of a linear functional:

Corollary 2.39 (Dual conic norm and semi-Lipschitz constant). *Let $(C, \|\cdot|)$ be a normed cone, and $\varphi \in C^*$. Then $\|\varphi|^* = \|\varphi|_S$ and the subcone of linear functionals of* $\mathrm{SLip}_0(C)$ (*linear semi-Lipschitz functions*) *is isometrically isomorphic to* $(C^*, \|\cdot|^*)$ (*linear usc functions*).

Proof. The inequality $\|\varphi|_S \leq \|\varphi|^*$ follows from Proposition 2.35 ((ii)⇒(iii)). For the opposite inequality, as φ is semi-Lipschitz and $\varphi(0) = 0$, we get

$$\varphi(x) = \varphi(x) - \varphi(0) \leq \|\varphi|_S\, d_e(0, x) = \|\varphi|_S \|x|,$$

which by Proposition 2.38 implies that $\|\varphi|^* \leq \|\varphi|_S$. ∎

2.5. Duality of asymmetric normed spaces. In this subsection we consider the particular case that the normed cone is an asymmetric normed space $(X, \|\cdot|)$.

Proposition 2.40 (Dual of a finite-dimensional space). *Let $(X, \|\cdot|)$ be a finite-dimensional vector space endowed with an asymmetric norm $\|\cdot|$. Then there exists $M > 0$ such that*

$$\|{-x}| \leq M\|x| \quad \text{for all } x \in X. \tag{2.13}$$

Furthermore, $(X, \|\cdot|)^*$ is also an asymmetric normed space such that for every $\varphi \in (X, \|\cdot|)^*$ we have $-\varphi \in (X, \|\cdot|)^*$ and $\|-\varphi|^* \leq M\|\varphi|^*$. In particular, $(X^*, \|\cdot|)$ is a linear space (*not only a normed cone*).

Proof. Let $B = \{x \in X : \|x| \leq 1\}$ be the unit ball of X. Since in finite dimensions all asymmetric norms inducing a T_1-topology are equivalent (see [17, Corollary 11] or [6, Theorem 3] for example), it follows that B is closed convex and $0 \in \operatorname{int} B$. Thus we can ensure the existence of $M > 0$ such that $\left\|\frac{-x}{\|x|}\right| \leq M$ for all $x \in X$ with $\|x| \neq 0$, which yields $\|-x| \leq M\|x|$ for all $x \in X$. Now, if $\varphi \in (X, \|\cdot|)^*$ then

$$-\varphi(x) = \varphi(-x) \leq \|\varphi|^* \|-x| \leq M\|\varphi|^* \|x| \quad \text{for all } x \in X$$

and

$$\|-\varphi|^* \Big(= \sup_{\|x| \leq 1} -\varphi(x)\Big) \leq M\|\varphi|^* \quad \text{for all } \varphi \in (X, \|\cdot|)^*. \ ∎$$

Remark 2.41 (An infinite-dimensional counterexample). If X is infinite-dimensional, then (2.13) may not be fulfilled. For example, let

$$X = \left\{ f \in \mathcal{C}([0,1]) : \int_0^1 f(t)\,dt = 0 \right\}$$

and $\|f| := \max_{t\in[0,1]} \max\{f(t), 0\}$. Let $\{f_n\}_n \subset X$ be defined as

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if } 0 \le x < \frac{1}{n^2}, \\ \frac{n}{2-n^2}x + \frac{1-n^2}{2n-n^3} & \text{if } \frac{1}{n^2} \le x < 1 - \frac{1}{n^2}, \\ -n^3 x - n(1-n^2) & \text{if } 1 - \frac{1}{n^2} \le x \le 1, \end{cases} \qquad (n \in \mathbb{N}).$$

Then $\|f_n| = 1/n$ for $n \ge 2$ and $\|-f_n| = n \to \infty$, which contradicts (2.13). In addition, X^* is a normed cone (and not a vector space). To see this, let $\delta_1 : \mathcal{C}([0,1]) \to \mathbb{R}$ be defined as $\delta_1(f) = f(1)$. Then $\{f_n\}_n \to 0$, $\delta_1(0) = 0$ and $\delta_1(f_n) = -n \to -\infty$, which shows that the linear functional δ_1 is not lower semicontinuous in $(X, \|\cdot|)$.

Remark 2.42 (Continuity of evaluation functionals). Let $(X, \|\cdot|)$ be an asymmetric normed space with dual X^*. For every $x \in X$, the evaluation functional $\widehat{x} : X^* \to \mathbb{R}$ defined as $\widehat{x}(\varphi) = \varphi(x)$ is linear and $\|\cdot|^*$-continuous. Indeed, we have

$$\widehat{x}(\varphi) = \varphi(x) \le \|\varphi|^* \|x| \quad \text{and} \quad -\widehat{x}(\varphi) = -\varphi(x) = \varphi(-x) \le \|\varphi|^* \|-x|,$$

hence $|\widehat{x}(\varphi)| \le \max\{\|x|, \|-x|\}\, \|\varphi|^*$, i.e. $\widehat{x}$ is Lipschitz and thus continuous.

Lemma 2.43 ($(\mathcal{L}^1(\mathbb{R}), \|\cdot|_{1,+})^* = (\mathcal{L}^\infty_+(\mathbb{R}), \|\cdot\|_\infty)$). *Let $\mathcal{L}^1(\mathbb{R})$ be endowed with the asymmetric norm*

$$\|f|_{1,+} := \int_{\mathbb{R}} f^+\, d\lambda,$$

where $f^+(x) = \max\{f(x), 0\}$ and λ denotes the Lebesgue measure. Then the dual of $(\mathcal{L}^1(\mathbb{R}), \|\cdot|_{1,+})$ is isometrically isomorphic to $(\mathcal{L}^\infty_+(\mathbb{R}), \|\cdot\|_\infty)$, where $\mathcal{L}^\infty_+(\mathbb{R})$ denotes the cone of non-negative functions in $\mathcal{L}^\infty(\mathbb{R})$.

Proof. The facts that $(\mathcal{L}^1(\mathbb{R}), \|\cdot|_{1,+})$ is an asymmetric normed space and $(\mathcal{L}^\infty_+(\mathbb{R}), \|\cdot\|_\infty)$ is a normed cone are straightforward. Let $\varphi \in (\mathcal{L}^1(\mathbb{R}), \|\cdot|_{1,+})^*$. Then $\varphi : \mathcal{L}^1(\mathbb{R}) \to \mathbb{R}$ is linear and ($\|\cdot|_{1,+}$-u)-continuous (see Example 2.6). Hence, by Remark 2.33, φ is continuous for the symmetrized norms in both spaces, therefore

$$|\varphi(f)| \le \|\varphi\|^* \max\{\|f^+|_{1,+}, \|-f^+|_{1,+}\} \le \|\varphi\|^* \|f\|_1,$$

where $\|\cdot\|^*$ denotes the dual norm of the normed space $(\mathcal{L}^1(\mathbb{R}), (\|\cdot|_{1,+})^s)$ and $\|\cdot\|_1$ is the usual norm on $\mathcal{L}^1(\mathbb{R})$. It follows that φ is ($\|\cdot\|_1$-$|\cdot|$)-continuous, and therefore there exists $g \in \mathcal{L}^\infty(\mathbb{R})$ such that $\varphi(f) = \int g f\, d\lambda$ for all $f \in \mathcal{L}^1(\mathbb{R})$. We claim that $g \ge 0$ almost everywhere.

Indeed, suppose, towards a contradiction, that there exists a set E of measure $0 < \lambda(E) < \infty$ such that $g < 0$ on E. Consider the sequence $f_n = -n\mathbb{1}_E$ (where $\mathbb{1}_E$ is the characteristic function of E), which clearly belongs to $\mathcal{L}^1(\mathbb{R})$. On the other hand, since $\|f_n|_{1,+} = 0$ for all $n \in \mathbb{N}$, the function f_n belongs to the unit ball of the asymmetric norm $\|\cdot|_{1,+}$. Then, as $n \to +\infty$, we deduce

$$\varphi(f_n) = \int g f_n \, d\lambda = \int_{E^c} g f_n \, d\lambda + \int_E g f_n \, d\lambda = n \int_E (-g) \, d\lambda \to +\infty.$$

Therefore, φ cannot be $(\|\cdot|_{1,+}\text{-}u)$-continuous, a contradiction.

Notice now that any $g \in \mathcal{L}^\infty_+(\mathbb{R})$ defines a linear $(\|\cdot|_{1,+}\text{-}u)$-continuous functional φ in the same manner:

$$\varphi(f) = \int_{\mathbb{R}} g f \, d\lambda \le \int_{\mathbb{R}} g f^+ \, d\lambda \le \|g\|_\infty \int_{\mathbb{R}} f^+ = \|g\|_\infty \|f|_{1,+},$$

which yields $\|\varphi|^* \le \|g\|_\infty$. On the other hand, take $\varepsilon > 0$ and a set E of finite measure such that $g(x) \ge \|g\|_\infty - \varepsilon$ on E. Then consider the function

$$f = \frac{\operatorname{sgn}(g)}{\lambda(E)} \, \mathbb{1}_E,$$

where $\operatorname{sgn}(g)$ denotes the sign of g, and note that $\|f|_{1,+} \le 1$. Then

$$\varphi(f) = \frac{1}{\lambda(E)} \int_E g \, d\lambda \ge \frac{1}{\lambda(E)} \int_E [\|g\|_\infty - \varepsilon] \, d\lambda = \|g\|_\infty - \varepsilon.$$

It follows that $\|\varphi|^* = \|g\|_\infty$, and therefore we can identify the dual of $(\mathcal{L}^1(\mathbb{R}), \|\cdot|_{1,+})$ with $(\mathcal{L}^\infty_+(\mathbb{R}), \|\cdot\|_\infty)$ by an isometric isomorphism. ■

Let us now give the following definition.

Definition 2.44 (Asymmetric weak topologies). Let X be an (asymmetric) normed space with dual X^*.

(i) The *weak* topology w on X is defined as the coarsest topology for which every $\phi \in X^*$ remains upper semicontinuous.

(ii) The *weak-star* topology w^* on X^* is defined as the coarsest topology that makes every evaluation functional $\{\widehat{x} : X^* \to (\mathbb{R}, |\cdot|), x \in X\}$ continuous (notice by Remark 2.42 that $\widehat{x}$ is always $\|\cdot|^*$-continuous, where $\|\cdot|^*$ is the conic hemi-norm of X^*).

Therefore the weak-star topology w^* on X^* is weaker than the forward $\|\cdot|^*$-topology. In what follows, we shall use the notation $\langle y^*, y\rangle = y^*(y)$.

Lemma 2.45. *Let X be an asymmetric normed space with dual X^*, and $\varphi : X^* \to \mathbb{R}$ a w^*-continuous linear functional. Then there exists $x_\varphi \in X$ such that $\varphi(x^*) = x^*(x_\varphi)$ for all $x^* \in X^*$.*

Proof. Since φ is w^*-continuous, the set $\varphi^{-1}(-1,1)$ is a w^*-neighbourhood of 0, so there exist $x_1,\dots,x_n \in X$ such that

$$\Big\{x_i^* \in X^* : \max_{i=1,\dots,n} |\langle x^*, x_i\rangle| < 1\Big\} \subseteq \varphi^{-1}(-1,1),$$

which yields

$$\bigcap_{i=1}^{n} \operatorname{Ker}(\widehat{x}_i) \subseteq \operatorname{Ker}(\varphi). \tag{2.14}$$

The above kernels are contained in the cone X^*. We can linearly extend φ and the evaluation functionals $\widehat{x}_1,\dots,\widehat{x}_n$ from the normed cone X^* to the linear space $\operatorname{span}(X^*) \subseteq \mathbb{R}^X$. This operation preserves the inclusion (2.14) on $\operatorname{span}(X^*)$. Consequently, the extension $\widehat{x}_\varphi$ of φ is a linear combination of the extensions of $\widehat{x}_1,\dots,\widehat{x}_n$. ■

The following result is analogous to the classical one in operator theory (see [37, Theorem 4.10]).

Lemma 2.46. *Let $(X, \|\cdot|_X)$, $(Y, \|\cdot|_Y)$ be asymmetric normed spaces, X^* and Y^* their respective dual cones and $T : Y^* \to X^*$ a linear bounded operator (meaning that there exists $K \geq 0$ such that $\|Ty^*|_Y \leq K\|y^*|_X$ for all $x \in X$). If T is (w^*-w^*)-continuous, then there exists a linear bounded operator $S : X \to Y$ such that $T = S^*$, in the sense that*

$$\langle y^*, Sx\rangle = \langle Ty^*, x\rangle \quad \text{for all } x \in X \text{ and } y^* \in Y^*.$$

Furthermore, if T is a bijective isometry, so is S.

Proof. Let $x \in X$, and define $f : Y^* \to \mathbb{R}$ as $f(y^*) = \widehat{x}(Ty^*)$, which is w^*-continuous, and therefore by Lemma 2.45 there exists y_x such that $\widehat{x} \circ T = \widehat{y}_x$ and $y^*(y_x) = \widehat{x}Ty^*$, and define $Sx = y_x$, which is linear and bounded, since

$$\|Sx|_Y = \|y_x|_Y = \|\widehat{y}_x| = \|\widehat{x} \circ T| = \sup_{\|y^*|\leq 1} (\widehat{x} \circ T)(y^*) \leq \|x|_X \|T|.$$

Moreover, $S^* = T$, because

$$\langle S^*y^*, x\rangle = \langle y^*, Sx\rangle = \langle \widehat{x} \circ T, y^*\rangle = \langle Ty^*, x\rangle$$

for all $x \in X$ and $y^* \in Y^*$. Finally, if T is an isometry then

$$\|Sx|_Y = \sup_{\|y^*|\leq 1} \langle y^*, Sx\rangle = \sup_{\|y^*|\leq 1} \langle Ty^*, x\rangle = \sup_{\|y^*|\leq 1} \langle x^*, x\rangle = \sup_{\|x^*|\leq 1} \langle Ty^*, x\rangle,$$

where the first equality follows as a corollary of the Hahn–Banach theorem for asymmetric normed spaces [11, Corollary 2.2.4]. ■

The following proposition shows that an asymmetric normed space and its bicompletion have the same dual. This fact will be relevant for our main result.

Proposition 2.47 (Unique extension of a linear usc functional). *Let $(X, \|\cdot|)$ be an asymmetric normed space, $D \subseteq X$ a subspace that is dense in the symmetrization of the induced quasi-metric, and $\varphi : D \to \mathbb{R}$ a linear usc functional. Then φ has a unique linear usc extension to X.*

Proof. Thanks to the Hahn–Banach theorem [11, Theorem 2.2.1], φ has at least one linear usc extension to X. Let us assume, towards a contradiction, that φ has two different extensions ϕ_1 and ϕ_2, with $\phi_1(x) < \phi_2(x)$ for some $x \in X$. Since D is dense for the symmetrized extended quasi-metric (see Definition 2.7), there is a sequence $\{x_n\}_n \subseteq D$ such that $x_n \to x$ in both d_e and $\bar{d}_e$. Since ϕ_1 and ϕ_2 are usc for d_e, we deduce that they are also lsc for $\bar{d}_e$ (see Remark 2.36). Moreover, both functionals coincide on the sequence $\{x_n\}_n$. We deduce:

$$\limsup_n \phi_2(x_n) \le \phi_1(x) < \phi_2(x) \le \liminf_n \phi_2(x_n),$$

which is a contradiction. Therefore $\phi_1 = \phi_2$. ∎

Proposition 2.48 (Dual of an asymmetric normed space). *Let $(X, \|\cdot|)$ be an asymmetric normed space and $(\tilde{X}, \|\cdot|_\sim)$ its bicompletion. Then the respective dual cones are isometrically isomorphic.*

Proof. We already know that the extension mapping from X^* to $\tilde{X}^*$ is a bijection, in virtue of Proposition 2.47. To check that it is an isometry, we only need to check that $\|\phi_{|X}|^* \ge \|\phi|^*$ for any $\phi \in \tilde{X}^*$, as the reverse inequality is obvious. Let $B_{\tilde{X}}$ be the unit ball of $\tilde{X}$ for the forward distance, and consider $\phi \in \tilde{X}^*$ and a sequence $\{z_n\}_n$ on $B_{\tilde{X}}$ such that $\phi(z_n) \to \|\phi|^* := \sup_{z \in B_{\tilde{X}}} \phi(z)$. Since X is dense for the symmetrized topology in $\tilde{X}$ (by definition), for each $n \in \mathbb{N}$ there exists a sequence $\{x_n^j\}_j \subseteq B_X$ such that $\{x_n^j\}$ converges to z_n in the symmetrized distance of $\tilde{X}$. In particular, $\{x_n^j\}_j$ converges for both quasi-metrics d_e and $\bar{d}_e$. Since ϕ is lsc for $\bar{d}_e$, we see that $\phi(z_n) \le \liminf_j \phi(x_n^j)$ for every $n \in \mathbb{N}$. Then, for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\|\phi|^* < \varepsilon + \phi(z_{n_0})$, and consequently

$$\|\phi|^* < \varepsilon + \liminf_j \phi(x_{n_0}^j) \le \varepsilon + \|\phi_{|X}|^*. \blacksquare$$

3. The semi-Lipschitz free space. Throughout this section, (X, d) will denote a quasi-metric space, with d being possibly a quasi-hemi-metric, and with a base point $x_0 \in X$.

3.1. Construction of $\mathcal{F}_a(X)$. We are ready to proceed to the construction of the (asymmetric) semi-Lipschitz free space. For every $x \in X$ we consider the corresponding *evaluation* mapping

$$\delta_x : \mathrm{SLip}_0(X) \to \mathbb{R}, \qquad \delta_x(f) = f(x),\ \forall f \in \mathrm{SLip}_0(X).$$

Notice that δ_x is a linear mapping over the cone $\mathrm{SLip}_0(X)$ (see Definition 2.14). We can also define the linear mapping $-\delta_x$ by $-\delta_x(f) := -f(x)$ for all $f \in \mathrm{SLip}_0(X)$.

Proposition 3.1 (δ_x belongs to the linear part of $(\mathrm{SLip}_0(X))^*$). *For each $x \in X$, both the evaluation functional $\delta_x : \mathrm{SLip}_0(X) \to \mathbb{R}$ and its opposite $-\delta_x$ belong to the dual cone* $(\mathrm{SLip}_0(X), \|\cdot|_S)^*$.

Proof. Let $x \in X$. Since δ_x is linear, we only need to check that it is bounded from above on the unit ball of $\mathrm{SLip}_0(X)$. Indeed, for any f in $\mathrm{SLip}_0(X)$, we have $f(x) = f(x) - f(x_0) \leq d(x_0, x)\,\|f|_S$, and therefore $\delta_x \in \mathrm{SLip}_0(X)^*$. Using the same argument, we get $-f(x) \leq d(x, x_0)\|f|_S$. ∎

Remark 3.2. The fact that both δ_x and $-\delta_x$ are semi-Lipschitz shows that δ_x is actually a Lipschitz function on $(\mathrm{SLip}_0(X), \|\cdot|)$ of constant $\|\delta_x\|_{\mathrm{Lip}} = \max\{d(x, x_0), d(x_0, x)\}$.

Proposition 3.3 (Isometric injection of X into $\mathrm{SLip}_0(X)^*$). *The mapping*

$$\delta : (X, d) \to (\mathrm{SLip}_0(X)^*, \|\cdot|^*)$$

defined by $\delta(x) = \delta_x$ is (injective and) an isometry *onto its image. Therefore, for any $x, y \in X$, we have*

$$d(x, y) = \|\delta_y - \delta_x|^*.$$

Proof. Let $x, y \in X$. First of all, it is worth noting that the quasi-metric generated by the conic norm is extended (Proposition 2.20) and that $\|\delta_y - \delta_x|^*$ is well defined (Proposition 3.1). Note also that any dual cone is cancellative, since it is contained in a linear space of real-valued functions. To prove injectivity of δ, consider $x, y \in X$ such that $\delta_x = \delta_y$. Then we take the functions $f(\cdot) = d(x, \cdot) - d(x, x_0)$ and $g(\cdot) = d(y, \cdot) - d(y, x_0)$. Since $\delta_x(f) = \delta_y(f)$ and $\delta_x(g) = \delta_y(g)$, we conclude that both $d(x, y)$ and $d(y, x)$ must be zero, therefore $x = y$ (Definition 2.2(ii)$'$).

By Remark 2.21(ii), for any $x, y \in X$ we have $d_e(\delta_x, \delta_y) = \|\delta_y - \delta_x|^*$. Then, for any $x, y \in X$,

$$\begin{aligned} d_e(\delta_x, \delta_y) &= \sup_{\|f|_S \leq 1} (\delta_y - \delta_x)(f) = \sup_{\|f|_S \leq 1} \{f(y) - f(x)\} \\ &\leq \sup_{\|f|_S \leq 1} \|f|_S\, d(x, y) = d(x, y). \end{aligned}$$

Conversely, by taking $f(\cdot) = d(x, \cdot) - d(x, x_0)$ it follows, as in Example 2.34, that

$$f(y) - f(x) = d(x, y) \text{ and } f(y) - f(x) = (\delta_y - \delta_x)(f) \leq \|\delta_y - \delta_x|_* = d_e(\delta_x, \delta_y).$$

Then the assertion holds. ∎

We now take the asymmetric normed space $(\mathrm{span}(\delta(X)), \|\cdot|^*)$ (which is contained in the normed cone $(\mathrm{SLip}_0(X), \|\cdot|^*))$, and we define the (asymmetric) *semi-Lipschitz free space* to be the bicompletion of $(\mathrm{span}(\delta(X)), \|\cdot|^*)$.

DEFINITION 3.4 (The semi-Lipschitz free space). Let (X, d) be a quasi-metric space with a base point x_0. The *semi-Lipschitz free space over* (X, d), denoted by $\mathcal{F}_a(X)$, is the (unique) bicompletion of the asymmetric normed space $(\mathrm{span}(\delta(X)), \|\cdot|^*)$, where $\|\cdot|^*$ is the restriction of the norm of the space $\mathrm{SLip}_0(X)^*$.

We are now ready to establish our main result, which is analogous of the fundamental property of the Lipschitz free space of a metric space: being a predual of the space of Lipschitz functions vanishing at the base point.

THEOREM 3.5 (The equality $\mathcal{F}_a(X)^* = \mathrm{SLip}_0(X)$). *Let (X, d) be a quasi-metric space with a base point x_0. Then the dual cone of $\mathcal{F}_a(X)$ is isometrically isomorphic to* $\mathrm{SLip}_0(X)$.

Proof. Thanks to Proposition 2.48, we only need to check that the dual cone of $(\mathrm{span}(\delta(X)), \|\cdot|^*)$ is isometrically isomorphic to $\mathrm{SLip}_0(X)$. To this end, we define the mapping

$$\Phi : \mathrm{SLip}_0(X) \to (\mathrm{span}(\delta(X)), \|\cdot|^*)^*,$$

with

$$\Phi(f)\Big(\sum_i \lambda_i \delta_{x_i}\Big) = \sum_i \lambda_i f(x_i)$$

for any linear combination of evaluation functionals. First, we check that Φ is well defined: $\Phi(f)$ is obviously linear, so let us demonstrate the condition (iii) of Proposition 2.35. For any $f \in \mathrm{SLip}_0(X)$ and any $\sum_i \lambda_i \delta_{x_i} \in \mathrm{span}(\delta(X))$,

$$\Phi(f)\Big(\sum_i \lambda_i \delta_{x_i}\Big) = \sum_i \lambda_i f(x_i) = \Big(\sum_i \lambda_i \delta_{x_i}\Big)(f) \le \Big\|\sum_i \lambda_i \delta_{x_i}\Big|^* \|f|_S.$$

Therefore $\|f|_S \ge \|\Phi(f)|^{**}$, where $\|\cdot|^{**}$ is the norm on $(\mathrm{span}(\delta(X)), \|\cdot|^*)^*$. Conversely, consider $f \in \mathrm{SLip}_0(X)$. Then, by Proposition 2.28,

$$\begin{aligned}\|f|_S &= \sup_{d(y,x)>0} \frac{\max\{f(x) - f(y), 0\}}{d(y,x)} \\ &= \sup_{d(y,x)>0} \frac{\max\{\Phi(f)(\delta_x - \delta_y), 0\}}{\|\delta_x - \delta_y|^*} \le \|\Phi(f)|^{**},\end{aligned}$$

from which we deduce that Φ is an isometry. Since Φ is obviously linear and injective, it remains to establish surjectivity. This follows from the fact that any $\varphi \in (\mathrm{span}(\delta(X)), \|\cdot|^*)^*$ can be seen as $\Phi(\varphi \circ \delta)$, with $\varphi \circ \delta$ being semi-Lipschitz on X: indeed, for all $x, y \in X$,

$$\varphi(\delta(x)) - \varphi(\delta(y)) = \varphi(\delta_x - \delta_y) \le \|\varphi|^{**}\|\delta_x - \delta_y|^* = \|\varphi|^{**} d(y,x).$$

This shows that $\varphi \circ \delta$ belongs to $\mathrm{SLip}_0(X)$ and Φ is surjective. ■

Remark 3.6 (Compatibility with the classical theory of metric free spaces). If (X,d) is a metric space, then $\mathrm{SLip}_0(X) = \mathrm{Lip}_0(X)$. Moreover, every linear usc functional on a normed space is continuous; thus, the dual cone of a normed linear space is the same as the usual dual. We deduce that $\mathcal{F}_a(X) = \mathcal{F}(X)$.

Remark 3.7. For a quasi-metric space (X,d), it is easy to check that the space $\mathrm{SLip}_0(X,\bar{d})$ of semi-Lipschitz functions for the reverse quasi-metric is exactly $-\mathrm{SLip}_0(X,d)$, and that $\|f|_S = \|-f|_{\bar{S}}$ for any $f \in \mathrm{SLip}_0(X,d)$, where $\|-f|_{\bar{S}}$ denotes the semi-Lipschitz constant of $-f$ on $(X,\bar{d})$. Using this isometry, we can identify the dual cones of $\mathrm{SLip}_0(X,\bar{d})$ by the isometry Ψ defined by $\Psi(\mu)(f) = \mu(-f)$ for all $f \in \mathrm{SLip}_0(X,d)$, and therefore $\mathcal{F}_a(X,d) = \Psi(\mathcal{F}_a(X,\bar{d}))$ and $\|\Psi(\mu)|_{\bar{d}}^* = \|-\mu|^*$, where $\|\cdot|_{\bar{d}}^*$ is the norm of $\mathcal{F}_a(X,\bar{d})$.

3.2. Relation to molecules. Given a quasi-metric space (X,d) (always with a base point $x_0 \in X$), we next give a description of the closed unit ball of $\mathcal{F}_a(X)$ by means of the semi-Lipschitz evaluation functionals (often called *molecules*)

$$M_{(x,y)} = \frac{\delta(x) - \delta(y)}{d(y,x)}, \qquad \text{where } x,y \in X \text{ with } d(x,y) > 0.$$

Let $\widehat{\mathcal{M}}_X := \{M_{(x,y)} : x,y \in X \text{ with } d(y,x) > 0\}$.

Before going to this, it is worth noting that if (X,d) is an asymmetric locally convex space, the *asymmetric polar* of a subset $Y \subset X$ in the case of the asymmetric dual X^* can be defined as [11, p. 161]

$$Y^\alpha = \{\varphi \in X^* : \varphi(y) \le 1 \text{ for all } y \in Y\}.$$

Analogously, we can define the *asymmetric polar* of a subset W of the dual X^* by [11, p. 165]

$$W_\alpha = \{x \in X : \varphi(x) \le 1 \text{ for all } \varphi \in W\}.$$

Proposition 3.8. *Let (X,d) be a quasi-metric space with a base point x_0. The closed unit ball of $\mathcal{F}_a(X)$ coincides with*

$$(\{M_{(x,y)} : x,y \in X,\ d(y,x) > 0\}^\alpha)_\alpha.$$

Proof. Let $B_{\mathrm{SLip}_0(X)}$, $B_{\mathcal{F}_a(X)}$ and $B_{\mathcal{F}_a(X)^*}$ denote respectively the closed unit balls of $\mathrm{SLip}_0(X)$, $\mathcal{F}_a(X)$ and $\mathcal{F}_a(X)^*$, and consider the isometry $\Phi : \mathrm{SLip}_0(X) \to (\mathrm{span}(\delta(X)), \|\cdot|^*)^*$ defined in the proof of Theorem 3.5 as

$$\Phi(f)\Big(\sum_i \lambda_i \delta_{x_i}\Big) = \sum_i \lambda_i f(x_i)$$

for any linear combination of evaluation functionals. If $f \in \mathrm{SLip}_0(X)$, the condition $\|f|_S \le 1$ is equivalent to $\frac{f(x)-f(y)}{d(y,x)} \le 1$ for all $x,y \in X$ with

$d(y,x) > 0$ (by Proposition 2.28). Since Φ is an isometry, $\|f|_S \le 1$ also yields $\Phi(f)(M_{(x,y)}) \le 1$ for all $M_{(x,y)} \in \widehat{\mathcal{M}}_X$. Hence

$$\begin{aligned} B_{\mathcal{F}_a(X)^*} &= \{\Phi(f) : f \in \mathrm{SLip}_0(X),\, \Phi(f)(M_{(x,y)}) \le 1,\, \forall M_{(x,y)} \in \widehat{\mathcal{M}}_X\} \\ &= \{F \in \mathcal{F}_a(X)^* : F(M_{(x,y)}) \le 1,\, \forall M_{(x,y)} \in \widehat{\mathcal{M}}_X\} = (\widehat{\mathcal{M}}_X)^\alpha \end{aligned}$$

and thus

$$\Phi(B_{\mathrm{SLip}_0(X)})_\alpha = ((\widehat{\mathcal{M}}_X)^\alpha)_\alpha.$$

Moreover,

$$\begin{aligned} ((\widehat{\mathcal{M}}_X)^\alpha)_\alpha &= \Phi(B_{\mathrm{SLip}_0(X)})_\alpha = \{\gamma \in \mathcal{F}_a(X) : \Phi(f)(\gamma) \le 1,\, \forall f \in B_{\mathrm{SLip}_0(X)}\} \\ &= \{\gamma \in \mathcal{F}_a(X) : \gamma(f) \le 1,\, \forall f \in B_{\mathrm{SLip}_0(X)}\} \\ &= \Big\{\gamma \in \mathcal{F}_a(X) : \|\gamma|^* \,\Big(= \sup_{\|f|_S \le 1} \gamma(f)\Big) \le 1\Big\} = B_{\mathcal{F}_a(X)}. \ \blacksquare \end{aligned}$$

Remark 3.9. Let (X,d) be a quasi-metric space and $\overline{x} \notin X$. Then setting $\tilde{X} = X \cup \{\overline{x}\}$ and extending d from $X \times X$ to $\tilde{X} \times \tilde{X}$ by $\tilde{d}(x,\overline{x}) = \tilde{d}(\overline{x},x) = 1$ and $\tilde{d}(\overline{x},\overline{x}) = 0$, we obtain a new quasi-metric space $(\tilde{X},\tilde{d})$ with base point $x_0 \equiv \overline{x}$. The above construction corresponds to an asymmetric version of the Arens–Eells approach (cf. [4]).

3.3. Relation to asymmetrizations. Let $X = (X,D)$ be a metric space with a base point $x_0 \in X$ and denote by

$$L = (\mathrm{Lip}_0(X,D), \|\cdot\|_L)$$

its non-linear dual. Let $P \subseteq L$ be a cone satisfying (2.3), that is, for every $\phi \in L$ there exist $\phi_1, \phi_2 \in P$ with $\phi = \phi_1 - \phi_2$ and $\max\{\|\phi_1\|_L, \|\phi_2\|_L\} \le \|\phi\|_L \le \|\phi_1\|_L + \|\phi_2\|_L$. Let us denote by D_P the P-asymmetrization of X (see Definition 2.5). We also denote by

$$SL = (\mathrm{SLip}_0(X,D_P), \|\cdot|_S)$$

the non-linear asymmetric dual of (X,D_P), that is, the normed cone of semi-Lipschitz functions on (X,D_P).

Lemma 3.10 (Isometric injection of P into SL). *For every metric space (X,D) and every P-asymmetrization (X,D_P):*

(i) *there exists an isometric injection of P into SL;*
(ii) *there is a non-expansive injection of SL into L.*

Proof. Let $\phi \in SL$ and $x,y \in X$. Then

$$\begin{aligned} \phi(y) - \phi(x) &\le \|\phi|_S D_P(x,y) = \|\phi|_S \|\delta_y - \delta_x|_{\mathcal{F}_P} \\ &\le \|\phi|_S \|\delta_y - \delta_x\|_{\mathcal{F}} = \|\phi|_S D(x,y), \end{aligned}$$

which implies that $\phi \in \mathrm{Lip}_0(X,D)$ and $\|\phi\|_L \le \|\phi|_S$. This proves (ii).

Let now $\phi : (X, D) \to \mathbb{R}$ be in P with $\|\phi\|_L \neq 0$. Then $\phi_1 = \phi/\|\phi\|_L$ is also in P and $\|\phi_1\|_L = 1$. Given $x, y \in X$, we deduce

$$\begin{aligned} D_P(x,y) = \|\delta_y - \delta_x|_{\mathcal{F}_P} &= \sup_{\substack{\psi\in P\\ \|\psi\|_L\le 1}} (\psi(y) - \psi(x)) \\ &\geq \phi_1(y) - \phi_1(x) = \frac{1}{\|\phi\|_L}(\phi(y) - \phi(x)), \end{aligned}$$

which yields $\phi(y) - \phi(x) \leq \|\phi\|_L D_P(x, y)$. Hence, $\phi \in SL$ and $\|\phi|_S \leq \|\phi\|_L$. Combining with (ii) leads to $\|\phi\|_L = \|\phi|_S$, and (i) follows. ∎

Let us set

$$F = \operatorname{span}\{\delta(x) : x \in X\} \subset SL^*, \qquad \widehat{F} = \operatorname{span}\{\widehat{\delta}(x) : x \in X\} \subset L^* \tag{3.1}$$

where δ (respectively, $\widehat{\delta}$) is the canonical injection of (X, D_P) into SL^* (respectively, of (X, D) into L^*). There is a canonical bijection between F and $\widehat{F}$, under which a general element $Q = \sum_{i=1}^n \lambda_i \delta(x_i)$ of F is identified with the element $\widehat{Q} = \sum_{i=1}^n \lambda_i \widehat{\delta}(x_i)$ of $\widehat{F}$. Using this bijection, we get the following result.

Proposition 3.11 ($\|\cdot\|_{\mathcal{F}}$ is equivalent to the symmetrization of $\|\cdot|_{\mathcal{F}_a}$). *For any $Q \in F$,*

$$\begin{aligned} \max\{\|Q|_{\mathcal{F}_a}, \|-Q|_{\mathcal{F}_a}\} \leq \|\widehat{Q}\|_{\mathcal{F}} &\leq \|\widehat{Q}|_{\mathcal{F}_P} + \|-\widehat{Q}|_{\mathcal{F}_P} \\ &\leq 2\max\{\|Q|_{\mathcal{F}_a}, \|-Q|_{\mathcal{F}_a}\}. \end{aligned}$$

Proof. Let $Q \in F$. Since $F = \operatorname{span}(\delta(X))$, we have $Q = \sum_{i=1}^n \lambda_i \delta(x_i)$ for some $n \in \mathbb{N}$, $\lambda_i \in \mathbb{R}$ and $x_i \in X$, $i = 1, \ldots, n$, and

$$\|\widehat{Q}|_{\mathcal{F}_P} = \sup_{\substack{\phi\in P\\ \|\phi\|_L\le 1}} \langle \phi, \widehat{Q}\rangle = \sup_{\substack{\phi\in P\\ \|\phi\|_L\le 1}} \sum_{i=1}^n \lambda_i \phi(x_i) \leq \sup_{\substack{\varphi\in SL\\ \|\phi|_S\le 1}} \sum_{i=1}^n \lambda_i \varphi(x_i) =: \|Q|_{\mathcal{F}_a}.$$

We also obtain $\|-\widehat{Q}|_{\mathcal{F}_P} \leq \|-Q|_{\mathcal{F}_a}$. Now, if $\varphi \in SL$ satisfies $\|\phi|_S \leq 1$, then by Lemma 3.10(ii) we deduce that $\varphi \in L$ and $\|\varphi\|_L \leq \|\varphi|_S \leq 1$. Hence

$$\|Q|_{\mathcal{F}_a} \leq \sup_{\substack{\phi\in L\\ \|\phi\|_L\le 1}} \sum_{i=1}^n \lambda_i \phi(x_i) = \|\widehat{Q}\|_{\mathcal{F}}$$

and $\|-Q|_{\mathcal{F}_a} \leq \|\widehat{Q}\|_{\mathcal{F}}$, which yields

$$\max\{\|Q|_{\mathcal{F}_a}, \|-Q|_{\mathcal{F}_a}\} \leq \|\widehat{Q}\|_{\mathcal{F}} \leq \|\widehat{Q}|_{\mathcal{F}_P} + \|-\widehat{Q}|_{\mathcal{F}_P},$$

where the last inequality is a consequence of (2.5). The result follows. ∎

Below, we shall identify F with $\widehat{F}$, defined in (3.1). Under this identification, the norm $\|\cdot\|_{\mathcal{F}}$ can be considered to be also defined on F. With this convention, the statement of Proposition 3.11 reads as follows: the norm $\|\cdot\|_{\mathcal{F}}$ is equivalent to the symmetrization of $\|\cdot|_{\mathcal{F}_a}$ and hence

$$\mathcal{F}_a(X, D_P) = \overline{F}^{\|\cdot\|_{\mathcal{F}_a^s}} = \overline{F}^{\|\cdot\|_{\mathcal{F}}} = \mathcal{F}(X, D), \tag{3.2}$$

which implies that $\mathcal{F}_a(X, D_P)$ and $\mathcal{F}(X, D)$ can be identified as sets. Moreover,

$$D_P(x, y) = \|\delta(y) - \delta(x)|_{\mathcal{F}_P} = \|\delta(y) - \delta(x)|_{\mathcal{F}_a}.$$

Hence the following result holds.

THEOREM 3.12 (Compatibility I). *Let (X, D) be a metric space with a P-asymmetrization. Then the symmetrizations of $(\mathcal{F}(X, D), \|\cdot|_{\mathcal{F}_P})$ and of $(\mathcal{F}_a(X), \|\cdot|_{\mathcal{F}_a})$ are both isomorphic to $(\mathcal{F}(X, D), \|\cdot\|_{\mathcal{F}})$.*

The following diagram illustrates the situation described by Theorem 3.12:

$$\begin{array}{ccccc}
F = \operatorname{span}(\delta(X)) & \sqsubseteq & \mathcal{F}_a(X) & \sqsubseteq & (SL)^* \\
\updownarrow & & \uparrow & & \\
 & & \|\cdot\|_{\mathcal{F}_a^s}\text{-dense} & & \\
\widehat{F} = \operatorname{span}(\widehat{\delta}(X)) & \sqsubseteq & \mathcal{F}(X) & \sqsubseteq & L^* \\
 & & \uparrow & & \\
 & & \|\cdot\|_{\mathcal{F}}\text{-dense} & &
\end{array}$$

Let us now study the inverse procedure: we start with a quasi-metric space (X, d) and consider a symmetrization D of its distance (where D is either d^s or d^{s_0}, see Definition 2.7). It is easily seen that every $\phi \in \mathrm{SLip}_0(X, d)$ satisfies $\phi \in \mathrm{Lip}_0(X, D)$ and $\|\phi\|_L \leq \|\phi|_S$. Therefore, $P := \mathrm{SLip}_0(X, d)$ can be viewed as a cone in $\mathrm{Lip}_0(X, D)$ and used to define an asymmetric norm $\|\cdot|_{\mathcal{F}_P}$ on $\mathcal{F}(X, D)$ and consequently a quasi-metric D_P on X. In this setting, Proposition 3.13 below establishes a compatibility result under the assumption that there exists $\alpha \geq 1$ such that for every $\phi \in \mathrm{SLip}_0(X, d)$,

$$(\|\phi\|_L \leq)\ \|\phi|_S \leq \alpha \|\phi\|_L. \tag{3.3}$$

PROPOSITION 3.13 (Compatibility II). *Let (X, d) be a quasi-metric space with symmetrized distance D; assume* (3.3) *holds. Set $P := \mathrm{SLip}_0(X, d)$ and define, for every $Q \in \mathcal{F}(X, D)$,*

$$\|Q|_{\mathcal{F}_P} := \sup_{\substack{\phi \in P \\ \|\phi\|_L \leq 1}} \langle Q, \phi \rangle .$$

Then for all $Q \in \operatorname{span}(\delta(X))$,

$$\|Q|_{\mathcal{F}_a} \leq \|Q|_{\mathcal{F}_P} \leq \alpha \|Q|_{\mathcal{F}_a}. \tag{3.4}$$

In particular, if for $x, y \in X$ we set

$$D_P(x, y) := \|\delta_y - \delta_x|_{\mathcal{F}_P},$$

then for all $x, y \in X$,

$$d(x,y) \le D_P(x,y) \le \alpha d(x,y)\,. \tag{3.5}$$

TERMINOLOGY (Equivalence of asymmetric norms/quasi-metrics). We interpret (3.4) as an equivalence relation for the asymmetric norms $\|Q|_{\mathcal{F}_a}$ and $\|Q|_{\mathcal{F}_P}$. Similarly, (3.5) means that the quasi-distances d and D_P are equivalent.

Proof of Proposition 3.13. The equivalence between the asymmetric norms $\|\cdot|_{\mathcal{F}_a}$ and $\|\cdot|_{\mathcal{F}_P}$ on the vector space $\operatorname{span}(\delta(X))$ follows directly from their definitions and the inequalities $\|\phi\|_L \le \|\phi|_S \le \alpha\|\phi\|_L$. ∎

REMARK 3.14. The equivalence between the quasi-metric d and the canonical asymmetrization D_P of the symmetrized distance D yields an equivalence between D and the symmetrization $(D_P)^s$ of D_P.

If in addition to (3.3), we assume that $P = \mathrm{SLip}_0(X,d)$ induces an asymmetrization on the free space $\mathcal{F}(X,D)$, that is, for every $\phi \in \mathrm{Lip}_0(X,D)$ there exist $\phi_1, \phi_2 \in P$ such that

$$\phi = \phi_1 - \phi_2 \quad \text{and} \quad \max\{\|\phi_1\|_L, \|\phi_2\|_L\} \le \|\phi\|_L \le \|\phi_1\|_L + \|\phi_2\|_L,$$

then the equivalence between D and $(D_P)^s$ extends to the corresponding free spaces (see Remark 2.1). In particular, the following result holds.

PROPOSITION 3.15 (Compatibility III). *Let (X,d) be a quasi-metric space and $D = d^s$ or $D = d^{s_0}$. Assume that the cone $P = \mathrm{SLip}_0(X,d)$ of $\mathrm{Lip}_0(X,D)$ induces an asymmetrization in $\mathcal{F}(X,D)$ and (3.3) holds. Then the asymmetric free spaces $\mathcal{F}_a(X,d)$ and $\mathcal{F}_a(X,D_P)$ coincide (as sets) with the free space $\mathcal{F}(X,D)$:*

$$\mathcal{F}_a(X,d) = \mathcal{F}(X,D) = \mathcal{F}_a(X,D_P).$$

Moreover:

- (i) *The quasi-metrics d and D_P are equivalent, as also are the (symmetric) metrics D, $(D_P)^s$ and $(D_P)^{s_0}$ (symmetrizations of D_P).*
- (ii) *The asymmetric norms $\|\cdot|_{\mathcal{F}_a(X,d)}$, $\|\cdot|_{\mathcal{F}_P}$ and $\|\cdot|_{\mathcal{F}_a(X,D_P)}$ are equivalent.*
- (iii) *The symmetrizations of $\|\cdot|_{\mathcal{F}_a(X,d)}$, $\|\cdot|_{\mathcal{F}_P}$ and $\|\cdot|_{\mathcal{F}_a(X,D_P)}$ are equivalent to $\|\cdot\|_{\mathcal{F}}$.*

Proof. We have seen in (3.2) that $\mathcal{F}(X,D) = \mathcal{F}_a(X,D_P)$ (as sets). By Proposition 3.13, the asymmetric norms $\|\cdot|_{\mathcal{F}_a(X,d)}$ and $\|\cdot|_{\mathcal{F}_P}$ are equivalent on $F = \operatorname{span}\{\delta(x) : x \in X\}$, therefore

$$\mathcal{F}_a(X,d) = \overline{F}^{\|\cdot\|^s_{\mathcal{F}_a(X,d)}} = \overline{F}^{\|\cdot\|^s_{\mathcal{F}_P}} = \mathcal{F}(X,D).$$

Assertions (i) follow directly from Proposition 3.13. For (ii) it remains to prove that $\|\cdot|_{\mathcal{F}_a(X,d)}$ and $\|\cdot|_{\mathcal{F}_a(X,D_P)}$ are equivalent. We established in (3.5) that the quasi-distances d and D_P are equivalent. This implies that the normed cones $\mathrm{SLip}_0(X,d)$ and $\mathrm{SLip}_0(X,D_P)$ are isomorphic, which leads to an isomorphism of the corresponding semi-Lipschitz free spaces. The equivalence between the symmetrizations of the asymmetric norms asserted in (iii) now follows from (ii). Thanks to Theorem 3.12 they are also equivalent to $\|\cdot\|_{\mathcal{F}}$. ∎

REMARK 3.16. If the value of α associated to the assumption (3.3) is equal to 1, all of the aforementioned equivalences of Proposition 3.15 become equalities.

3.4. Properties (S) and (S*). We have shown that the P-asymmetrization of a metric space (X,D) gives rise to a quasi-metric space, for which the symmetrization of its asymmetric free space is isomorphic to the free space $(\mathcal{F}(X),\|\cdot\|_{\mathcal{F}})$. In this subsection we shall be interested in cases in which the aforementioned isomorphism is in fact an isometry.

DEFINITION 3.17. Let (X,D) be a metric space, $L=\mathrm{Lip}_0(X,D)$ and $P\subset L$ be a cone. We say that the metric space (X,D) satisfies:

(i) *property* $(\mathbf{S})$ *with respect to* P if P induces a non-trivial asymmetrization D_P on X and
$$SL=\mathrm{SLip}_0(X,D_P)=P;$$
(ii) *property* $(\mathbf{S}^*)$ (respectively, $(\mathbf{S}_0^*)$) *with respect to* P if, in addition to (i),
$$\|Q\|_{\mathcal{F}}=\|Q|_{\mathcal{F}_P}+\|-Q|_{\mathcal{F}_P}\ \text{(respectively, } \|Q\|_{\mathcal{F}}=\max\{\|Q|_{\mathcal{F}_P},\|-Q|_{\mathcal{F}_P}\})$$
for every $Q\in\mathcal{F}(X,D)$.

The following proposition is straightforward.

PROPOSITION 3.18. *Let (X,D) be a metric space.*

(i) *If (X,D) satisfies $(\mathbf{S})$ with respect to P, then $(\mathcal{F}(X,D),\|\cdot|_{\mathcal{F}_P})$ and $(\mathcal{F}_a(X,D_P),\|\cdot|_{\mathcal{F}_a})$ are identical.*
(ii) *If (X,D) satisfies $(\mathbf{S}^*)$ (resp. $(\mathbf{S}_0^*)$) with respect to P, then the d^s-symmetrization (resp. d^{s_0}-symmetrization) of $(\mathcal{F}_a(X,D_P),\|\cdot|_{\mathcal{F}_a})$ given in (2.8) is isometrically isomorphic to $(\mathcal{F}(X,D),\|\cdot\|_{\mathcal{F}})$.*

Before we proceed, let us give examples of metric spaces for which the above properties fail.

EXAMPLE 3.19. (i) (Property $(\mathbf{S})$ fails) Let $X=\mathbb{R}$ with the usual distance $D(t,s)=|s-t|$ for $t,s\in\mathbb{R}$. Let L be the space of Lipschitz functions on $\mathbb{R}$ vanishing at 0 and set
$$P:=\Big\{\phi\in L:\int_{\mathbb{R}}\phi\in[0,+\infty]\Big\}.$$

Then P contains the cone L_+ of non-negative Lipschitz functions vanishing at 0, and consequently $L = P - P$ and (2.3) holds. It is easy to see that

$$D_P(t,s) = \sup_{\substack{\phi\in P\\ \|\phi\|_L\le 1}} (\phi(s)-\phi(t)) = |s-t| = D(t,s).$$

Therefore, $SL = L \neq P$ and (**S**) fails.

(ii) (Property (**S**) holds but properties ($\mathbf{S}^*$) and ($\mathbf{S}_\mathbf{0}^*$) fail) We consider again $X = \mathbb{R}$ equipped with its usual distance D, and L the space of Lipschitz functions on $\mathbb{R}$ vanishing at 0. We now set

$$P = L_+ := \{\phi \in L : \phi \ge 0\}.$$

It follows easily that

$$D_+(s,t) = \sup_{\substack{\phi\in L_+\\ \|\phi\|_L\le 1}} (\phi(t)-\phi(s)) = \begin{cases} |t-s| & \text{if } 0\le s\le t \text{ or } s\le t\le 0,\\ \min\{t, s-t\} & \text{if } 0\le t\le s,\\ \min\{|s|, s-t\} & \text{if } t\le s\le 0,\\ |t| & \text{if } t\le 0\le s \text{ or } s\le 0\le t.\end{cases}$$

Let us show that property (**S**) holds. Indeed, for $s \neq 0$ we have $D_+(0,s) = s$ and $D_+(s,0) = 0$. By Lemma 3.10(i), $P \subset SL \subset L$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be any function vanishing at 0 and assume that for some $s \neq 0$ we have $\varphi(s) < 0$. Then $\varphi(0) - \varphi(s) > 0$ and $D_+(s,0) = 0$ reveals that φ cannot be in SL, showing that (**S**) holds.

Taking any two integers $n,k \ge 2$ we have $D_+(n,-k) = k$, $D_+(-k,n) = n$ and $D(n,-k) = n+k$, which shows that ($\mathbf{S}_0^*$) fails. On the other hand, $D_+(1,n) = n-1 = D(1,n)$ and $D_+(n,1) = 1$, which shows that ($\mathbf{S}^*$) fails.

A typical example of a metric space for which ($\mathbf{S}^*$) holds is the set of real numbers $\mathbb{R}$ viewed as a pointed metric space, for the cone $P = \{\phi \in L : \phi' \ge 0\}$ (see Lemma 5.2). To obtain additional examples of metric spaces satisfying ($\mathbf{S}^*$), let us first recall definitions and results due to Godard [20], regarding $\mathbb{R}$-trees.

Definition 3.20 ($\mathbb{R}$-tree). An $\mathbb{R}$-*tree* is a metric space T satisfying:

(i) for any $x,y \in T$, there exists a unique isometry $\phi =: \phi_{xy}$ of the closed interval $[0, d(x,y)]$ into T such that $\phi(0) = x$ and $\phi(d(x,y)) = y$ (the range of this isometry is called the *segment* and is denoted by $[x,y]$);

(ii) any one-to-one continuous mapping $\varphi : [0,1] \to T$ has the same range as the isometry $\phi_{a,b}$ associated to the points $a = \varphi(0)$ and $b = \varphi(1)$.

Our aim is to prove that subsets of (pointed) $\mathbb{R}$-trees have property ($\mathbf{S}^*$). The base point of an $\mathbb{R}$-tree is denoted by 0. Then one defines a partial order $\preccurlyeq$ on T by setting $x \preccurlyeq y$ if $x \in [0,y]$.

A subset A of T is said to be *measurable* whenever $\phi_{xy}^{-1}(A)$ is Lebesgue-measurable for any x and y in T. If A is measurable and S is the segment $[x,y]$, we write $\lambda_S(A)$ for $\lambda(\phi_{xy}^{-1}(A))$, where λ is the Lebesgue measure on $\mathbb{R}$. Let $\mathcal{R}$ be the family of all subsets of T which can be written as a finite union of disjoints segments, and for $R = \bigcup_{k=1}^{n} S_k \in \mathcal{R}$, we set

$$\lambda_R(A) = \sum_{k=1}^{n} \lambda_{S_k}(A).$$

Then

$$\lambda_T(A) = \sup_{R\in\mathcal{R}} \lambda_R(A)$$

defines a measure (called the *length measure*) on the σ-algebra of T-measurable sets such that

$$\int_{[x,y]} f(u)\, d\lambda_T(u) = \int_0^{d(x,y)} f(\phi_{xy}(t))\, dt$$

for any $f \in L_1(T)$ and $x, y \in T$.

Definition 3.21 (Measure on an $\mathbb{R}$-tree). Let T be a pointed $\mathbb{R}$-tree, and let A be a closed subset of T. We denote by μ_A the measure defined by

$$\mu_A = \lambda_A + \sum_{a\in A} L(a)\delta_a,$$

where λ_A is the restriction of the length measure λ_T to A, δ_a is the Dirac measure on a, and $L(a) = \inf_{x\in A\cap[0,a)} d(a,x)$.

Proposition 3.22 ([20, Proposition 2.3]). *Let T be a pointed $\mathbb{R}$-tree, and let A be a closed subset of T containing* 0. *Then $\mathcal{L}^1(\mu_A)^*$ is isometrically isomorphic to $\mathcal{L}^\infty(\mu_A)$.*

Definition 3.23 (Differentiation on an $\mathbb{R}$-tree). Let T be a pointed $\mathbb{R}$-tree, A a closed subset of T containing 0, and $f : A \to \mathbb{R}$. For $a \in A$, let $\tilde{a}$ be the unique point in $[0,a]$ such that $d(a,\tilde{a}) = L(a)$. If $L(a) > 0$, we say that f is *differentiable at a with derivative*

$$f'(a) = \frac{f(a) - f(\tilde{a})}{L(a)}.$$

If $L(a) = 0$, we say that f is *differentiable at a* whenever the limit

$$\lim_{\substack{x\to\tilde{a} \\ x\in[0,a)\cap A}} \frac{f(a) - f(x)}{d(x,a)}$$

exists, and we denote by $f'(a)$ the value of this limit.

Proposition 3.24 ([20, pp. 4313–4314]). *Let f be a real-valued Lipschitz function defined on an $\mathbb{R}$-tree T. Then f is differentiable almost everywhere* *on T and*

$$f(x) - f(0) = \int_{[0,x]} f' \, d\lambda_T$$

for all $x \in T$.

The following theorem characterizes subsets of $\mathbb{R}$-trees in terms of their Lipschitz free spaces.

Theorem 3.25 ([20, Theorem 4.2]). *Let (X, D) be a complete pointed metric space. Then the following assertions are equivalent:*

(i) *$\mathcal{F}(X)$ is isometrically isomorphic to a subspace of an $\mathcal{L}^1$-space;*
(ii) *(X, D) isometrically embeds into an $\mathbb{R}$-tree.*

We are now ready to prove our result on $\mathbb{R}$-trees.

Proposition 3.26. *Let (X, D) be a subset of an $\mathbb{R}$-tree T. Then (X, D) satisfies property $(\mathbf{S}^*)$ with respect to the cone*

$$P = \{\phi \in \mathrm{Lip}_0(X, D) : \phi' \geq 0\}.$$

Proof. Thanks to Theorem 3.25, we may use Godard's embedding, denoted by Φ_*, to isometrically identify $\mathcal{F}(X, D)$ with a subspace Y of $\mathcal{L}^1(T)$, by sending $\delta_x \in \mathcal{F}(X, D)$ to $\Phi_*(\delta_x) = \mathbb{1}_{[0,x]} \in \mathcal{L}^1(T)$. This embedding is the restriction to $\mathcal{F}(X, D)$ of the pre-adjoint of the (weak-star to weak-star continuous) isometry $\Phi : \mathcal{L}^\infty(T) \to \mathrm{Lip}_0(T)$ defined by $\Phi(g)(x) = \int_{[0,x]} g \, d\mu_X$ for any $x \in T$.

Let $\iota : (X, D) \to (Y, \|\cdot\|_1)$ be the isometric injection induced by Godard's embedding Φ. We keep the same notation $\|\cdot|_{\mathcal{F}_P}$ for the asymmetric hemi-norm induced in Y by this embedding. The P-asymmetrization of the norm of Y is given by

$$\|f|_{\mathcal{F}_P} = \sup_{\substack{\phi \geq 0 \\ \|\phi\|_\infty \leq 1}} \langle \phi, f \rangle = \sup_{\substack{\phi \geq 0 \\ \|\phi\|_\infty \leq 1}} \int_X f\phi \, d\mu_X = \int_X f^+ := \|f|_{1,+}$$

for all $f \in Y$, where $f^+(t) = \max\{f(t), 0\}$ for any $t \in T$. Therefore, $D_P(y, x) = \|\iota(x) - \iota(y)|_{\mathcal{F}_P} = 0$ whenever $\iota(x) \leq \iota(y)$ almost everywhere, which is equivalent to $x \preccurlyeq y$ in the order of T. Then, for $\varphi \in SL = \mathrm{SLip}_0(X, D_P)$ and $x, y \in X$ such that $x \preccurlyeq y$, we have $\varphi(x) - \varphi(y) \leq \|\varphi|_S D_P(y, x) = 0$, and therefore $x \preccurlyeq y$ yields $\varphi(x) \leq \varphi(y)$.

It is easy to check that $\Phi^{-1}(\varphi) = \varphi' \in \mathcal{L}^\infty(T)$ for all $\varphi \in L$. The monotonicity property of semi-Lipschitz functions proved above yields $\varphi' \geq 0$, so φ belongs to the cone P. Therefore, $SL \subset P$ and in view of Lemma 3.10(i) we deduce that $SL = P$, hence (X, D) satisfies property $(\mathbf{S})$.

Let $g \in \mathcal{F}(X,D)$, and $f = \Phi_*(g)$. Then

$$\|g\|_{\mathcal{F}} = \|f\|_1 = \sup_{\|\phi\|_\infty \le 1} \langle \phi, f\rangle = \langle f, \operatorname{sgn}(f)\rangle = \langle f^+, \operatorname{sgn}(f)\rangle - \langle f^-, \operatorname{sgn}(f)\rangle$$
$$= \|f|_{1,+} + \|-f|_{1,+} = \|g|_{\mathcal{F}_P} + \|-g|_{\mathcal{F}_P},$$

where $\operatorname{sgn}(f)$ denotes the sign of f. Thus (X,D) satisfies property $(\mathbf{S}^*)$. ∎

Combining Propositions 3.18 and 3.26, we obtain

PROPOSITION 3.27. *Let (X,D) be a subset of an $\mathbb{R}$-tree. Then there exists a canonical asymmetrization D_P of D such that the symmetrization of the semi-Lipschitz free space $\mathcal{F}_a(X,D_P)$ is isometrically isomorphic to $\mathcal{F}(X,D)$.*

4. Linearization of semi-Lipschitz functions: a universal property. As was pointed out in Remark 2.31, Definition 2.26 (as well as Definitions 2.27 and 2.32) can be readily generalized to functions between quasi-metric spaces, as well as to the case of semi-Lipschitz functions with values in a normed cone.

Let $(C, \|\cdot|)$ be a normed cone, and denote by $d_e^c(u,v)$ its corresponding extended quasi-metric (cf. Proposition 2.20). We next introduce the notion of semi-Lipschitz function with values in C.

DEFINITION 4.1 (Semi-Lipschitz function with values in a normed cone). Let (X,d) be a quasi-metric space. A function $f : X \to C$ is said to be a *semi-Lipschitz function* if there exists $L \ge 0$ such that for all $x, y \in X$,

$$d_e^c(f(y), f(x)) \le L d(y,x). \tag{4.1}$$

In this case, the *semi-Lipschitz conic norm* of a function $f : X \to C$ is defined by

$$\|f|_S := \inf\{L > 0 : (4.1) \text{ holds}\}.$$

The class of semi-Lipschitz functions on X with values in C is denoted as $\mathrm{SLip}(X,C)$. Also, if $x_0 \in X$ is a base point, we define the *asymmetric pivot space*

$$\mathrm{SLip}_0(X,C) := \{f \in \mathrm{SLip}(X,C) : f(x_0) = 0\}.$$

As in Proposition 2.28, a function $f : X \to C$ is semi-Lipschitz if and only if $\|f|_S < \infty$. Moreover, if d is a quasi-metric and $f : X \to C$ is semi-Lipschitz, then

$$\|f|_S = \sup_{x \ne y} \frac{\max\{d_e^c(f(y), f(x)), 0\}}{d(y,x)} = \sup_{x \ne y} \frac{d_e^c(f(y), f(x))}{d(y,x)} < \infty.$$

If d is a quasi-hemi-metric, we use the same expression for $\|f|_S$ as that in Proposition 2.28(ii).

Given a quasi-metric space (X,d) with a base point x_0, consider the isometric injection $\delta : (X,d) \to (\mathrm{SLip}_0(X)^*, \|\cdot|^*)$ of Proposition 3.3. We next show that the semi-Lipschitz free space over a quasi-metric space (X,d) with a base point x_0 is characterized by the following universal property, which is an analog of the Lipschitz case (see [22, Lemma 2.2]).

Theorem 4.2 (Linearization of semi-Lipschitz functions). *Let (X,d) be a quasi-metric space with a base point x_0. Suppose that $(C,\|\cdot|)$ is a normed cone and $f\in \mathrm{SLip}_0(X,C)$. Then there exists a unique linear map $T_f:\mathcal{F}_a(X)\to C$ extending f, i.e. $T_f\circ\delta=f$ and $\|T_f|=\|f|_S$.*

Proof. If $f\in\mathrm{SLip}_0(X,C)$, then $T_f:\mathcal{F}_a(X)\to C^{**}$ defined by

$$T_f(\gamma)(\phi)=\gamma(\phi\circ f)\quad(\gamma\in\mathcal{F}_a(X),\,\phi\in C^*)$$

belongs to the set of bounded linear mappings from $\mathcal{F}_a(X)$ into C^{**}, and

$$\begin{aligned}\|T_f| &= \sup_{\|\gamma|^*\le 1}\|T_f(\gamma)|^{**}=\sup_{\|\gamma|^*\le 1}\sup_{\|\phi|^*\le 1}T_f(\gamma)(\phi)\\ &=\sup_{\|\phi|^*\le 1}\sup_{\|\gamma|^*\le 1}\gamma(\phi\circ f)=\sup_{\|\phi|^*\le 1}\|\phi\circ f|_S\le\|f|_S.\end{aligned}$$

Observe that the last inequality is achieved by taking into account that ϕ is linear and

$$\begin{aligned}\sup_{\|\phi|^*\le 1}\|\phi\circ f|_S &= \sup_{\|\phi|^*\le 1}\sup_{d(y,x)>0}\left\{\frac{(\phi\circ f)(x)-(\phi\circ f)(y)}{d(y,x)}\right\}\\ &=\sup_{\|\phi|^*\le 1}\sup_{d(y,x)>0}\left\{\frac{\phi(f(x)-f(y))}{d(y,x)}\right\}\le\sup_{\|\phi|^*\le 1}\|\phi|^*\|f|_S=\|f|_S.\end{aligned}$$

(By abuse of notation, we still denote by $\|T_f|=\sup_{\|\gamma|^*\le 1}\|T_f(\gamma)|^*$ the conic norm of the linear function $T_f:\mathcal{F}_a(X)\to C^{**}$.) Furthermore, if $i_C:C\to C^{**}$ is the canonical injection, we have

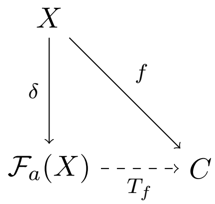
$$\begin{aligned}\langle T_f(\delta(x)),\phi\rangle &= T_f(\delta(x))(\phi)=\delta(x)(\phi\circ f)\\ &=\phi(f(x))=i_C(f(x))(\phi)=\langle i_C(f(x)),\phi\rangle\end{aligned}$$

for every $x\in X$ and $\phi\in C^*$, and hence $T_f(\delta(x))=i_C(f(x))\in i_C(C)$ for every $x\in X$. This yields that $T_f(\gamma)\in i_C(C)$ for every $\gamma\in\mathcal{F}_a(X)$. Identifying $i_C(f(x))\in i_C(C)$ with $f(x)\in C$, we have $T_f\in\mathcal{L}(\mathcal{F}_a(X),C)$ and $T_f\circ\delta=f$. So, since $T_f\circ\delta=f$ and δ is an isometry (Proposition 3.3), we deduce that

$$\begin{aligned}\|f|_S &= \sup_{d(y,x)>0}\left\{\frac{d_e^c(f(y),f(x))}{d(y,x)}\right\}\\ &=\sup_{d(y,x)>0}\left\{\frac{\|T_f(\delta(x))-T_f(\delta(y))|}{d(y,x)}\right\}=\sup_{d(y,x)>0}\left\{\frac{\|T_f(\delta(x)-\delta(y))|}{d(y,x)}\right\}\\ &\le\sup_{d(y,x)>0}\left\{\frac{\|T_f|\|\delta(x)-\delta(y)|^*}{d(y,x)}\right\}=\|T_f|\sup_{d(y,x)>0}\left\{\frac{\|\delta(x)-\delta(y)|^*}{\|\delta(x)-\delta(y)|^*}\right\}=\|T_f|.\end{aligned}$$

Thus $\|T_f| = \|f|_S$. Assume now that there exists a bounded linear mapping $S_f : \mathcal{F}_a(X) \to C$ such that $S_f \circ \delta = f$. Then it is clear that $S_f(\delta(x)) = T_f(\delta(x))$ for all $x \in X$ and, by the definition of $\mathcal{F}_a(X)$, it follows that $S_f = T_f$. ■

Remark 4.3 (Universal property). Equivalently, the condition $T_f \circ \delta = f$ means that the following diagram commutes:



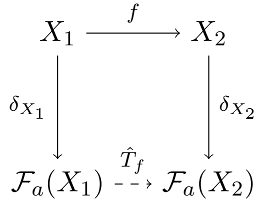
Furthermore, as a consequence of the universal property that we have just proved, it is not difficult to establish that the mapping $f \mapsto T_f$ is an isometric isomorphism of $\mathrm{SLip}_0(X, C)$ into the cone of bounded linear mappings $L(\mathcal{F}_a(X), C)$, which constitutes another proof of Theorem 3.5 for the particular case $C = \mathbb{R}$. Indeed, we already know that the mapping $f \mapsto T_f$ is an isometry of $\mathrm{SLip}_0(X, \mathbb{R})$ onto $\mathcal{F}_a(X)^*$. Now, given $T \in L(\mathcal{F}_a(X), C)$, we can define a mapping $f : X \to C$ by $f(x) = T(\delta(x))$ for all $x \in X$. Since

$$d_e^c(f(y), f(x)) = d_e^c\big(T(\delta(y)), T(\delta(x))\big) \le \|T| \, \|\delta(x) - \delta(y)|^* = \|T| d(y, x)$$

for all $x, y \in X$, the function f is in $\mathrm{SLip}_0(X, C)$. By the universal property of $\mathcal{F}_a(X)$, there is a unique operator $T_f \in L(\mathcal{F}_a(X), C)$ such that $T_f \circ \delta = f$. Hence $T = T_f$ and thus the mapping $f \mapsto T_f$ is a surjective isometry.

The proof of the following result is immediate from Theorem 4.2.

Corollary 4.4 (Linearization of quasi-metric morphisms). *Let (X_1, d_1) and (X_2, d_2) be pointed quasi-metric spaces, and $f \in \mathrm{SLip}_0(X_1, X_2)$. Then there is a unique linear map $\hat{T}_f : \mathcal{F}_a(X_1) \to \mathcal{F}_a(X_2)$ such that $\hat{T}_f \circ \delta_{X_1} = \delta_{X_2} \circ f$, i.e. the diagram*



commutes, and $\|\hat{T}_f| = \|f|_S$, where δ_{X_1} and δ_{X_2} are the isometric injections of the quasi-metric spaces (X_1, d_1) and (X_2, d_2) to their free spaces (cf. Proposition 3.3).

For the following proposition, we refer to the reader to [31] for a survey on the extensions of semi-Lipschitz functions on quasi-metric spaces.

Proposition 4.5 (The free space of a quasi-metric subspace). *Let (X,d) be a quasi-metric space with a base point x_0, and consider a subspace (M,d) of (X,d) such that $x_0 \in M$. Then $\mathcal{F}_a(M)$ is isometrically isomorphic to a subspace of $\mathcal{F}_a(X)$.*

Proof. Let $\hat{T}_i : \mathcal{F}_a(M) \to \mathcal{F}_a(X)$ be the linearization given by Corollary 4.4 of the identity mapping $i : M \to X$. Since $\|\hat{T}_i| = \|i|_S = 1$, we know that $\|\hat{T}_i(Q)|^*_{\mathcal{F}_a(X)} \le \|Q|^*_{\mathcal{F}_a(M)}$. For the opposite inequality, consider $Q \in \operatorname{span}(\delta(M))$. Clearly, $\hat{T}_i(Q) = Q \in \operatorname{span}(\delta(X))$. Then, for any $f \in \operatorname{SLip}(M)$, the expression $\tilde{f}(x) = \inf_{m\in M}\{f(m) + \|f|_S d(m,x)\}$, $x \in X$ (which is an adaptation of the McShane extension of Lipschitz maps), provides a semi-Lipschitz extension with the same associated conic norm $\|f|_S$. It follows that

$$\|Q|^*_{\mathcal{F}_a(M)} = \sup_{\substack{\|f|_S\le 1\\ f\in\operatorname{SLip}(M)}} \langle Q,f\rangle \le \sup_{\substack{\|f|_S\le 1\\ f\in\operatorname{SLip}(X)}} \langle Q,f\rangle = \|Q|^*_{\mathcal{F}_a(X)} = \|\hat{T}_i(Q)|^*_{\mathcal{F}_a(X)}.$$

By continuity of $\hat{T}_i$ (and density of $\operatorname{span}(\delta(M))$ in $\mathcal{F}_a(M)$), we can extend the previous inequality to any $Q \in \mathcal{F}_a(M)$. ∎

Let us consider another conic norm on $\operatorname{span}(\delta(X))$ (and on $\mathcal{F}_a(X)$), which is based on a variant of the so-called Kantorovich–Rubinstein norm (see [12, Section 8.4.5]).

Example 4.6 (Kantorovich–Rubinstein conic norm). Let $X \neq \emptyset$ be a set equipped with a quasi-metric d and a base point x_0. For $\gamma, \overline{\gamma} \in \operatorname{span}(\delta(X))$ consider all representations $\gamma - \overline{\gamma} = \sum_{i=1}^n \lambda_i(\widehat{y}_i - \widehat{z}_i)$, where $\lambda_i \ge 0$ and possibly some $\widehat{y}_i$ or $\widehat{z}_i$ are equal to $\widehat{x}_0 = 0$, and set

$$d_{\mathrm{KR}}(\gamma,\overline{\gamma}) := \inf\{\lambda_1 d(z_1,y_1) + \cdots + \lambda_n d(z_n,y_n)\}.$$

Then $\|\gamma|_{\mathrm{KR}} := d_{\mathrm{KR}}(\widehat{x}_0,\gamma)$ is an asymmetric norm on $\operatorname{span}(\delta(X))$ and

$$d_{\mathrm{KR}}(\widehat{x},\widehat{y}) = d(y,x) \quad \text{for all } x,y \in X.$$

Moreover, $\|\gamma|_{\mathrm{KR}}$ coincides with the restriction of the conic norm $\|\cdot|^*$ of $\operatorname{SLip}_0(X)^*$ to $\operatorname{span}(\delta(X))$ and thus extends to $\mathcal{F}_a(X)$. Indeed, if $\|\cdot|'$ is a conic norm on $\operatorname{span}(\delta(X))$ satisfying $\|\delta(x)-\delta(y)|' \le d(y,x)$ for all $x,y \in X$, then every $\gamma = \lambda_1(\widehat{y}_1 - \widehat{z}_1) + \cdots + \lambda_n(\widehat{y}_n - \widehat{z}_n)$ accomplishes

$$\begin{aligned}\|\gamma|' &= \|\lambda_1(\widehat{y}_1 - \widehat{z}_1) + \cdots + \lambda_n(\widehat{y}_n - \widehat{z}_n)|' \\ &\le \|\lambda_1(\widehat{y}_1 - \widehat{z}_1)|' + \cdots + \|\lambda_n(\widehat{y}_n - \widehat{z}_n)|' \\ &\le \lambda_1 d(z_1,y_1) + \cdots + \lambda_n d(z_n,y_n),\end{aligned}$$

which shows that $\|\gamma|' \le \|\gamma|_{\mathrm{KR}}$. In particular, $\|\gamma|^* \le \|\gamma|_{\mathrm{KR}}$ (since the conic norm $\|\cdot|^*$ on $\mathcal{F}_a(X)$ satisfies $\|\delta(x)-\delta(y)|^* = d(y,x)$ for all $x,y \in X$). Hence

$d(y,x) = \|\delta(x) - \delta(y)|^* \leq \|\delta(x) - \delta(y)|_{\mathrm{KR}} \leq d(y,x)$ for all $x, y \in X$, which implies that

$$\|\delta(x) - \delta(y)|_{\mathrm{KR}} = d(y,x) \quad \text{for all } x, y \in X.$$

Consider now the mapping $L : X \to (\mathrm{span}(\delta(X)), \|\cdot|_{\mathrm{KR}})$ sending x to $\delta(x)$, which is clearly an isometric embedding. By the universality property of $\mathcal{F}_a(X)$ (see Theorem 4.2), L extends to $\tilde{L} : \mathcal{F}_a(X) \to (\mathrm{span}(\delta(X)), \|\cdot|_{\mathrm{KR}})$ and $\|\cdot|_{\mathrm{KR}} \leq \|\cdot|^*$, so the conic norms $\|\cdot|_{\mathrm{KR}}$ and $\|\cdot|^*$ are the same.

5. Examples of semi-Lipschitz free spaces. Let us now illustrate the semi-Lipschitz free space for three concrete examples of quasi-metric spaces: a finite quasi-metric space consisting of three points, the set of natural numbers $\mathbb{N}$ with a discrete quasi-metric and the set of real numbers $\mathbb{R}$ under the quasi-hemi-metric defined by the canonical conic hemi-norm u. We also include an example-scheme stemming from canonical asymmetrizations of subsets of $\mathbb{R}$-trees.

5.1. A 3-point quasi-metric space. Let $X = \{x_0, x_1, x_2\}$ be a set of three points, endowed with the following quasi-metric (in a general form):

$$\rho(x_0, x_1) = a_{01}, \quad \rho(x_1, x_0) = a_{10}, \quad \rho(x_0, x_2) = a_{02},$$
$$\rho(x_2, x_0) = a_{20}, \quad \rho(x_1, x_2) = a_{12}, \quad \rho(x_2, x_1) = a_{21}.$$

Taking x_0 as a base point, it is clear that the set of semi-Lipschitz functions vanishing at x_0 can be algebraically identified with $\mathbb{R}^2$, i.e. any function $g : X \to \mathbb{R}$ with $g(x_0) = 0$ is in $\mathrm{SLip}_0(X)$, with associated semi-Lipschitz norm equal to

$$\|g|_S = \max\left\{\frac{g_1 - g_2}{a_{21}}, \frac{g_2 - g_1}{a_{12}}, \frac{g_1}{a_{01}}, \frac{g_2}{a_{02}}, \frac{-g_1}{a_{10}}, \frac{-g_2}{a_{20}}\right\},$$

where $g_1 = g(x_1)$ and $g_2 = g(x_2)$. Hence, the unit ball B of $\mathrm{SLip}_0(X, \rho) \simeq \mathbb{R}^2$ is the polygon generated by the linear inequalities defined in terms of the asymmetric norm. The dual cone of $(\mathrm{SLip}_0(X), \|\cdot|_S)$ is the vector space $\mathbb{R}^2$ endowed with the asymmetric norm determined by the Minkowski gauge of the asymmetric polar B^o of the unit ball B of $\mathrm{SLip}_0(X, \rho)$, that is,

$$B^o = \{X \in \mathbb{R}^2 : \langle g, X\rangle \leq 1, \forall g \in B\}.$$

As the evaluation functionals $\delta(x_1), \delta(x_2)$ generate the vector space $\mathbb{R}^2$, it follows that $\mathcal{F}_a(X, \rho)$ is isomorphic to $\mathbb{R}^2$, with the asymmetric norm determined by the aforementioned Minkowski gauge. Furthermore, for any $g \in \mathrm{SLip}_0(X)$, its linearization $T_g : \mathcal{F}_a(X) \to \mathbb{R}$ is given by

$$T_g(\lambda_1 \widehat{x}_1 + \lambda_2 \widehat{x}_2) = \lambda_1 g(x_1) + \lambda_2 g(x_2),$$

with $\lambda_i \in \mathbb{R}$, $i = 1, 2$. Notice that the unit balls of $\mathrm{SLip}_0(X, \rho)$ and its dual cone have at most six extreme points (see Figure 2).

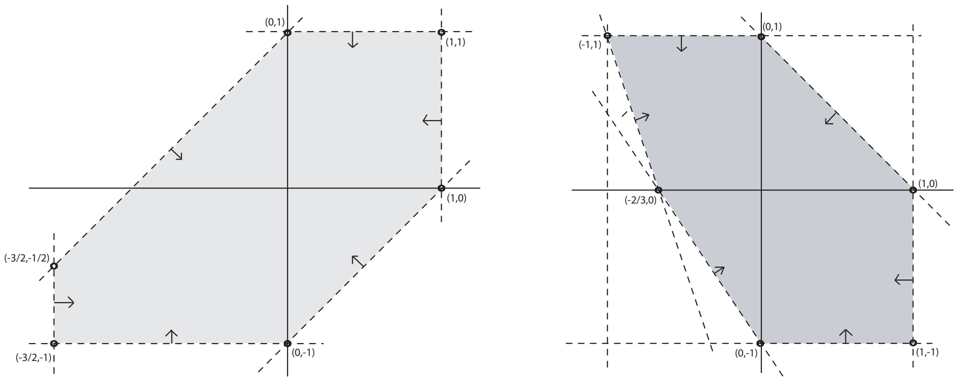


Fig. 2. Representation of the unit ball of $\mathrm{SLip}_0(X,\rho)$ and its asymmetric polar, respectively, with $X=\{x_0,x_1,x_2\}$, $\rho(x_1,x_0)=\frac{3}{2}$ and $\rho(x_i,x_j)=1$ for $i\neq j$ with $(i,j)\neq(1,0)$

5.2. $\mathbb{N}$ as a metric or quasi-metric space. We now consider the set $\mathbb{N}$ of natural numbers (including 0) endowed with the quasi-metric defined by

$$d(n,m)=\begin{cases}1 & \text{if } m\notin\{0,n\},\\ 0 & \text{if } m\in\{0,n\}.\end{cases}$$

We fix $x_0=0$ as a base point. Let $y=(y(n))_n\in\mathrm{SLip}_0(\mathbb{N},d)$. Then $y(0)=0$ and the semi-Lipschitz condition implies that the sequence $(y(n))_n$ is non-negative: indeed,

$$y(0)-y(n)=-y(n)\le \|y|_S\,d(n,0)=0$$

and

$$y(n)-y(0)=y(n)\le \|y|_S\,d(0,n)=\|y|_S.$$

Therefore we have $(y(n))_n\in\ell^\infty(\mathbb{N})$ and $\|y|_S\ge\|y\|_\infty$. Moreover,

$$\|y|_S=\sup_{d(n,m)>0}\frac{y(m)-y(n)}{1}\le\sup_{d(n,m)>0}y(m)=\|y\|_\infty,$$

since $y(n)\ge 0$ for all $n\in\mathbb{N}$. It is easy to check than any bounded non-negative sequence satisfies the semi-Lipschitz condition, so it follows that $\mathrm{SLip}_0(\mathbb{N},d)$ is $(\ell^\infty_+(\mathbb{N}),\|\cdot\|_\infty)$, the positive cone of $\ell^\infty(\mathbb{N})$. The dual norm on $\ell^\infty_+(\mathbb{N})^*$ is given by

$$\|\varphi|^*=\sup_{\substack{(y_n)\in\ell^\infty_+(\mathbb{N})\\ \|(y_n)\|_\infty\le 1}}\varphi((y_n)).$$

The set $\{\delta(n):n\in\mathbb{N}\}\subset\ell^\infty_+(\mathbb{N})^*$ of evaluation functionals can be identified with the canonical basis of $\ell^1(\mathbb{N})$, so the linear span of $\delta(\mathbb{N})$ is the set $c_{00}(\mathbb{N})$ of finitely supported sequences. On this set, the dual norm of

$\mathrm{SLip}_0(\mathbb{N},d)^*$ becomes

$$\|(x_n)|^* = \sum_{n\in\mathbb{N}} \max\{x_n, 0\} = \sum_{n\in\mathbb{N}} x_n^+ =: \|(x_n)|_{1,+},$$

since the supremum on the dual norm is taken over the positive cone of $\ell^\infty(\mathbb{N})$ (and it is attained at the sequence $(\mathrm{sgn}(x_n)\vee 0)$). It is easy to check that the symmetrization of the asymmetric norm $\|\cdot|_{1,+}$ is equivalent to the usual norm of $\ell^1(\mathbb{N})$, and therefore the asymmetric normed space $(\ell^1(\mathbb{N}), \|\cdot|_{1,+})$ satisfies the conditions required to make it the bicompletion of $(c_{00}(\mathbb{N}), \|\cdot|_{1,+})$. Therefore, the semi-Lipschitz free space $\mathcal{F}_a(\mathbb{N},d)$ is isometrically isomorphic to $(\ell^1(\mathbb{N}), \|\cdot|_{1,+})$ and the linearization T_y of a function $y=(y(n))_n \in \mathrm{SLip}_0(\mathbb{N},d)$ can be obtained from

$$T_y(e_n) = y(n), \qquad n = 1, 2, \ldots,$$

where e_n is the nth element of the canonical basis of $\ell^1(\mathbb{N})$.

It is well known that the free space $\mathcal{F}(\mathbb{N},D)$ of $\mathbb{N}$ equipped with the distance

$$D(m,n) = \begin{cases} 2 & \text{if } n \notin \{0,m\},\\ 1 & \text{if } n=0 \text{ or } m=0,\\ 0 & \text{if } n=m \end{cases}$$

is isometric to $\ell^1(\mathbb{N})$ (see, for instance, [20, 21, 39]), and

$$L = \mathrm{Lip}_0(\mathbb{N},D) = \left\{ y = y_n \in \mathbb{R}^{\mathbb{N}} : \|y\|_L := \frac{y(n)-y(m)}{D(m,n)} < \infty \right\}$$

is isometric to $\ell^\infty(\mathbb{N})$. If $m,n\in\mathbb{N}$, then the canonical asymmetrization of D (Definition 2.5) is

$$D_+(m,n) = \|\widehat{\delta}(n) - \widehat{\delta}(m)|_{\mathcal{F}_+} = \sup_{\substack{y\in\ell^\infty(\mathbb{N})_+\\ \|y\|_\infty\le 1}} \langle y, e_n - e_m\rangle = \sup_{0\le y_n\le 1} \sum_{k\ge 0} y_k x_k,$$

where $x_n = 1$, $x_m = -1$, and $x_k = 0$, for $k \notin \{n,m\}$. According to Theorem 3.12, $\mathcal{F}(\mathbb{N},D) = \mathcal{F}(\mathbb{N},d)$ (as sets), with $d = D_+$, $\mathrm{SLip}_0(\mathbb{N},d) = \ell^\infty_+(\mathbb{N}) = \mathrm{SLip}_0(\mathbb{N},D_+)$ and $\|x|_{\mathcal{F}_a} = \|x|_{\mathcal{F}_+} = \sum_{n\ge 0} x_n^+$.

5.3. The quasi-metric space $(\mathbb{R},u)$**.** Note that since the symmetrized distance d_u^s is equal to the usual metric of $\mathbb{R}$ (which can be seen as a pointed $\mathbb{R}$-tree), $\mathcal{F}(\mathbb{R},u)$ can be obtained from Proposition 3.18. We include a direct self-contained proof, which does not rely on Godard's work on $\mathbb{R}$-trees. Let us start with some preliminary results.

LEMMA 5.1 (Semi-Lipschitz functions in $(\mathbb{R},u)$). *Let $f \in \mathrm{SLip}_0(\mathbb{R},u)$. Then f is a non-decreasing function in* $\mathrm{Lip}_0(\mathbb{R})$.

Proof. By Remark 2.33, f is Lipschitz on $(\mathbb{R},u^s) = (\mathbb{R},|\cdot|)$, and therefore is differentiable almost everywhere. Note that if $x\le y$, then $d_u(y,x)=0$, so $f(x)\le f(y)$. As f is non-decreasing, $f'\ge 0$. ∎

Lemma 5.2. *The normed cone* $(\mathrm{SLip}_0(\mathbb{R},u),\|\cdot|_S)$ *is isometrically isomorphic to* $(\mathcal{L}^\infty_+(\mathbb{R}),\|\cdot\|_\infty)$.

Proof. Consider the mapping $T:(\mathcal{L}^\infty_+(\mathbb{R}),\|\cdot\|_\infty)\to(\mathrm{SLip}_0(\mathbb{R},u),\|\cdot|_S)$ defined by

$$Tg(x)=\int_0^x g\,d\lambda=\int \mathbb{1}_{[0,x]}g,$$

which is surjective by the previous analysis. This mapping is well defined since for $x\ge y$ we have

$$Tg(x)-Tg(y)=\int_y^x g\,d\lambda\le\|g\|_\infty(x-y)=\|g\|_\infty d_u(y,x).$$

If $x<y$ then

$$Tg(x)-Tg(y)=-\int_x^y g\,d\lambda\le 0=d_u(y,x).$$

This also proves that $\|Tg|_S\le\|g\|_\infty$. On the other hand, let $x\in\mathbb{R}$ be a point of differentiability of Tg. Then

$$Tg'(x)=\lim_{y\searrow x}\frac{Tg(y)-Tg(x)}{y-x}\le\sup_{x<y}\frac{Tg(y)-Tg(x)}{y-x}=\|Tg|_S,$$

and since clearly $(Tg)'=g$, we conclude that $\|g\|_\infty\le\|Tg|_S$ and that T is an isometric isomorphism. ∎

For the following result, if $f\in\mathcal{L}^1(\mathbb{R})$, recall the notation $\|f|_{1,+}=\int_{\mathbb{R}}f^+d\lambda$, where $f^+(x)=\max\{f(x),0\}$ and λ denotes the Lebesgue measure, which was used in Lemma 2.43.

Theorem 5.3. *The semi-Lipschitz free space* $\mathcal{F}_a((\mathbb{R},u))$ *of the asymmetric hemi-normed space* $(\mathbb{R},u)$ *is isometrically isomorphic to* $(\mathcal{L}^1(\mathbb{R}),\|\cdot|_{1,+})$.

Proof. By Lemma 2.43, we know that $(\mathcal{L}^1(\mathbb{R}),\|\cdot|_{1,+})$ is the asymmetric predual of $(\mathcal{L}^\infty_+(\mathbb{R}),\|\cdot\|)$. Therefore we only need to check that the isometry $T:(\mathcal{L}^\infty_+(\mathbb{R}),\|\cdot\|_\infty)\to(\mathrm{SLip}_0(\mathbb{R},u),\|\cdot|_S)$ defined in the previous proof is (w^*-w^*)-continuous; then Lemma 2.46 will give us an isometry between the preduals $\mathcal{F}_a(\mathbb{R},u)$ and $(\mathcal{L}^1(\mathbb{R}),\|\cdot|_{1,+})$. So, let (g_α) be a net on $\mathcal{L}^\infty_+(\mathbb{R})$ converging to g in the w^*-topology induced by the predual $(\mathcal{L}^1(\mathbb{R}),\|\cdot|_{1,+})$, and take $x\in\mathbb{R}$ and the corresponding $\widehat{x}\in\mathcal{F}_a(\mathbb{R},u)$. Then

$$\langle Tg_\alpha,\widehat{x}\rangle=\int_0^x g_\alpha=\langle g_\alpha,\mathbb{1}_{[0,x]}\rangle\to\langle g,\mathbb{1}_{[0,x]}\rangle, \tag{5.1}$$

by the w^*-convergence of (g_α). Now, for an arbitrary $\mu\in\mathcal{F}_a(X)$ we can take a sequence $(\mu_n)\subset\mathrm{span}(\delta(\mathbb{R}))$ such that $\mu_n\to\mu$ in the symmetrized topology of $\mathrm{SLip}_0(\mathbb{R},u)^*$, and therefore

$$\langle Tg_\alpha,\mu\rangle = \lim_n \langle Tg_\alpha,\mu_n\rangle, \tag{5.2}$$

where the last convergence is with respect to the usual norm on $\mathbb{R}$, thanks to the symmetrized $|\cdot|$-continuity of semi-Lipschitz functions. Equations (5.1) and (5.2) yield $\langle Tg_\alpha,\mu\rangle \to \langle Tg,\mu\rangle$ for the norm topology in $\mathbb{R}$, so T is (w^*-w^*)-continuous, and by Lemma 2.46 there exists an isometric isomorphism between $(\mathcal{F}_a(\mathbb{R},u),\|\cdot|^*)$ and $(\mathcal{L}^1(\mathbb{R}),\|\cdot|_{1,+})$. ■

As we show in Example 2.6, $d_u(x,y) = u(y-x)$ is a canonical asymmetrization of $D(x,y) = |y-x|$ for the cone $P = \{\phi \in L : \phi' \geq 0\}$. Notice that the canonical asymmetrization D_+, based on the cone $P = L_+$, gives a different asymmetrization.

5.4. Canonical asymmetrization of subsets of $\mathbb{R}$-trees. Propositions 3.18 and 3.26 provide a variety of examples of quasi-metric spaces (X,d) whose corresponding semi-Lipschitz free spaces are isometrically isomorphic to subspaces of $(\mathcal{L}^1(T),\|\cdot|_{1,+})$, where T is an $\mathbb{R}$-tree containing the symmetrized space (X,d^s). We can obtain more specific examples by applying the following recent result from [3, Theorem 1.1], which gives a characterization of all complete metric spaces whose Lipschitz free space is isometric to a subspace of $\ell^1(\Gamma)$ for some set Γ.

THEOREM 5.4. *Let (X,D) be a complete pointed metric space. Then the following are equivalent:*

(i) *$\mathcal{F}(X)$ is isometrically isomorphic to a subspace of $\ell^1(\Gamma)$ for some set Γ;*
(ii) *(X,D) is a subset of an $\mathbb{R}$-tree such that $\lambda(X)=0$ and $\lambda(\overline{\mathrm{Br}(X)})=0$, where λ is the length measure and $\mathrm{Br}(X)$ is the set of branching points of X.*

Since every metric space as above satisfies property $(\mathbf{S}^*)$ (see Proposition 3.26), we deduce that the corresponding semi-Lipschitz free spaces are isometrically isomorphic to $(\ell^1(\Gamma),\|\cdot|_{1,+})$ for some set Γ.

A careful reader might have observed that in all examples presented in this section, the semi-Lipschitz free space of the given quasi-metric space can be easily obtained from the Lipschitz free space of its symmetrization. We shall now show that this is always the case, provided assumption $(\mathcal{H})$ below holds. (This is the case in all of the aforementioned examples.)

Using the same notation as in the second part of Subsection 3.3, let (X,d) be a quasi-metric space and (X,D) its symmetrization (D is either d^s or d^{s_0}). Then $P := \mathrm{SLip}_0(X,d)$ is a cone in $\mathrm{Lip}_0(X,D)$ and $\|\phi\|_L \leq \|\phi|_S$ for all $\phi \in P$. Let us assume:

($\mathcal{H}$) For every $\phi \in \mathrm{Lip}_0(X, D)$ there exist $\phi_1, \phi_2 \in \mathrm{SLip}_0(X, d)$ such that

$$\phi = \phi_1 - \phi_2 \quad \text{and} \quad \max\{\|\phi_1|_S, \|\phi_2|_S\} \leq \|\phi\|_L.$$

Since $\|\phi_i\|_L \leq \|\phi_i\|_S$ for $i \in \{1, 2\}$ and in view of the triangular inequality

$$\|\phi\|_L = \|\phi_1 - \phi_2\| \leq \|\phi_1\|_L + \|\phi_2\|_L,$$

we deduce that ($\mathcal{H}$) implies in particular that P induces a canonical asymmetrization in $\mathcal{F}(X, D)$ (in the sense of Remark 2.1).

Proposition 5.5. *Let (X, d) be a quasi-metric space and assume ($\mathcal{H}$) holds. Then the semi-Lipschitz free space $\mathcal{F}_a(X, d)$ coincides (as a set) with the free space $\mathcal{F}(X, D)$ of the symmetrized space (X, D) and is endowed with the asymmetric norm*

$$\|Q| = \sup_{\substack{\|\phi|_S \leq 1 \\ \phi \in \mathrm{SLip}_0(X,d)}} \langle Q, \phi\rangle \quad \textit{for all } Q \in \mathcal{F}_a(X, d).$$

Proof. Following the method used in Subsection 3.3, we start by identifying the sets

$$F = \mathrm{span}\{\delta(x) : x \in X\} \subset \mathrm{SLip}_0(X, d)^*,$$
$$\widehat{F} = \mathrm{span}\{\widehat{\delta}(x) : x \in X\} \subset \mathrm{Lip}_0(X, D)^*$$

where δ and $\widehat{\delta}$ are the canonical injections of (X, d) into $\mathrm{SLip}_0(X, d)^*$ and of (X, D) into $\mathrm{Lip}_0(X, D)^*$, respectively.

It suffices to prove that the d^s-symmetrization $\|\cdot\|^s$ of the asymmetric norm $\|\cdot|$ is equivalent to $\|\cdot\|_{\mathcal{F}}$. Consider $Q \in F$. Since $\|\phi|_S \geq \|\phi\|_L$ for any $\phi \in \mathrm{SLip}_0(X, d)$, it follows (by the definition of each norm) that $\|Q| \leq \|Q\|_{\mathcal{F}}$, so $\|Q\|^s \leq 2\|Q\|_{\mathcal{F}}$. Conversely, take ϕ in the unit ball of $\mathrm{Lip}_0(X, D)$ such that $\|Q\|_{\mathcal{F}} = \langle Q, \phi\rangle$, and consider $\phi_1, \phi_2 \in \mathrm{SLip}_0(X, d)$ such that $\phi = \phi_1 - \phi_2$, with $\max\{\|\phi_1|_S, \|\phi_2|_S\} \leq \|\phi\|_L \leq 1$. Then

$$\|Q\|_{\mathcal{F}} = \langle Q, \phi\rangle = \langle Q, \phi_1\rangle + \langle -Q, \phi_2\rangle \leq \|Q| + \|-Q| := \|Q\|^s.$$

The result follows from $\mathcal{F}_a(X, d) = \overline{F}^{\|\cdot\|^s_{\mathcal{F}_a}} = \overline{F}^{\|\cdot\|_{\mathcal{F}}} = \mathcal{F}(X, D)$. ∎

6. Conclusions, future research. It seems to be of paramount importance to relate symmetric and asymmetric structures in a way that remains compatible with the embeddings to the corresponding free spaces. At the same time, a given asymmetric space might not be equal to a canonical asymmetrization of some (symmetric) metric space. It is even unknown if a given quasi-distance is always topologically equivalent to a quasi-distance with this property.

Let us observe that there are several ways to symmetrize a quasi-distance and obtain (symmetric) metric spaces that generate the same underlying topology. Indeed, let $\phi : \mathbb{R}^2_+ \to [0, \infty)$ be any non-negative continuous function satisfying $\phi(a, b) \geq \max\{a, b\}$ and assume further that ϕ is symmetric (i.e. $\phi(a, b) = \phi(b, a)$ for all $a, b \geq 0$), coercive (i.e. with bounded sublevel sets) and $\phi(a, b) = 0$ if and only if $a = b$ for all $a, b \geq 0$. Then for every quasi-distance d, we obtain a symmetric distance d^ϕ via

$$d^\phi(x, y) = \phi(d(x, y), d(y, x)) \quad \text{for all } x, y \in X.$$

(In Definition 2.7 we have only focused on the cases $\phi_0(a, b) = \max\{a, b\}$ and $\phi(a, b) = a+b$.) Concerning the inverse procedure (asymmetrization), we have mainly been based on the lattice structure of the non-linear dual of the metric spaces, which is used to induce a canonical asymmetrization on the initial metric space (X, D). However, in some cases, canonical asymmetrizations of the same space may look completely different (see Example 3.19). In a similar spirit, starting from an asymmetric space (X, d) and considering its symmetrization (X, D), it is not known whether or not the set $\mathrm{SLip}_0(X, d)$ of semi-Lipschitz functions, viewed as a cone in $\mathrm{Lip}_0(X, D)$, induces a canonical asymmetrization on (X, D) (cf. Proposition 3.15) and in particular when conditions (3.3) or $(\mathcal{H})$ hold. Therefore, many natural questions still remain unexplored and the whole panorama is far from being completely understood.

Another topic that merits to be further explored is the particular case of normed spaces. Indeed, considering a normed space $(X, \|\cdot\|)$ as a metric space leads to a canonical asymmetrization D_P of its distance $D(x, y) = \|y - x\|$. If $X = \mathbb{R}$ (therefore, $\mathcal{F}(\mathbb{R}) = \mathcal{L}^1(\mathbb{R})$), then taking $P = \mathcal{L}^\infty_+(\mathbb{R})$ we observe that the asymmetrized distance $D_P = u$ is associated to an asymmetric norm; see Subsection 5.3. It would be interesting to determine which normed spaces admit canonical asymmetrizations of their norms in this way, and inversely, characterize asymmetric norms that can be obtained via this procedure.

Let us finish this discussion with a more philosophical comment. Convexity is a fundamental notion of variational analysis whose definition relies on the linear structure. It particular, it is not distance-related, in sharp contrast with usual differential calculus, Lipschitz functions, Riemann/Finsler geometry and metric generalizations of convexity. In particular, the class of Lipschitz functions is clearly affected if we consider asymmetric distances or asymmetrizations of the space. In both cases semi-Lipschitz functions are appropriate morphisms to describe properties of the space (see [14] e.g.) and this work outlines a natural way to define a notion of a quasi-metric free space as well as of a canonical asymmetrization of a space. Let us recall that in a metric space (X, D), every Lipschitz function $\varphi : K \to \mathbb{R}$ can be extended to a Lipschitz function $\tilde{\varphi} : X \to \mathbb{R}$ without any increase in the Lipschitz constant, that is, $\mathrm{Lip}(\varphi, K) = \mathrm{Lip}(\tilde{\varphi}, X)$. Indeed, McShane [30] gave a concrete formula (based on inf-convolution) to obtain such an extension, called *minimal extension*. In a completely analogous way one constructs minimal semi-Lipschitz extensions for real-valued semi-Lipschitz functions defined on a quasi-metric space (X, d) maintaining the semi-Lipschitz constant (see proof of Proposition 4.5). An important instance of minimal Lipschitz extension is the so-called AMLE (absolutely minimal Lipschitz extension), which in the case of a Euclidean space corresponds to the solution of the infinite-Laplacian operator [26]. The notion of absolutely minimal semi-Lipschitz extension makes perfect sense in an asymmetric framework, but it is not known whether or not such an extension always exists and, in the case of a finite-dimensional asymmetric normed space, whether it can be identified to a solution of some differential type operator. Concerning this latter topic, for the time being there is no clear way to deal with differentiability in asymmetric structures. Doing this in a satisfactory manner seems to be somehow related to the fact that the asymmetric space is canonical, since in this case, and only there, one would expect to obtain a canonical asymmetric differential calculus. Formalizing and proving this meta-theorem is mathematically challenging, but at the same time it will shed new light on asymmetric analysis, a topic on which very little is known and which might be relevant in the future.

Acknowledgements. The authors wish to express their gratitude to the referee for his careful reading and comments, as well as for his suggestion to use Godard's characterization of isometric embeddings to an $\mathbb{R}$-tree, which led to Propositions 3.26 and 3.27.

Part of this work was realized during a research stay of A. Daniilidis (*Gaspard Monge* invited professor) and F. Venegas (Research trainee) at INRIA (Équipe Tropicale) and CMAP of École Polytechnique (France), from September to December 2018. The authors wish to thank G. Godefroy and S. Gaubert for insightful comments. Part of this work has been presented by the third author at the conference "Function Theory on Infinite Dimensional Spaces XVI", held at the Complutense University of Madrid (November 2019). This author is grateful to the organizing committee of the event for hospitality and financial support. He also thanks S. Tapia, A. Procházka and C. Petitjean for useful discussions.

Research of A. Daniilidis supported by the grants: CMM AFB170001, FONDECYT 1211217, ECOS-CONICYT C18E04 (Chile), and PGC2018-097960-B-C22 (MCIU/AEI/ERDF, UE).

Research of J. M. Sepulcre supported by the grant PGC2018-097960-B-C22 (MCIU/AEI/ERDF, UE).

Research of F. Venegas supported by the grants: CMM AFB170001, FONDECYT 1171854, ECOS-CONICYT C18E04, and CONICYT Doctorate Fellowship PFCHA 2019–21191167 (Chile).

Shortly before the conclusion of this work, it came to our knowledge that L. Candido, P. Kaufmann and J. Romão have been working on this topic, and independently obtained results which partially overlap with some of the ones presented here; see [10].

References

[1] M. Akian, S. Gaubert and A. Hochart, *Minimax representation of nonexpansive functions and application to zero-sum recursive games*, J. Convex Anal. 25 (2018), 225–240.

[2] F. Albiac and N. J. Kalton, *Lipschitz structure of quasi-Banach spaces*, Israel J. Math. 170 (2009), 317–335.

[3] R. J. Aliaga, C. Petitjean and A. Procházka, *Embeddings of Lipschitz-free spaces into* ℓ_1, J. Funct. Anal. 280 (2021), no. 6, art. 108916, 26 pp.

[4] R. Arens and J. Eells, *On embedding uniform and topological spaces*, Pacific J. Math. 6 (1956), 397–403.

[5] M. Bachir, *A non-convex analogue to Fenchel duality*, J. Funct. Anal. 181 (2001), 300–312.

[6] M. Bachir and G. Flores, *Index of symmetry and topological classification of asymmetric normed spaces*, Rocky Mountain J. Math. 50 (2020), 1951–1964.

[7] D. Bao, S.-S. Chern and Z. Shen, *An Introduction to Riemann–Finsler Geometry*, Grad. Texts in Math. 200, Springer, 2000.

[8] L. Borel-Mathurin, *Approximation properties and non-linear geometry of Banach spaces*, Houston J. Math. 38 (2012), 1135–1148.

[9] J. Cabello-Sánchez and J. A. Jaramillo, *A functional representation of almost isometries*, J. Math. Anal. Appl. 445 (2017), 1243–1257.

[10] L. Candido, P. Kaufmann and J. Romão, *Asymmetric Lipschitz-free spaces*, in preparation.

[11] Ş. Cobzaş, *Functional Analysis in Asymmetric Normed Spaces*, Birkhäuser, 2012.

[12] Ş. Cobzaş, R. Miculescu and A. Nicolae, *Lipschitz Functions*, Lecture Notes in Math. 2241, Springer, 2019.

[13] M. Cúth, M. Doucha and P. Wojtaszczyk, *On the structure of Lipschitz-free spaces*, Proc. Amer. Math. Soc. 144 (2016), 3833–3846.

[14] A. Daniilidis, J. A. Jaramillo and F. Venegas, *Smooth semi-Lipschitz functions and almost isometries between Finsler Manifolds*, J. Funct. Anal. 279 (2020), no. 8, art. 108662, 29 pp.

[15] S. Deng and Z. Hou, *The group of isometries of a Finsler space*, Pacific J. Math. 207 (2002), 149–155.

[16] L. C. García-Lirola and A. Procházka, *Pełczyński space is isomorphic to the Lipschitz free space over a compact set*, Proc. Amer. Math. Soc. 147 (2019), 3057–3060.

[17] L. M. García-Raffi, *Compactness and finite dimension in asymmetric normed linear spaces*, Topology Appl. 153 (2005), 844–853.

[18] L. M. García-Raffi, S. Romaguera, E. A. Sánchez-Pérez and O. Valero, *Metrizability of the unit ball of the dual of a quasi-normed cone*, Boll. Un. Mat. Ital. B (8) 7 (2004), 483–492.

[19] S. Gaubert and G. Vigeral, *A maximin characterization of the escape rate of non-expansive mappings in metrically convex spaces*, Math. Proc. Cambridge Philos. Soc. 152 (2012), 341–363.

[20] A. Godard, *Tree metrics and their Lipschitz-free spaces*, Proc. Amer. Math. Soc. 138 (2010), 4311–4320.
[21] G. Godefroy, *A survey on Lipschitz-free Banach spaces*, Comment. Math. 55 (2015), 89–118.
[22] G. Godefroy and N. J. Kalton, *Lipschitz-free Banach spaces*, Studia Math. 159 (2003), 121–141.
[23] P. Hájek and E. Pernecká, *On Schauder bases in Lipschitz-free spaces*, J. Math. Anal. Appl. 416 (2014), 629–646.
[24] L. Hanin, *Kantorovich–Rubinstein norm and its application in the theory of Lipschitz spaces*, Proc. Amer. Math. Soc. 115 (1992), 345–352.
[25] M. A. Javaloyes, L. Lichtenfelz and P. Piccione, *Almost isometries of non-reversible metrics with applications to stationary spacetimes*, J. Geom. Phys. 89 (2015), 38–49.
[26] R. Jensen, *Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient*, Arch. Ration. Mech. Anal. 123 (1993), 51–74.
[27] L. Kantorovich, *On mass transfer*, Dokl. Akad. Nauk SSSR 37 (1942), 227–229 (in Russian).
[28] L. V. Kantorovich and G. Sh. Rubinshteĭn, *On a functional space and certain extremal problems*, Dokl. Akad. Nauk SSSR 115 (1957), 1058–1061 (in Russian).
[29] V. L. Klee, *Some characterizations of compactness*, Amer. Math. Monthly 58 (1951), 389–393.
[30] E. J. McShane, *Extension of range of functions*, Bull. Amer. Math. Soc. 40 (1934), 837–842.
[31] C. Mustăţa, *Extensions of semi-Lipschitz functions on quasi-metric spaces*, Rev. Anal. Numér. Théor. Approx. 30 (2001), 61–67.
[32] S. Oltra and O. Valero, *Isometries on quasi-normed cones and bicompletion*, New Zealand J. Math. 33 (2004), 83–90.
[33] A. Papadopoulos and M. Troyanov, *Weak Finsler structures and the Funk weak metric*, Math. Proc. Cambridge Philos. Soc. 147 (2009), 419–437.
[34] V. G. Pestov, *Free Banach–Lie algebras, couniversal Banach–Lie groups, and more*, Pacific J. Math. 157 (1993), 137–144.
[35] D. A. Raĭkov, *Free locally convex spaces for uniform spaces*, Mat. Sb. (N.S.) 63 (1964), 582–590 (in Russian).
[36] S. Romaguera and M. Sanchis, *Properties of the normed cone of semi-Lipschitz functions*, Acta Math. Hungar. 108 (2005), 55–70.
[37] W. Rudin, *Functional Analysis*, 2nd ed., McGraw-Hill, 1991.
[38] O. Valero, *Quotient normed cones*, Proc. Indian Acad. Sci. Math. Sci. 116, 2006, 175–191.
[39] N. Weaver, *Lipschitz Algebras*, World Sci., 1999.

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