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# Habilitation à Diriger des Recherches

Mathématiques Appliquées

présentée par

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# Analyse convexe et quasi-convexe
# Applications en optimisation

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*A toi cher ami,*
*tu nous as quittés très tôt ...*


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# Analyse convexe et quasi-convexe ; applications en optimisation

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*Sommaire* :



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## I. Liste des travaux présentés

### Analyse convexe

[1] A. Daniilidis, Subdifferentials of convex functions and sigma-cyclic monotonicity, *Bull. Austral. Math. Soc.* **61** (2000), 269-276.

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### Analyse quasi-convexe (approche via l'analyse non-lisse)

[6] A. Daniilidis, N. Hadjisavvas, Characterization of nonsmooth semistrictly quasiconvex and strictly quasiconvex functions, *J. Optim. Th. Appl.* **102** (1999), 525-536.

[7] A. Daniilidis, N. Hadjisavvas, On the subdifferentials of quasiconvex and pseudoconvex functions and cyclic monotonicity, *J. Math. Anal. Appl.* **237** (1999), 30-42.

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[13] A. Daniilidis, N. Hadjisavvas, On generalized cyclically monotone operators and proper quasimonotonicity, *Optimization* **47** (2000), 123-135.

**Intégration d'opérateurs multivoques**

[14] M. Bachir, A. Daniilidis & JP Penot, Lower subdifferentiability and Integration, *Set-Valued Anal.* **10** (2002), 89-108.

[15] A. Daniilidis, P. Georgiev, JP Penot, Integration of multivalued operators and cyclic submonotonicity, *Trans. Amer. Math. Soc.* **355** (2003), 177-195.

**Optimisation multi-critère**

[16] A. Daniilidis, N. Hadjisavvas & S. Schaible, Connectedness of the Efficient Set for Three Objective Quasiconcave Maximization Problems, *J. Optim. Th. Appl.* **97** (1997), 517-524.

[17] A. Daniilidis, "Arrow-Barankin-Blackwell theorems and related results in cone duality: a survey", in: *Optimization (Namur, 1998)*, 119-131, Lecture Notes in Econom. and Math. Systems **481**, Springer, Berlin, 2000.

## II. Synthèse de l'activité scientifique

**Introduction :** Ce document de synthèse s'articule autour de l'analyse convexe, de l'analyse quasi-convexe et des applications en optimisation. Dans le premier domaine on aborde les thèmes de la continuité, de la différentiabilité et des critères de coïncidence pour les fonctions convexes, puis la convexification des fonctions semi-continues inférieurement. Pour l'étude des fonctions quasi-convexes deux approches sont adoptées : une approche analytique, via un sous-différentiel généralisé, et une approche géométrique, basée sur les normales aux tranches. La dernière partie est consacrée à des applications à l'intégration d'opérateurs multivoques, aux inéquations variationnelles et à des problèmes d'optimisation multi-critères en dimension finie et infinie. Parmi les nouveautés de ce travail, on trouve la notion de monotonie fortement cyclique, qui caractérise le sous-différentiel d'une fonction convexe dont la restriction à son domaine est continue, la quasi-monotonie cyclique, qui est une propriété intrinsèque du sous-différentiel d'une fonction quasi-convexe, et la notion de quasi-monotonie propre, qui caractérise les opérateurs pour lesquels l'inéquation variationnelle associée a toujours des solutions sur toute sous-partie convexe et faiblement compacte de leur domaine. Notons encore une nouvelle caractérisation de la propriété de Radon-Nikodym, et une extension à la dimension infinie d'un résultat de Janin concernant l'intégration d'un opérateur maximal cycliquement sous-monotone, résultat qui généralise le théorème classique de Rockafellar pour les opérateurs maximaux cycliquement monotones.

## Analyse convexe

Le thème principal de cette section est l'étude des propriétés des fonctions convexes. Les travaux effectués dans ce domaine se décomposent en quatre parties :

- une étude de classification des propriétés de continuité des fonctions convexes s.c.i.,
- une nouvelle caractérisation de la propriété de Radon-Nikodym en relation avec la différentiabilité au sens de Gâteaux,
- une étude de critères de coïncidence de deux fonctions convexes s.c.i.,
- une étude sur la convexification des fonctions s.c.i. basée sur le sous-différentiel de Fenchel-Moreau.

### Continuité des fonctions convexes

Dans certains problèmes d'optimisation on doit souvent considérer des fonctions qui prennent leurs valeurs dans l'espace $\widehat{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. On considère dans la suite que $\widehat{\mathbb{R}}$ est muni de la topologie étendue de $\mathbb{R}$, i.e. la topologie engendrée par les ouverts habituels de $\mathbb{R}$ et les parties de la forme $(a, +\infty]$, où $a \in \mathbb{R}$.

Soit $X$ un espace de Banach et $f : X \to \mathbb{R} \cup \{+\infty\}$ une fonction convexe, semi-continue inférieurement (s.c.i.). Notons $\mathrm{dom} f$ l'ensemble $\{x \in X : f(x) < +\infty\}$. Il est bien connu ([50, page 37]) que

$$f \text{ est continue en } x \text{ si et seulement si } x \in \mathrm{intdom} f. \tag{1}$$

On peut également ajouter à cette équivalence l'assertion suivante :

$$\partial f \text{ est localement borné} \tag{2}$$

[Rappelons ici ([52] e.g.) que pour tout $x \in \mathrm{dom} f$, le sous-différentiel au sens de Fenchel-Moreau $\partial f$ est défini par :

$$\partial f(x) = \{x^* \in X^* : f(y) \geq f(x) + \langle x^*, y - x \rangle, \forall y \in X\}.] \tag{3}$$

Il suit de (1) que $f$ est discontinue en $x \in \mathrm{dom} f$, si et seulement si $x \in \mathrm{dom} f \setminus \mathrm{intdom} f$. En particulier, $f$ est discontinue en tout point, lorsque $\mathrm{intdom} f = \emptyset$.

Citons trois exemples - typiques de notre étude - où cela est le cas :

- $f$ est la fonction indicatrice d'une partie convexe, fermée et d'intérieur vide.
- $f$ est le résultat d'un prolongement (en ajoutant la valeur $+\infty$) d'une fonction $g$ convexe continue à un plus grand espace sur lequel $\mathrm{intdom} g = \emptyset$ (ceci est une opération standard en optimisation).
- $f$ est la norme $\|\cdot\|_1$ sur l'espace de Hilbert $\ell^2(\mathbb{N})$.

Dans les deux premiers exemples, on note que la discontinuité de la fonction $f$ n'est pas un défaut intrinsèque, mais elle est due à la valeur $+\infty$ que l'on a imposée en dehors de son domaine. Par contre, dans le troisième exemple, la discontinuité a des raisons bien plus profondes : on remarque aisément qu'au voisinage de chaque point $x \in \mathrm{dom} f$ la fonction a des valeurs à la fois finies et arbitrairement grandes. Comment alors distinguer ces deux différents types de discontinuité ?

Commençons par une simple remarque : la restriction $f|_{\mathrm{dom}\, f}$ de la fonction $f$ à son domaine ne prend que des valeurs réelles. Dans les deux premiers exemples cette restriction est continue, alors que dans le troisième elle est discontinue en tout point. Pour répondre donc à la question ci-dessus, il suffit de trouver un moyen de concrétiser cette information.

Dans cette partie, on propose une réponse complète basée sur une approche duale. Avant de donner les détails, rappelons d'abord une propriété liée au sous-différentiel, la *monotonie cyclique* ([53]) :

Un opérateur $T : X \rightrightarrows X^*$ est dit cycliquement monotone, si pour tout $n \in \mathbb{N}\setminus\{0\}$, tous $x_0, x_1, \ldots, x_n$ dans $X$ et tous $x_0^* \in T(x_0)$, $x_1^* \in T(x_1)$ ,...,$x_n^* \in T(x_n)$ on a :

$$\sum_{i=0}^{n} \langle x_i^*, x_{i+1} - x_i \rangle \leq 0. \tag{4}$$

(où $x_{n+1} := x_0$).

Cette propriété fondamentale caractérise, parmi les opérateurs monotones, ceux qui sont inclus dans le sous-différentiel d'une fonction convexe s.c.i. Dans [22], on introduit une variante de la monotonie cyclique, appelée "*monotonie $\sigma-$cyclique*", où les sommes finies sont remplacées par des sommes de séries :

Un opérateur $T : X \rightrightarrows X^*$ est dit $\sigma$-cycliquement monotone en $x_0$, si pour toute suite $\{x_n\}_{n=0}^{+\infty}$ de $X$ satisfaisant $\lim_{n\to\infty} x_n = x_0$ et pour tous $x_i^* \in T(x_i)$ $(i = 0, 1, 2, \ldots)$ on a :

$$\limsup_{n\to+\infty} \sum_{i=0}^{n} \langle x_i^*, x_{i+1} - x_i \rangle \leq 0 \tag{5}$$

On peut montrer que (5) est vérifié pour tout opérateur monotone dans $\mathbb{R}$ ainsi que pour tout sous-différentiel localement borné, ce qui n'est évidemment pas le cas du sous-différentiel d'une fonction ayant des discontinuités. En effet, dans [10], on montre que la monotonie $\sigma$-cyclique en $x_0$ caractérise le sous-différentiel d'une fonction convexe, s.c.i., dont la restriction à son domaine de sous-différentiabilité est continue en $x_0$.

Une propriété plus restrictive, adaptée à la continuité de la restriction au domaine, est aussi introduite :

Un opérateur $T : X \rightrightarrows X^*$ est dit fortement cycliquement monotone en $x_0$, si pour tout $\varepsilon > 0$ il existe $\delta > 0$ tel que pour tout $x_1 \in \mathrm{dom}\, T \cap B(x_0, \delta)$, pour toute suite $\{x_i\}_{i=2}^{+\infty}$ de $\mathrm{dom}\, T$ satisfaisant $\lim_{i\longrightarrow+\infty} x_i = x_0$, et pour toute suite $\{x_i^*\}_{i\geq 1}$ de $X^*$ satisfaisant $x_i^* \in T(x_i)$ pour tout $i \geq 1$, on a :

$$\limsup_{n\to+\infty} \sum_{i=1}^{n} \langle x_i^*, x_{i+1} - x_i \rangle \leq \varepsilon. \tag{6}$$

Le schéma suivant résume les résultats de notre étude ($f$ désigne une fonction convexe s.c.i. et $\partial f$ son sous-différentiel, et l'on dit qu'un opérateur $T$ est localement borné sur une partie $D$ si tout point de $D$ a un voisinage sur lequel $T$ est borné) :

$$\begin{array}{ccc}
\partial f \text{ localement borné sur dom} f & \Longleftrightarrow & f \text{ continue} \\
\Downarrow & & \Downarrow \\
\partial f \text{ fort. cycliquement monotone} & \Longleftrightarrow & f|_{\text{dom } f} \text{ continue} \\
\Downarrow & & \Downarrow \\
\partial f \ \sigma\text{-cycliquement monotone} & \Longleftrightarrow & f|_{\text{dom } \partial f} \text{ continue}
\end{array}$$

Notons que toute implication non affichée ci-dessus est en général fausse.

**Différentiabilité des fonctions convexes et propriété de Radon-Nikodym**

On dit qu'un espace de Banach $X$ est de Radon-Nikodym, si toute mesure vectorielle $m : \mathcal{B} \to X$ de variation bornée et absolument continue par rapport à la mesure de Lebesgue $\lambda$, peut être représentée comme une intégrale (au sens de Bochner) d'une fonction $g := [\frac{dm}{d\lambda}] \in L^1([0,1], X)$ ; ($\mathcal{B}$ désigne la tribu borélienne de $[0,1]$). D'après les travaux classiques de Rieffel, Phelps, Stegall et d'autres (voir [50, Chap. 5] e.g.) cette propriété sur un espace dual se trouve étroitement liée à la différentiabilité au sens de Fréchet des fonctions convexes continues sur le prédual. Plus précisément on a :

$$X \text{ est un espace d'Asplund} \Longleftrightarrow X^* \text{ est un espace de Radon-Nikodym} \tag{7}$$

[Rappelons que $X$ est un espace d'Asplund, si toute fonction convexe continue sur un ouvert de $X$ est Fréchet différentiable sur une partie $G_\delta$-dense de son domaine.]

Réciproquement, la caractérisation des espaces de Radon-Nikodym par la Fréchet-différentiabilité générique des fonctions $w^*$-s.c.i. convexes continues sur le dual, a été montrée par J. Collier dans [17] :

$$X \text{ est de Radon-Nikodym} \Longleftrightarrow X^* \text{ est préfaiblement Asplund.} \tag{8}$$

Ici le terme "préfaiblement Asplund" signifie que toute fonction $w^*$-s.c.i. convexe continue sur $X^*$ est Fréchet-différentiable dans une partie $G_\delta$ dense de son domaine.

Dans [7], on donne une nouvelle caractérisation des espaces de Radon-Nikodym :

$X$ est de Radon-Nikodym si et seulement si toute fonction $w^*$-s.c.i. convexe continue sur l'espace dual $X^*$ est Gâteaux-différentiable en un certain point, sa Gâteaux-dérivée étant dans le prédual $X$.

Puisque les Fréchet-dérivées des fonctions $w^*$-s.c.i. sont toujours dans le prédual ([3, Proposition 2.1]), l'intérêt du résultat montré est de remplacer "Fréchet" par "Gâteaux" et de passer de la différentiabilité générique à la différentiabilité en un point. D'autre part, si l'espace $X$ n'est pas de Radon-Nikodym, alors il peut être possible d'avoir une fonction $w^*$-s.c.i. convexe continue sur $X^*$ qui est nulle part Fréchet-différentiable, mais qui a des points de Gâteaux-différentiabilité et des Gâteaux dérivées dans le prédual $X$. En effet, on construit une fonction $f$ $w^*$-s.c.i. convexe continue sur $c_0(\mathbb{N})^* = \ell^1(\mathbb{N})$, nulle part Fréchet différentiable mais Gâteaux-différentiable en tout point d'un ensemble dense dans $\ell^1(\mathbb{N})$, avec dérivées dans le prédual $c_0(\mathbb{N})$.

**Critères de coïncidence de fonctions convexes**

Dans ce paragraphe, on s'intéresse aux questions suivantes :

- Une fonction convexe s.c.i. est-elle déterminée d'une manière unique à partir de ses valeurs sur une partie dense ?
- Si $f$ est une fonction s.c.i., si dom$f$ est une partie convexe et si $\partial f$ est non-vide sur une partie dense de dom$f$, peut-t-on conclure que $f$ est convexe ?

Concernant la première conjecture, on montre dans [12] qu'en dimension infinie la réponse est négative. En particulier, il est même possible d'avoir deux fonctions distinctes $f_1$ et $f_2$ convexes, s.c.i., et positivement homogènes, qui prennent les mêmes valeurs sur une partie dense de $X$ et qui satisfont $f_1 \leq f_2$ en tout point (notons ici que ce même exemple montre que la réponse à la deuxième question est aussi négative).

On étudie ensuite la classe $\mathcal{G}_1(X)$ des fonctions convexes s.c.i. $g$ pour lesquelles il n'existe aucune fonction majorante (différente de $g$) s.c.i. qui coïncide avec $g$ sur une partie dense de dom$g$. On obtient alors le résultat suivant :

Pour toute fonction convexe s.c.i. $g$, de domaine dense dans $X$, on a

$$g \in \mathcal{G}_1(X) \Longleftrightarrow \text{dom} g = X.$$

A partir de la deuxième conjecture, une question bien plus pertinente se pose : si $g$ est une fonction convexe et s.c.i. et si $f$ est une fonction s.c.i. telle que $f^{**} = g$ et que dom$(\partial f)$ est dense dans dom$g$, est-il vrai que $f = g$ ?

Pour aborder cette question, notons $\mathcal{G}_2(X)$ la classe des fonctions $g$ pour lesquelles la conclusion ci-dessus est vraie. Il s'ensuit que $\mathcal{G}_2(X)$ contient strictement la classe $\mathcal{G}_1(X)$. Pour cette nouvelle classe on obtient alors le résultat suivant :

Pour toute fonction convexe s.c.i. et positivement homogène $g$ avec dom$g$ dense dans $X$, on a :

$$g \in \mathcal{G}_2(X) \Longleftrightarrow \text{dom} g = X.$$

**Etude de la convexification d'une fonction s.c.i.**

Il est bien connu ([53], [55] e.g.) qu'une fonction s.c.i. est convexe si (et seulement si) son sous-différentiel de Fenchel-Moreau est un opérateur maximal cycliquement monotone. (Il est clair que l'hypothèse de maximalité est essentielle, puisque le sous-différentiel au sens de Fenchel-Moreau d'une fonction quelconque est toujours cycliquement monotone.)

Dans [8], on démontre qu'en dimension finie, si l'hypothèse de maximalité du sous-différentiel est remplacée par le fait que ses valeurs sont non vides sur un ensemble dense de l'espace, on peut toujours conclure que la fonction est convexe (et partout définie). Notons que ce résultat n'est plus vrai en dimension infinie (voir aussi le paragraphe précédent), sauf si le sous-différentiel possède une sélection localement bornée dans son domaine.

On s'intéresse ensuite au résultat classique de Rockafellar ([53]) qui affirme que tout opérateur maximal cycliquement monotone est en fait le sous-différentiel d'une fonction convexe s.c.i. (unique à une constante près). Pour ce faire, Rockafellar a introduit un processus d'intégration pour les opérateurs multivoques cycliquement monotones. Rappelons ici ce processus :

Etant donné $T : X \rightrightarrows X^*$ et $x \in \text{dom}T$, la fonction $f_T$ définie ci-dessous est propre (lorsque l'opérateur est cycliquement monotone) convexe s.c.i. et elle satisfait $T(x) \subset \partial f(x)$ pour tout $x \in X$ :

$$f_T(x) := \sup \left\{ \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle \ + \ \langle x_n^*, x - x_n \rangle \right\} \tag{9}$$

où le supremum est pris sur tout $n \geq 1$, tous $x_1, x_2, \ldots, x_n$ dans dom$T$ et tous $x_0^* \in T(x_0), x_1^* \in T(x_1), \ldots, x_n^* \in T(x_n)$.

Rappelons maintenant que le sous-différentiel $\partial f$ (au sens de Fenchel et Moreau) d'une fonction $f$ s.c.i. (non nécessairement convexe) est toujours un opérateur cycliquement monotone (mais éventuellement à valeurs vides partout). Cependant, dans le cas où la fonction $f$ (ou une perturbation linéaire d'icelle) a un minimiseur global, la partie dom$(\partial f)$ n'est pas vide et on peut alors considérer la fonction convexe s.c.i. $\hat{f} = f_{\partial f}$, définie par (9) pour $T = \partial f$. Il s'ensuit que $\hat{f}$ est un minorant de $f$, donc elle minore l' enveloppe convexe s.c.i. $f^{**}$ de $f$ ([36, page 218] e.g.).

Une question naturelle alors se pose :

Quand les fonctions $\hat{f}$ et $f^{**}$ sont-elles égales ?

Le résultat classique de Rockafellar ([53]) affirme que cela est le cas, si la fonction $f$ est elle-même convexe et s.c.i. car on aura alors $f = \hat{f} = f^{**}$. D'autre part, un exemple relativement simple dans $\mathbb{R}^2$ (voir [11]) montre qu'une telle conclusion – bien que toujours vraie en dimension 1 – est en général fausse si $f$ n'est pas convexe.

Dans [8], on démontre que $\hat{f} = f^{**}$, pourvu que $f$ soit *1-coercive* (ou *super-coercive*, selon certains auteurs), i.e.

$$\lim_{\|x\|\to\infty} \frac{f(x)}{\|x\|} = +\infty. \tag{10}$$

Ce résultat a été généralisée dans [11] pour la classe des fonctions *epi-pointées*, i.e. des fonctions satisfaisant $\mathrm{intdom} f^* \neq \emptyset$, où $f^*$ désigne la conjuguée de $f$.

## Analyse quasi-convexe

Le thème de recherche dans cette section est l'analyse quasi-convexe, c'est-à-dire la convexité et la monotonie généralisées et leurs applications en optimisation. Dans cette partie on trouve :

- une étude dans le cadre de l'analyse non-lisse, c'est-á-dire faisant appel à des notions de sous-différentiel généralisé,
- une étude géométrique, avec la notion de "normales aux tranches".

**Approche analytique : sous-différentiel généralisé**

La convexité généralisée est étroitement liée à l'économie mathématique. La quasi-convexité, souvent même stricte ou semi-stricte, est une hypothèse standard sur (l'opposé de) la fonction d'utilité dans les modèles micro-économiques. Cette propriété décrit plus ou moins correctement le comportement des consommateurs (voir [1] e.g.). Rappelons ici qu'une fonction $f : X \to \mathbb{R} \cup \{+\infty\}$ est dite quasi-convexe, si pour tout $\lambda \in \mathbb{R}$, la partie $S_f(\lambda) := \{x \in X : f(x) \leq \lambda\}$ est convexe.

Une autre notion importante en économie mathématique est la fonction de demande, qui correspond à la dérivée $f'$ (si elle existe) de la fonction d'utilité $f$. L'opposée $g$ d'une telle fonction est quasi-monotone ([1] e.g.), i.e. elle satisfait la relation suivante, pour tous $x_1, x_2 \in X$ :

$$\langle g(x_1), x_2 - x_1\rangle > 0 \Longrightarrow \langle g(x_2), x_2 - x_1\rangle \geq 0. \tag{11}$$

Notons tout d'abord que cette notion peut être étendue d'une manière naturelle aux applications (i.e. opérateurs) multivoques $T : X \rightrightarrows X^*$ de la façon suivante ([42]) : pour tous $x_1, x_2 \in X$ et tous $x_1^* \in T(x_1)$, $x_2^* \in T(x_2)$ :

$$\langle x_1^*, x_2 - x_1\rangle > 0 \Longrightarrow \langle x_2^*, x_2 - x_1\rangle \geq 0. \tag{12}$$

(Dans [31] une autre manière de définir la quasi-monotonie avait été introduite pour la classe des opérateurs multivoques qui proviennent d'un sous-différentiel. L'équivalence de ces deux définitions a été établie dans [48].)

Une notion plus restrictive est la pseudo-monotonie (au sens de Karamardian [39]) : pour tous $x_1, x_2 \in X$ et tous $x_1^* \in T(x_1)$, $x_2^* \in T(x_2)$ :

$$\langle x_1^*, x_2 - x_1\rangle \geq 0 \Longrightarrow \langle x_2^*, x_2 - x_1\rangle \geq 0. \tag{13}$$

Cette dernière notion a été utilisée dans des problèmes de complémentarité, d'équilibre ou d'inéquations variationnelles ([39], [54], [35] e.g.).

Dans les années 80-90, avec les développements de l'analyse non-lisse, plusieurs notions de sous-différentiel généralisé ont été proposées pour pallier l'absence de différentiabilité dans le cas des fonctions non régulières (c'est-à-dire, non différentiables, voire même non continues). Ensuite, et pendant quelques années, plusieurs auteurs ([42], [48], [47], [45] e.g.) se sont intéressés à établir une correspondance entre la convexité généralisée et la monotonie généralisée dans ce cadre non-lisse. Il a été alors établi que, si $f$ est une fonction s.c.i. et $\partial^* f$ un sous-différentiel abstrait satisfaisant le théorème de la valeur moyenne approchée ([4], [46] e.g.), alors :

$$f \text{ est quasi-convexe} \Longleftrightarrow \partial^* f \text{ est quasi-monotone.} \tag{14}$$

Si l'on suppose aussi que $f$ est continue, alors :

$$f \text{ est pseudo-convexe} \Longleftrightarrow \partial^* f \text{ est pseudo-monotone.} \tag{15}$$

Les deux principaux piliers de l'analyse quasi-convexe, la convexité généralisée et la monotonie généralisée, après avoir suivi un développement indépendant, se sont trouvés désormais étroitement liés par l'analyse non lisse.

*(i) Caractérisation de la stricte (resp. semi-stricte) quasi-convexité*

L'objectif de ce paragraphe est d'élargir la correspondance non lisse mentionnée ci-dessus et de caractériser aussi des sous-classes de fonctions quasi-convexes : concrètement, dans [25], on caractérise les fonctions localement lipschitziennes $f$ qui sont semi-strictement (resp. strictement) quasi-convexes par la semi-stricte (resp. stricte) quasi-monotonie de leur sous-différentiel de Clarke $\partial^o f$ ([16]). On obtient alors le diagramme suivant :

$$\begin{array}{ccc} f \text{ strict. quasi-convexe} & \Longleftrightarrow & \partial^o f \text{ strict. quasi-monotone} \\ \Downarrow & & \Downarrow \\ f \text{ semi-strict. quasi-convexe} & \Longleftrightarrow & \partial^o f \text{ semi-strict. quasi-monotone} \\ \Downarrow & & \Downarrow \\ f \text{ quasi-convexe} & \Longleftrightarrow & \partial^o f \text{ quasi-monotone} \end{array}$$

Après la publication de l'article [25], on s'est aperçu que la définition de la semi-stricte (resp. stricte) quasi-monotonie d'un opérateur multivoque avait antérieurement été introduite par D.T. Luc dans [43]. Dans le même travail on trouve d'ailleurs certaines des implications mentionnées ci-haut, ainsi qu'une caractérisation de la semi-stricte (resp. stricte) quasi-convexité dans le cas de dimension 1. Bien attendu cette référence aurait figurée dans [25] si on en avait eu connaissance.

*(ii) Dualité entre convexité et monotonie généralisées et notion de cyclicité*

Les équivalences exprimées en (14) et (15) tirent leurs racines du résultat classique suivant ([18]) : si $f$ est une fonction s.c.i. et $\partial^* f$ est un sous-différentiel abstrait qui satisfait le théorème de la valeur moyenne approchée, alors :

$$f \text{ est convexe} \Longleftrightarrow \partial^* f \text{ est monotone.} \tag{16}$$

Il s'ensuit que $\partial^* f$ sera égal à $\partial f$, le sous-différentiel de Fenchel-Moreau de l'analyse convexe. De ce fait, on peut également ajouter à (16) une assertion équivalente :

$$\partial^* f \text{ est cycliquement monotone.} \tag{17}$$

Une question naturelle alors se pose :

Existe-il une notion analogue en convexité généralisée ?

L'objectif des travaux [26] et [28] est d'introduire des notions de quasi-monotonie (respectivement, pseudo-monotonie) cyclique et d'établir qu'elles sont vérifiées par tout sous-différentiel $\partial^* f$ d'une fonction $f$ s.c.i. et quasi-convexe (respectivement, continue et pseudo-convexe).

Présentons alors ces définitions (voir (4) pour la définition d'un opérateur cycliquement monotone).

- Un opérateur $T : X \rightrightarrows X^*$ est dit *cycliquement quasi-monotone*, si pour tout $n \geq 1$ et tous $x_1, x_2, \dots, x_n \in X$, il existe $i \in \{1, 2, \dots, n\}$ tel que :

$$\langle x_i^*, x_{i+1} - x_i \rangle \leq 0, \forall x_i^* \in T(x_i) \tag{18}$$

(où $x_{n+1} := x_1$).

- Un opérateur $T : X \rightrightarrows X^*$ est dit *cycliquement pseudo-monotone*, si pour tout $n \geq 1$, tous $x_1, x_2, \dots, x_n \in X$ et tous $x_i^* \in T(x_i)$, $i = 1, 2, \dots, n$ on a :

$$\forall i \in \{1, 2, ..., n-1\},\ \langle x_i^*, x_{i+1} - x_i \rangle \geq 0 \Longrightarrow \langle x_n^*, x_1 - x_n \rangle \leq 0. \tag{19}$$

A partir de ces définitions, il est facile à vérifier les implications suivantes :

$$\begin{array}{ccc} \text{monotonie cyclique} & \Longrightarrow & \text{monotonie} \\ \Downarrow & & \Downarrow \\ \text{pseudo-monotonie cyclique} & \Longrightarrow & \text{pseudo-monotonie} \\ \Downarrow & & \Downarrow \\ \text{quasi-monotonie cyclique} & \Longrightarrow & \text{quasi-monotonie} \end{array}$$

Ensuite, on renforce les caractérisations (14) et (15) comme suit :

- pour toute fonction s.c.i.

$$f \text{ est quasi-convexe} \Longleftrightarrow \partial^* f \text{ est cycliquement quasi-monotone}$$

- pour toute fonction continue :

$$f \text{ est pseudo-convexe} \Longleftrightarrow \partial^* f \text{ est cycliquement pseudo-monotone.}$$

Cependant, il faut souligner que la cyclicité est une propriété étroitement liée aux sous-différentiels : la monotonie seule (et même la forte monotonie), n'entraîne pas forcément la quasi-monotonie cyclique.

La quasi-monotonie et la pseudo-monotonie cycliques sont des notions qui apparaissent également en économie mathématique. En particulier, pour une fonction de demande, la quasi-monotonie cyclique est un prérequis pour la construction de la fonction d'utilité.

*(iii) Sous-différentiel adapté à l'analyse quasi-convexe*

Rappelons d'abord certaines propriétés du sous-différentiel de Fenchel-Moreau (le sous-différentiel de l'analyse convexe) :

- Pour toute fonction $f$, $\partial f$ est cycliquement monotone.
- Si $f$ est continue et $\mathrm{dom} f$ est convexe, alors :

$$f \text{ est convexe } \Longleftrightarrow \mathrm{dom}(\partial f) \text{ est dense dans } \mathrm{dom} f.$$

- Si $f$ est convexe, alors pour tout sous-différentiel abstrait $\partial^*$ satisfaisant le théorème de la valeur moyenne approchée, on a

$$\partial^* f = \partial f. \tag{20}$$

Quel serait alors l'analogue en analyse quasi-convexe ?

Dans [29], on a proposé la notion suivante : à partir d'un sous-différentiel abstrait $\partial^*$ (par exemple, Dini, Clarke-Rockafellar etc) on définit le sous-différentiel "quasi-convexe" $\partial^q f : X \rightrightarrows X^*$ d'une fonction s.c.i. $f$ en $x \in \mathrm{dom} f$ comme suit :

$$\partial^q f(x) = \begin{cases} \partial^* f(x) \cap N_f(x), & \text{si } N_f^<(x) \neq \{0\} \\ \emptyset, & \text{si } N_f^<(x) = \{0\}, \end{cases} \tag{21}$$

où

$$N_f(x) = \{x^* \in X^* : \langle x^*, y - x \rangle \leq 0, \forall y \in S_f(f(x))\} \tag{22}$$

et

$$N_f^<(x) = \{x^* \in X^* : \langle x^*, y - x \rangle \leq 0, \forall y \in S_f^<(f(x))\}. \tag{23}$$

Ici $S_f(f(x))$ (resp. $S_f^<(f(x))$) désigne le sous-niveau (resp. sous-niveau strict) de la fonction $f$.

On montre ensuite que $\partial^q$ possède simultanément les trois propriétés recherchées :

- Pour toute fonction $f$, $\partial^q f$ est cycliquement quasi-monotone.
- Si $f$ est continue et $\mathrm{dom} f$ est convexe, alors :

$$f \text{ est quasi-convexe } \Longleftrightarrow \mathrm{dom}(\partial^q f) \text{ est dense dans } \mathrm{dom} f.$$

- Si $f$ est convexe, alors

$$\partial^q f = \partial f.$$

On établit ensuite des règles de calcul adaptées à l'analyse quasi-convexe.

**Approche géométrique : la normale aux tranches**

La notion de cône normal aux tranches de la fonction (voir (22)) a été utilisée à plusieurs reprises dans l'étude des conditions d'optimalité des fonctions quasi-convexes, et s'est révélée très fructueuse dans les problèmes de minimisation de fonctions quasi-convexes ([13], [20] e.g.).

Cependant, on peut facilement voir que, indifféremment de la fonction choisie, l'opérateur $N_f$ est toujours cycliquement quasi-monotone et par conséquent, il ne peut pas servir pour une caractérisation des fonctions quasi-convexes – rôle toujours considéré comme important pour un sous-différentiel.

Dans [5], on propose une variante $\mathcal{N}_f$ de la notion initiale, que l'on appelle "la normale aux tranches" et qui remédie à cet inconvénient. En effet, les résultats obtenus dans [5] et [6] montrent qu'une classification est alors possible, et que de plus, en adoptant cette démarche on peut obtenir des résultats bien plus pertinents. (On peut - à posteriori - justifier cela en évoquant le caractère géométrique de la normale, qui exploite directement la convexité des tranches d'une fonction quasi-convexe).

La notion de normale proposée consiste à considérer d'abord le cône tangent de la tranche, et de prendre ensuite le cône normal (i.e. le cône polaire) du cône tangent. Cela n'entraîne aucune modification sur $N_f$, si $f$ est une fonction quasi-convexe, ce qui signifie en particulier que tout résultat établi dans les travaux cités reste valable. D'autre part, la nouvelle notion s'avère efficace pour filtrer la convexité généralisée. Elle permet, en fait, une caractérisation géométrique des principales classes de fonctions quasi-convexes continues en terme de monotonie généralisée :

$$\begin{array}{ccc} f \text{ str. quasi-convexe} & \Longleftrightarrow & \mathcal{N}_f \text{ str. quasi-monotone} \\ \Downarrow & & \Downarrow \\ f \text{ semi-str. quasi-convexe} & \Longleftrightarrow & \mathcal{N}_f \text{ semi-str. quasi-monotone} \\ \Downarrow & & \Downarrow \\ f \text{ quasi-convexe} & \Longleftrightarrow & \mathcal{N}_f \text{ quasi-monotone} \end{array}$$

On souligne en particulier que les caractérisations ci-dessus sont d'une nature différente et ne peuvent pas être déduites des résultats évoqués dans les paragraphes précédents. D'ailleurs, elle sont établies pour la classe des fonctions continues (et non pas seulement pour les fonctions localement lipschitziennes).

Dans [6], on s'intéresse aussi au problème d'intégration de la normale, i.e. déterminer, parmi les fonctions d'une certaine classe $\mathcal{C}$, celles qui ont le même opérateur normal.

Notre résultat est le suivant :

Soit $\mathcal{C}$ la classe des fonctions quasi-convexes continues telles que:

(i) tout minimum local est global et

(ii) l'ensemble $\arg\min f$ est toujours inclus dans un hyperplan fermé.

Alors, pour $f, g \in \mathcal{C}$ on a :

$$\mathcal{N}_f = \mathcal{N}_g \Longleftrightarrow f \text{ est } (\mathcal{N}_g \setminus \{0\})\text{-pseudo-convexe.}$$

## Applications aux inéquations variationnelles

Dans cette section, on s'intéresse au problème d'inéquation variationnelle $\mathrm{PIV}(T,K)$ d'un opérateur multivoque $T : X \rightrightarrows X^*$ sur une partie convexe fermée et non-vide $K$ telle que $K \subseteq \mathrm{dom}T$. Cela consiste à trouver un $x \in K$ tel que pour tout $y \in K$, il existe $x^* \in T(x)$

$$\langle x^*, y - x \rangle \geq 0. \tag{24}$$

Ce problème a été introduit par Stampacchia (voir [40]) à partir d'un problème d'E.D.P. Des théorèmes d'existence dans le cas d'un opérateur (univoque) continu, puis monotone et hémi-continu, puis pseudomonotone (au sens de Brézis [15]) ont été établis ([40], [34] e.g.).

Cependant, $\mathrm{PIV}(T,K)$ présente aussi un intérêt en optimisation ; on reconnaît aisément que (24) forme une condition nécessaire d'optimalité lorsque $T$ est le sous-différentiel une fonction-objectif. Il est alors naturel de s'intéresser aux théorèmes d'existence quand $T$ a une propriété de monotonie généralisée.

**Conditions de coercivité optimales**

On avait vu dans la section précédente que la pseudo-monotonie (au sens de Karamardian) caractérise les sous-différentiels des fonctions continues pseudo-convexes. De plus, dans le cas où $T = \partial^* f$, avec $f$ pseudo-convexe, la condition (24) est non seulement nécessaire mais aussi suffisante pour que le problème de minimisation de $f$ ait une solution. Cela justifie l'intérêt d'étudier $\mathrm{PIV}(T,K)$ pour un opérateur $T$ multivoque et pseudomonotone au sens de Karamardian (voir [35], [54], [59], [19]).

Dans [27], on considère ce problème sur des parties non bornées. L'hypothèse classique faite sur l'opérateur, afin d'obtenir des résultats d'existence, est alors une condition de coercivité. Le théorème principal de [27] montre que, pour les opérateurs pseudomonotones et semi-continus supérieurement sur un espace réflexif, trois des conditions de coercivité récemment utilisées dans la littérature sont équivalentes à l'existence d'une solution du problème $\mathrm{PIV}(T,K)$. Dans ce cadre, ces trois conditions, alors équivalentes, sont optimales pour établir l'existence de solutions. Un résultat récent de Crouzeix [19] en dimension finie se trouve ainsi généralisé et complété.

**Quasi-monotonie propre et problème associé**

Une méthode standard pour résoudre le problème (24) est de résoudre d'abord un problème associé ([44]). Ce dernier (noté par PIVA$(T, K)$) consiste à trouver un $x \in K$ tel que, pour tout $y \in K$, et tout $y^* \in T(y)$ on ait :

$$\langle y^*, y - x \rangle \geq 0. \tag{25}$$

En effet, si l'on établit l'existence d'une telle solution, une conséquence immédiate sera l'existence de solutions du PIV$(T, K)$, pourvu que l'opérateur $T$ soit radialement semi-continu supérieurement, hypothèse habituelle dans toutes les applications.

Notre principale contribution dans cette théorie a été la définition de la notion de *quasimonotonie propre* (voir [26], [28]) :

- Un opérateur $T : X \rightrightarrows X^*$ est dit proprement quasi-monotone si, pour tous $x_1, x_2, \ldots, x_n \in X$ et tout $y = \sum_{i=1}^n \lambda_i x_i$, où $\sum_{i=1}^n \lambda_i = 1$ et $\lambda_i > 0$, il existe $i \in \{1, 2, \ldots, n\}$ tel que :

$$\forall x_i^* \in T(x_i) : \langle x_i^*, y - x_i \rangle \leq 0. \tag{26}$$

Cette nouvelle notion se situe entre la pseudo-monotonie (ou la quasi-monotonie cyclique) et la quasi-monotonie, comme le révèle le tableau suivant :

$$\begin{array}{ccc} & & \text{pseudo-monotonie} \\ & & \Downarrow \\ \text{quasi-monotonie cyclique} & \Longrightarrow & \textbf{quasi-monotonie propre} \\ & & \Downarrow \\ & & \text{quasi-monotonie} \end{array}$$

Toutes les implications ci-dessus sont strictes. Cependant, si $T = \partial^* f$ est quasi-monotone, alors $T$ est aussi proprement quasi-monotone (voir [28]).

On a montré dans [26], [28] que, si $T$ est proprement quasimonotone, alors (25) a au moins une solution sur toute partie $K$ faiblement compacte, convexe et non vide (notons qu'aucune hypothèse de continuité n'a été faite sur $T$). Récemment R. John [38] a complété ce résultat en montrant que la réciproque est vraie : si un opérateur $T$ est tel que, pour toute partie compacte, convexe, non vide, (25) a toujours une solution, alors l'opérateur est proprement quasimonotone :

$$\begin{array}{c} T \text{ est proprement} \\ \text{quasi-monotone} \end{array} \iff \begin{array}{c} \text{PIVA}(T, K) \text{ a des solutions,} \\ \forall K \neq \emptyset, \text{ convexe, } w\text{-compacte.} \end{array}$$

La quasi-monotonie propre est donc une hypothèse minimale sur l'opérateur pour assurer l'existence de solutions du problème associé. Dans de nombreuses applications des inéquations variationnelles (à la mécanique, physique etc), l'opérateur est supposé maximal monotone (en particulier égal au sous-différentiel d'une fonction convexe). En fait, d'après nos résultats, l'existence des solutions du problème associé est assurée lorsque l'opérateur est proprement quasimonotone (ce qui est le cas pour les sous-différentiels des fonctions quasiconvexes).

**Inéquations variationnelles vectorielles**

Dans [24], on s'intéresse aux inéquations variationnelles vectorielles qui sont notamment liés à l'optimisation multicritère. On obtient des théorèmes d'existence en dimension infinie. On donne aussi un contre-exemple qui révèle une erreur dans les résultats principaux de trois articles publiés entre 1990 et 1994. La preuve du théorème principal est basée sur l'existence de "points internes" (inner points) qui est une hypothèse faible (en particulier satisfaite pour un espace séparable).

## Intégration d'opérateurs multivoques

Intégrer un opérateur $T : X \to X^*$, c'est-à-dire trouver une fonction dérivable $f$ telle que $T = f'$, a attiré beaucoup d'attention. Lorsque l'opérateur $T$ est multivoque, cette question devient : trouver une fonction $f$ telle que $T \subseteq \partial^* f$, pour une certaine notion de sous-différentiel $\partial^*$ ([57] e.g.). Ce problème a été entièrement résolu par Rockafellar ([53]) dans le cas où la fonction $f$ est convexe et $\partial^* = \partial$ est le sous-différentiel de Fenchel-Moreau de l'analyse convexe. Plus précisement, il a été établi que la monotonie cyclique maximale est une condition nécessaire et suffisante pour qu'un opérateur $T : X \rightrightarrows X^*$ puisse s'écrire sous la forme $T = \partial f$.

Notons que la méthode développée dans [53] fait appel au caractère global du sous-différentiel $\partial$ - voir aussi sa définition dans (3). Le grand avantage et une particularité caractéristique de l'analyse convexe est le fait qu'une telle approche reste toute de même équivalente à des approches locales (voir aussi (20) par exemple). Notons ainsi que dans [14, Corollary 2.1], on trouve une autre méthode pour retrouver le résultat de Rockafellar.

Dans ce document, les travaux effectués sur ce sujet se répartissent en deux directions :

- une méthode à caractère global, proche de l'analyse quasi-convexe
- une méthode analytique, dans l'esprit de l'analyse non-lisse.

**Intégration via le sous-différentiel inférieur**

Dans [8] on a travaillé avec le sous-différentiel inférieur $\partial^<$ introduit par F. Plastria dans [51], qui est une adaptation au cas quasi-convexe du sous-différentiel de Fenchel-Moreau. On a montré qu'une fonction quasi-convexe lipschitzienne $f$ est caractérisée par l'existence d'une sélection bornée pour son sous-différentiel inférieur $\partial^< f$ dans un domaine dense.

Par ailleurs, on a considéré une classe d'opérateurs (notée $R(x_0)$ où $x_0$ est un point fixé) qui est strictement plus grande que la classe d'opérateurs cycliquement monotones. Si un opérateur $T$ satisfait $R(x)$ pour tout $x \in \mathrm{dom} T$, alors $T$ est monotone. D'autre part, on a montré qu'un opérateur $T$ satisfait $(R(x_0))$ en un point $x_0 \in \mathrm{dom} T$ si, et seulement si, il existe une fonction quasi-convexe $h_T$ (avec $\partial h_T(x_0) \neq \emptyset$) telle que $T(x_0) \subseteq \partial h_T(x_0)$ et $T(x) \subseteq \partial^< h_T(x)$, pour tout $x \in X$.

**Sous-monotonie cyclique et intégration**

La propriété de sous-monotonie a été introduite par Spingarn [56] pour caractériser le sous-différentiel de Clarke d'une fonction "sous-$C^1$". On rappelle qu'une fonction $f : U \to \mathbb{R}$ est dite sous-$C^1$ (où $U$ est une partie ouverte de $\mathbb{R}^n$), s'il existe une partie compacte $S$ et une fonction continue $F : U \times S \to \mathbb{R}$, telle que la dérivée $\nabla_x F$ (existe et) est continue sur $U \times S$ et que pour tout $x \in U$ on ait :

$$f(x) = \max_{s \in S} F(x, s). \tag{27}$$

Cette classe contient à la fois les fonctions convexes continues et les fonctions continûment dérivables. Ensuite Janin ([37]) a montré qu'en dimension finie, le sous-différentiel de Clarke $\partial^o f$ d'une fonction $f$ définie par (27) a une propriété bien plus forte, la sous-monotonie cyclique maximale, et que réciproquement, pour un tel opérateur $T$ il existe toujours une fonction unique (à une constante près) sous-$C^1$ $f$ telle que $T = \partial^o f$.

Dans [23], on généralise le résultat de Janin [37] en dimension infinie. Cela nous conduit à étudier la classe des fonctions sous-lisses (subsmooth), i.e. les fonctions localement lipchitziennes dont le sous-différentiel de Clarke est cycliquement sous-monotone. Pour de telles fonctions, le sous-différentiel de Clarke coïncide avec celui de Hadamard (donc la fonction est régulière) et, par conséquent, est génériquement un singleton. D'après des résultats récents de Borwein et Wang sur la taille du sous-différentiel de Clarke (voir [58] e.g.), on conclut aisément que l'ensemble des fonctions sous-lisses est maigre dans l'espace des fonctions localement lipchitziennes. Cependant cette classe contient toute fonction convexe continue et toute fonction continûment différentiable. Le résultat classique de Rockafellar pour les opérateurs maximaux cycliquement monotones et les fonctions convexes s.c.i. trouve son analogue pour les fonctions localement lipschitziennes.

## Optimisation multi-critère

Cette section comporte :

- une étude sur la connexité de l'ensemble des points Pareto-optimaux d'un problème multi-critère (de dimension finie),
- une étude sur les théorèmes de densité en optimisation vectorielle.

**Etude de la connexité du problème de maximisation de trois critères semi-strictement quasi-concaves.**

Dans [30], on démontre que l'ensemble des points optimaux (au sens de Pareto) pour trois fonctions objectif continues et semi-strictement quasiconvexes sur un convexe compact de l'espace euclidien, est connexe. Ce résultat répond par l'affirmative à une conjecture émise en 1985 par Schaible, Choo et Chew. Notons que la conjecture générale concernant $n \geq 1$ fonctions objectif a été récemment établie par Benoist ([9]).

**Théorie de densité de Arrow-Barankin-Blackwell**

Un théorème de Arrow, Barankin et Blackwell [2], relativement connu, assure la densité des points scalairement maximaux dans l'ensemble des points maximaux d'une partie convexe et compacte de $\mathbb{R}^n$. Ce théorème a une interprétation économique importante en termes de “panier de biens” et de “prix optimal”. Il a été généralisé à plusieurs reprises en dimension infinie pour des cônes convexes, fermés et pointés. Parmi les versions les plus significatives, on peut noter celle de Petschke [49] (donnée aussi indépendamment par Gallagher et Saleh [32]). Cinq ans plus tard, Gong [33] a proposé une amélioration de ce résultat, en affaiblissant l'hypothèse que le cône possède une base bornée. Cependant, aucun exemple ne montrait le caractère plus général de ce résultat. Dans [21], on établit une caractérisation des points “denting” pour des parties convexes et fermées d'un espace de Banach, qui généralise la caractérisation de Lin, Lin et Troyanski [41] établie sous l'hypothèse supplémentaire que ces parties soient bornées. Cette nouvelle caractérisation permet de démontrer que les hypothèses des théorèmes de Petschke et de Gong sont, en fait, équivalentes. Notre technique a aussi l'avantage de raccourcir les preuves originales de [49], [32] et [33].

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# III. Présentation des travaux

# PARTIE I

# Analyse convexe

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**Article [1]**

"Subdifferentials of convex functions and sigma-cyclic monotonicity"
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A. Daniilidis

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# SUBDIFFERENTIALS OF CONVEX FUNCTIONS AND SIGMA-CYCLIC MONOTONICITY

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The property of $\sigma$-cyclic monotonicity is proposed here to describe subdifferentials of lsc convex functions that are continuous in their domains. It is shown that all monotone operators in $R$ and all densely defined cyclically monotone operators in $R^n$ share this property. Examples of a densely defined maximal cyclically monotone operator in a Hilbert space and of a subdifferential of a convex lsc function in $R^2$ which are not $\sigma$-cyclically monotone operators are given.

## 1. INTRODUCTION AND PREREQUISITES.

In [**5**], Rockafellar introduced the class of cyclically monotone operators in order to describe the subdifferentials of convex lower semicontinuous (lsc) functions. His main result states that an operator $T : X \to 2^{X^*}$ is included in the subdifferential $\partial f$ of a lsc convex function $f$ if, and only if, it is cyclically monotone. In particular, $T$ coincides with $\partial f$ if, and only if, it is maximal cyclically monotone (with respect to the inclusion of graphs).

In this article we refine the notion of cyclic monotonicity by introducing an infinite variant of it that we call $\sigma$-cyclic monotonicity. We show that this property is always satisfied by the subdifferentials of convex lsc functions that are continuous in their domains, and hence by the subdifferentials of all convex lsc functions in $R$. A simple example shows that subdifferentials of discontinuous convex lsc functions in $R^2$ may fail to be $\sigma$-cyclically monotone. However, in finite dimensions, cyclic and $\sigma$-cyclic monotonicity coincide for the class of the densely defined operators. This is no longer true in infinite dimensions. In particular, there exists a convex lsc function $f$ in a separable Hilbert space, for which $\partial f$ is a densely defined maximal cyclically monotone operator, without being $\sigma$-cyclically monotone.

Before we proceed we fix our notation: $X$ will be a Banach space with dual space $X^*$. In particular, we denote by $\ell^2(N)$ the Hilbert space of square summable sequences,

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by $\ell^1(N)$ the Banach space of absolutely summable sequences and by $\ell^\infty(N)$ the Banach space of bounded sequences. We also denote by $c_{00}(N)$ the space of eventually null sequences. For any $x \in X$ and $x^* \in X^*$ we denote by $x^*(x)$ the value of $x^*$ at $x$. For $x \in X$ and $\varepsilon > 0$ we denote by $B_\varepsilon(x)$ the closed ball centred at $x$ with radius $\varepsilon > 0$. Finally for $x, y \in X$ we denote by $[x, y]$ the closed segment $\{tx + (1-t)y : t \in [0,1]\}$. The segments $(x, y]$, $[x, y)$ and $(x, y)$ are defined analogously. Throughout this article we always deal with multivalued operators $T$ defined on $X$ and taking values into $2^{X^*}$ (the space of subsets of $X^*$), and with proper convex lsc functions $f : X \to R \cup \{+\infty\}$ which are not identically equal to $\{+\infty\}$. We set $\mathrm{dom}(f) := \{x \in X : f(x) \in R\}$ for the domain of $f$ and $\mathrm{dom}(T) := \{x \in X : T(x) \neq \emptyset\}$ for the domain of the multivalued operator $T$.

A function $f : X \to R \cup \{+\infty\}$ is said to be *continuous* (respectively *lower semicontinuous*), if it is continuous (respectively lower semicontinuous) at every point $x \in X$, where $R \cup \{+\infty\}$ is equipped with the topology generated by the family $\Im_R \cup \{(x, +\infty],\ x \in R\}$ ($\Im_R$ being the usual topology of $R$). Note that such functions may take infinite values, as for instance the function $f : R \to R \cup \{+\infty\}$ with $f(x) = 1/x$ if $x > 0$ and $+\infty$ if $x \leqslant 0$. On the other hand, we say that a function $f$ is *continuous on a subset $S$ of its domain*, if its restriction to $S$ is a continuous (real-valued) function, see also [**4**, p.82]. The class of convex lsc functions which are continuous in their domain is much larger than the one of convex continuous functions. In particular, it contains the indicator functions of closed convex sets, as well as many other non-continuous functions, see [**3**, Example 3.8 (a)].

We recall that the subdifferential $\partial f$ of the function $f$ at the point $x_0 \in \mathrm{dom}(f)$ is given by the formula below:

$$\partial f(x_0) = \{x^* \in X^* : f(x) - f(x_0) \geqslant x^*(x - x_0),\ \forall x \in X\} \tag{1}$$

For a lsc convex function $f$, the above subdifferential coincides with the Clarke-Rockafellar subdifferential $\partial^\uparrow f(x_0)$ which was defined in [**6**] (see also [**2**]) as follows:

$$\partial^\uparrow f(x_0) = \{x^* \in X : x^*(d) \leqslant f^\uparrow(x_0, d),\ \forall d \in X\}. \tag{2}$$

where $f^\uparrow(x_0, d)$ is the generalised derivative of $f$ at $x_0$ in the direction $d \in X$:

$$f^\uparrow(x_0, d) = \sup_{\varepsilon > 0} \limsup_{\substack{x \to_f x_0 \\ t \searrow 0^+}} \inf_{d' \in B_\varepsilon(d)} \frac{f(x + td') - f(x)}{t} \tag{3}$$

where $t \searrow 0^+$ indicates the fact that $t > 0$ and $t \to 0$, and $x \to_f x_0$ means that both $x \to x_0$ and $f(x) \to f(x_0)$.

## 2. Main results

Let $T : X \to 2^{X^*}$ be a multivalued operator. We recall from [**3**] the following definition:

DEFINITION 1: $T$ is called cyclically monotone, if for any $n \in N$, for every $x_0, x_1, \ldots, x_n$ in $X$ and every $x_0^* \in T(x_0)$, $x_1^* \in T(x_1), \ldots,$ $x_n^* \in T(x_n)$ one has

$$\sum_{i=0}^{n} x_i^*(x_{i+1} - x_i) \leqslant 0 \tag{4}$$

where $x_{n+1} := x_0$.

We recall from [5] the following fundamental theorem.

**THEOREM 2.** *$T$ is cyclically monotone if, and only if, $T \subseteq \partial f$ for some convex lsc function $f$.*

As shown in [5], if we assume that $T$ is maximal cyclically monotone, then the above inclusion becomes an equality and the function $f$ turns out to be unique.

Let us now introduce the notion of $\sigma$-cyclic monotonicity.

DEFINITION 3: $T$ is called $\sigma$-cyclically monotone, if for every sequence $(x_n)_{n=0}^{+\infty}$ in $X$ such that $\lim\limits_{n\to\infty} x_n = x_0$ and for every $x_i^* \in T(x_i)$ (for $i = 0, 1, 2, \ldots$) one has

$$\limsup_{n\to\infty} \sum_{i=0}^{n} x_i^*(x_{i+1} - x_i) \leqslant 0. \tag{5}$$

It is easily seen that every $\sigma$-cyclically monotone operator is cyclically monotone. Indeed, given a multivalued operator $T$ and a finite sequence $\{x_0, x_1, \ldots, x_n\}$ in $\mathrm{dom}(T)$, one can define an infinite sequence $(x_k)_{k=0}^{+\infty}$ in $\mathrm{dom}(T)$, by setting $x_k = x_0$, for $k \geqslant n+1$. Then relation (5) clearly yields (4).

The following proposition shows that the class of $\sigma$-cyclically monotone operators contains the subdifferentials of functions which are continuous on their domain, thus in particular these of convex continuous functions. (Note also that the proof below does not require the convexity of $f$.)

**PROPOSITION 4.** *Suppose that $f : X \longrightarrow R \cup \{+\infty\}$ is continuous on $\mathrm{dom}(\partial f)$ and $T \subseteq \partial f$. Then $T$ is $\sigma$-cyclically monotone.*

PROOF: Consider any sequence $(x_n)_{n=0}^{+\infty}$ in $\mathrm{dom}(T)$ such that $\lim\limits_{n\to\infty} x_n = x_0$. For any $x_n^* \in T(x_n)$, the inclusion $T(x_n) \subseteq \partial f(x_n)$ together with (1) implies:

$$f(x_{n+1}) \geqslant f(x_n) + x_n^*(x_{n+1} - x_n) \tag{6}$$

for $n = 0, 1, 2, \ldots$. Adding the above inequalities we get

$$\sum_{i=0}^{n} x_i^*(x_{i+1} - x_i) \leqslant f(x_{n+1}) - f(x_0).$$

Taking the limit as $n \to +\infty$, and using the continuity of $f$ on $\mathrm{dom}(\partial f)$, we conclude that $\limsup\limits_{n\to\infty} \sum\limits_{i=0}^{n} x_i^*(x_{i+1} - x_i) \leqslant 0$. ∎

One deduces that the subdifferentials of indicator functions of convex sets are $\sigma$-cyclically monotone operators. The same is true for the subdifferentials of all lsc convex functions of one variable. More generally we show the following:

**COROLLARY 5.** *Let $T : R \to 2^R$ be a monotone operator (that is, $T$ satisfies (4) for $n = 1$). Then $T$ is $\sigma$-cyclically monotone.*

PROOF: It is known that every monotone operator in $R$ is cyclically monotone (see [**5**]), hence Theorem 2 ensures the existence of a lsc convex function $f : R \to R \cup \{+\infty\}$ such that $T \subseteq \partial f$. We shall show that $f$ is continuous on its domain (which is a segment of $R$), hence the result will follow from Proposition 4. Since $f$ is continuous at every interior point of its domain, it suffices to consider only the (eventual) case of a point $x_0$ in dom$(f)$ such that $x_0 \leqslant x$ for all $x \in$ dom$(f)$. (The case $x_0 \geqslant x$ for all $x \in$ dom$(f)$ can be treated analogously.) Suppose for simplicity that $x_0 = 0$ and $f(x_0) = 0$ and consider a sequence $(x_n)_n$ in dom$(f)$ such that $\lim\limits_{n\to+\infty} x_n = 0$. Since $f$ is lsc, one gets $\liminf\limits_{n\to+\infty} f(x_n) \geqslant 0$. Suppose now that there exists $x$ in dom$(f)$ with $f(x) > 0$. From convexity of $f$ it follows that $f(tx) \leqslant tf(x)$, for all $t \in [0,1]$, hence in particular $\lim\limits_{t\searrow 0^+} f(tx) = 0$. It follows easily that $\limsup\limits_{n\to+\infty} f(x_n) \leqslant 0$. ∎

The above result fails in $R^2$ even if $T$ is maximal cyclically monotone. This is shown by the following example:

EXAMPLE. Consider the function $f : R^2 \to R \cup \{+\infty\}$ given by the formula

$$f(x_1, x_2) = \begin{cases} x_2^2/x_1 & \text{if } x_1 > 0 \\ 0 & \text{if } x_1 = x_2 = 0 \\ +\infty & \text{elsewhere} \end{cases}$$

In [4, p.83] it has been shown that $f$ is convex and lower semicontinuous. This function is a classical example of a non asymptotically well behaved function and has been used several times in the past. We show that the operator $T = \partial f$ is not $\sigma$-cyclically monotone. Note first that $f$ is differentiable at every non-zero point of its domain, with gradient $\nabla f(x_1, x_2) = \big(-(x_2/x_1)^2, 2(x_2/x_1)\big)$. Set now $x_n = \big((1/(n+1))^3, 1/(n+1)\big)$ and $x_n^* \in \partial f(x_n) = \{\nabla f(x_n)\}$. It follows that $\lim\limits_{n\to+\infty} x_n = 0$ and that $x_n^*(x_{n+1} - x_n) > 0$, for $n \geqslant 1$. Since $f$ attains its minimum at $x_0 = 0$, it follows that $0 \in \partial f(x_0)$. Setting $x_0^* = 0$ we obtain:

$$\limsup_{n\to\infty} \left\{ x_0^*(x_1 - x_0) + \sum_{i=1}^{n} x_i^*(x_{i+1} - x_i) \right\} > 0$$

which shows that $T$ is not $\sigma$-cyclically monotone.

One may observe that in the previous example the domain of $T$ is included in a half-space. This condition is in fact indispensable for such examples in finite dimensional spaces. This is shown by the following proposition.

**PROPOSITION 6.** *Let $T : R^n \to 2^{R^n}$ be a densely defined cyclically monotone operator. Then $T$ is $\sigma$-cyclically monotone.*

PROOF: Since $T$ is cyclically monotone, we conclude from Theorem 2 that for some convex lsc function $f$ we have $T \subseteq \partial f$. Since $\mathrm{dom}(T)$ is dense in $R^n$ and $\mathrm{dom}(T) \subseteq \mathrm{dom}(\partial f) \subseteq \mathrm{dom}(f)$, it follows that $\mathrm{dom}(f)$ is also dense. Since the latter set is convex, we infer that $\mathrm{dom}(f) = R^n$, hence $f$ is continuous. The result now follows from Proposition 4. □

The statement of Proposition 6 is no longer true in infinite dimensional spaces, even if one assumes $T$ to be maximal (that is, equal to the subdifferential of a lsc convex function $f$). We shall use the following result, which is a version of the approximate mean value theorem for the Clarke-Rockafellar subdifferential, see [**1**, Corollary 4.3]:

**PROPOSITION 7.** *Let $f : X \to R \cup \{+\infty\}$ be a lsc function and $a, b \in X$ be such that $f(a) < f(b)$. Then there exist $c \in [a,b]$, $c \neq b$, and sequences $(x_n)_{n\in N}$ in $X$ and $(x_n^*)_{n\in N}$ in $X^*$ with $x_n^* \in \partial^\uparrow f(x_n)$, $x_n \to c$, $f(x_n) \to f(c)$ such that:*

$$x_n^*(x - x_n) > 0$$

*for all $n \in N$ and every $x = c + t(b-a)$, with $t > 0$.*

**PROPOSITION 8.** *There exists a lsc convex function $f$ on $\ell^2(N)$ such that its subdifferential $\partial f$ is densely defined without being $\sigma$-cyclically monotone.*

PROOF: Let $X = \ell^2(N)$. We consider the function $f : X \to R \cup \{+\infty\}$ defined for every $x = (x_i)_{i=1}^{+\infty} \in X$ as follows:

$$f(x) = \|x\|_1 := \sum_{i=1}^{+\infty} | x_i | . \tag{7}$$

Since $f$ is the pointwise supremum of the convex continuous functions $\varphi_n : X \to R$ given by $\varphi_n(x) = \sum_{i=1}^{n} | x_i |$, it follows that $f$ is lower semicontinuous and convex. From [**5**] (or [**3**]) we have that $\partial f : X \to 2^{X^*}$ is maximal cyclically monotone. We shall show that $\partial f$ is not $\sigma$-cyclically monotone.

Let us first observe that $\mathrm{dom}(f) = \ell^1(N)$. On the other hand one can easily see that $f$ is nowhere continuous. Using the inclusions $\ell^1(N) \subset \ell^2(N) \subset \ell^\infty(N) := [\ell^1(N)]^*$, the relation $[\ell^2(N)]^* = \ell^2(N)$ and the fact that $f(y) = +\infty$ for all $y \in \ell^2(N) \setminus \ell^1(N)$, it follows from (1) that for every $x \in \mathrm{dom}(f)$:

$$\partial f(x) = \partial(\|\cdot\|_1)(x) \cap \ell^2(N) \tag{8}$$

where $\partial(\|\cdot\|_1)(x)$ denotes the usual subdifferential of the norm $\|\cdot\|_1$ in $\ell^1(N)$, as a subset of $\ell^\infty(N)$.

Relation (8) shows that $\mathrm{dom}(\partial f)$ is dense in $X$, since it contains for instance the space $c_{00}(N)$. Setting now $x = 0$ in (8) we obtain in particular that:

$$\partial f(0) = \{y = (y_n)_n \in \ell^2(N) : |\, y_n \,| \leqslant 1\} \tag{9}$$

We shall need the following claim.

CLAIM. There exists a sequence $(y_n)_{n=1}^{+\infty}$ in $\mathrm{dom}(f)$ such that for all $n \geqslant 1$:

(i) $y_n \in B_{1/n}(0)$

(ii) $f(y_n) < f(y_{n+1})$

(iii) the restriction of $f$ to the segment $[y_n, y_{n+1}]$ is strictly increasing.

PROOF: We first note that for every $\varepsilon > 0$, the function $f$ takes arbitrarily large finite values in the ball $B_\varepsilon(x_0)$.

Set now $k_1 = 1$ and $y_1 = e_1 := (1, 0, 0, \ldots)$. For $\alpha_2 \in R$ and $k_2 \geqslant k_1 + 1$ we consider:

$$y_2 = \sum_{i=k_1+1}^{k_2} a_2 e_i := (0, a_2, a_2, \ldots, a_2, 0, 0, \ldots) \tag{10}$$

Imposing $\|y_2\| = 1/2$ and $f(y_2) = 2$ we conclude that $a_2 = 1/8$ and $k_2 - 1 = 16$. One can also check directly that condition (iii) is also satisfied for the segment $[y_1, y_2]$.

We shall now use induction to construct a sequence $(y_n)_{n=1}^{+\infty}$ in $c_{00}(N)$, such that for every $n \geqslant 1$, $\|y_n\| = 1/n$, $f(y_n) = n$ and the restriction of the function $f$ to the segment $[y_{n-1}, y_n]$ is strictly increasing. To this end, suppose that for $n \geqq 1$ we have defined $y_n \in c_{00}(N)$, $k_n \in N$ and $\alpha_n \in R$ such that

$$y_n = \sum_{i=k_{n-1}+1}^{k_n} \alpha_n e_i := (0, 0, \ldots, 0, \alpha_n, \alpha_n, \ldots, \alpha_n, 0, 0, \ldots) \tag{11}$$

where $(k_n - k_{n-1})\alpha_n = n$ and $(k_n - k_{n-1})\alpha_n^2 = 1/n^2$. We then define (for $n+1$)

$$y_{n+1} = \sum_{i=k_n+1}^{k_{n+1}} \alpha_{n+1} e_i := (0, 0, \ldots, 0, 0, \ldots, 0, \alpha_{n+1}, \alpha_{n+1}, \ldots, \alpha_{n+1}, 0, 0, \ldots) \tag{12}$$

where $k_{n+1}$ and $\alpha_{n+1}$ are such that $(k_{n+1} - k_n)\alpha_{n+1} = n + 1$ and $(k_{n+1} - k_n)\alpha_{n+1}^2 = 1/(n+1)^2$. It now suffices to check that the function

$$t \to \rho(t) = f\big(y_n + t(y_{n+1} - y_n)\big)$$

is increasing, for $t \in [0, 1]$. Indeed, using (7), (11) and (12), a direct computation gives:

$$\rho(t) = (1 - t)(k_n - k_{n-1})\alpha_n + t(k_{n+1} - k_n)\alpha_{n+1}.$$

It follows that $\rho'(t) = (k_{n+1} - k_n)\alpha_{n+1} - (k_n - k_{n-1})\alpha_n = (n + 1) - n = 1 > 0$. The proof of the claim is complete. ∎

We now construct a sequence $(x_n)_{n=0}^{+\infty} \subset \mathrm{dom}(\partial f)$ such that:

(a) $\|x_n - y_n\| \leqslant \|y_n\|$, $n = 1, 2, \dots$.

(b) $x_n^*(x_{n+1} - x_n) > 0$, for all $n \in N$.

Let $x_0 = 0$. Using (9) we can find some $x_0^* \in \partial f(0)$ such that $x_0^*(y_1 - x_0) > 0$. Consider now the open set $U_0 = \{x' \in X : x_0^*(x' - x_0) > 0\}$. Take any $z_2 \in U_0 \cap [y_1, y_2] \cap B_{\|y_1\|}(y_1)$. By the condition (iii) of the above claim we have $f(z_2) > f(y_1)$. Since the function $f$ is convex and lower semicontinuous, its convex subdifferential $\partial f$ coincides with the Clarke-Rockafellar subdifferential $\partial^\dagger f$. Hence, since $y_2 = y_1 + \lambda(z_2 - y_1)$ for some $\lambda > 1$, applying Proposition 7 (for $a = y_1$ and $b = z_2$) we conclude the existence of some $x_1 \in B_{\|y_1\|}(y_1) \cap U_0$ and some $x_1^* \in \partial f(x_1)$ such that $x_1^*(y_2 - x_1) > 0$. Note also that since $x_1 \in U_0$ we have $x_0^*(x_1 - x_0) > 0$.

We further set $U_1 = \{x' \in X : x_1^*(x' - x_1) > 0\}$. Since $U_1 \cap [y_2, y_3] \cap B_{\|y_2\|}(y_2)$ is nonempty, we may choose $z_3 \in U_1 \cap [y_2, y_3] \cap B_{\|y_2\|}(y_2)$. Condition (iii) again guarantees that $f(z_3) > f(y_2)$. Repeating the above arguments one inductively constructs a sequence $(x_n)_{n=0}^{+\infty}$ having properties (a) and (b).

Since $\|y_n\| \to 0$, it follows from (a) that $\lim_{n\to\infty} x_n = 0 := x_0$. It follows now directly from (b) that $\partial f$ is not $\sigma$-cyclically monotone. ∎

The above counterexample shows that, in infinite dimensional spaces, maximal cyclically monotone operators (having even a dense domain) may fail to be $\sigma$-cyclically monotone. However this can not happen if the operator $T$ is locally bounded on its domain. More generally we say that an operator $T$ has a *locally bounded selection on a subset $D$ of its domain*, if for every $x_0 \in X$ there exists $M > 0$ and $\delta > 0$ such that:

(13) $$\forall z \in D \cap B_\delta(x_0),\ \exists z^* \in T(z) :\ \|z^*\| \leqslant M$$

We are now ready to state the following result for infinite dimensional spaces.

**PROPOSITION 9.** *Let $T$ be a densely defined cyclically monotone operator that admits a locally bounded selection on its domain. Then $T$ is $\sigma$-cyclically monotone.*

PROOF: Using Theorem 2 we conclude that $T \subseteq \partial f$, for some convex lsc function $f$. It follows that $\partial f$ has a locally bounded selection on dom$(T)$ ($\subseteq$ dom $\partial f$). From Proposition 4 it suffices to show that $f$ is continuous. To this end, consider any $x_0 \in X$ and choose $M > 0$ and $\delta > 0$ so that (13) is satisfied. Since dom$(T)$ is dense on $X$, there exists a sequence $(x_n)_{n\in N}$ in dom$(\partial f) \cap B_\delta(x_0)$ such that $\lim_{n\to\infty} x_n = x_0$ and $x_n^* \in \partial f(x_n)$, with $\|x_n^*\| \leqslant M$. Take now any $x \in B_\delta(x_0) \cap \mathrm{dom}(f) \neq \emptyset$. It follows from (1) that $f(x_n) \leqslant f(x) + x_n^*(x_n - x)$. Since $f$ is lower semicontinuous, taking the limit as $n \to +\infty$ we conclude

$$f(x_0) \leqslant f(x) + M\|x_0 - x\|$$

It follows that $x_0 \in \mathrm{dom}(f)$, hence dom$(f) = X$ and $f$ is continuous. ∎

Final conclusion. Most arguments used in the proof of Proposition 8 are also valid for subdifferentials of lsc convex functions $f$ for which the restriction $f_{|\mathrm{dom}(\partial f)}$ has a discontinuity at some $x_0 \in \mathrm{dom}(\partial f)$. Consequently one expects that a subdifferential $\partial f$ of a lsc convex function $f$ is $\sigma$-cyclically monotone (if and) only if $f_{|\mathrm{dom}(\partial f)}$ is continuous. This conjecture would be positively answered if for example one could establish the validity of the claim in the proof of Proposition 8 for the general case.

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# DUAL CHARACTERIZATIONS OF RELATIVE CONTINUITY OF CONVEX FUNCTIONS

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**Abstract**

Various properties of continuity for the class of lower semicontinuous convex functions are considered and dual characterizations are established. In particular, it is shown that the restriction of a lower semicontinuous convex function to its domain (respectively, domain of subdifferentiability) is continuous if and only if its subdifferential is strongly cyclically monotone (respectively, $\sigma$-cyclically monotone).

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## 1. Introduction

Let $X$ be a Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous (in short lsc) function. A recent result of Correa, Jofre and Thibault [3] asserts that $f$ is convex if and only if its Clarke-Rockafellar subdifferential $\partial f$ is monotone. The same equivalence has also been established for abstract notions of subdifferentials (see [1], for example). In the aforementioned cases, since any notion of subdifferential of a convex function coincides with the classical Fenchel-Moreau subdifferential, it follows that $\partial f$ is not only monotone, but also cyclically monotone (see [6], for example). This latter property (that is, cyclicity) is not just a stronger property than mere monotonicity, but it expresses a behaviour of certain type. This behaviour has already been discussed in relation with integration problems ([2], for example) as well as in generalized convexity [5].

While cyclic monotonicity describes the behaviour of an operator around a 'cycle' of finite points, a variant of it—called $\sigma$-cyclic monotonicity—was introduced and studied in [4]. Compared with cyclic monotonicity, this new property carries additional information on the operator, since it describes its behaviour along infinite cycles formed by converging sequences. In particular, the fact that the subdifferential $\partial f$ of a function $f$ is $\sigma$-cyclically monotone guarantees a certain continuity property for the lsc convex function $f$.

In this article we show that the restriction of a lsc convex function to its domain of subdifferentiability is continuous if and only if its subdifferential is $\sigma$-cyclically monotone; (this result was conjectured in [4]). We also introduce the strong cyclic monotonicity, and show that this property characterizes the subdifferentials of the lsc convex functions having a continuous restriction to their domain, see Section 4.

The paper is organized as follows. In Section 2 we give some preliminary results and we fix our notation. In Section 3 we prove a local version of Rockafellar's formula [8] concerning the representation of the lsc convex functions. This local representation—apart from its independent interest—will be in use in Section 4, where we establish dual characterizations of the continuity of the functions $f|_{\mathrm{dom} f}$ and $f|_{\mathrm{dom}\,\partial f}$ for a lsc convex function $f$. Finally, in the same section, we give some criteria for an operator to be strongly cyclically monotone (respectively, $\sigma$-cyclically monotone) and we classify the various concepts of cyclic monotonicity.

## 2. Preliminaries

In the sequel, $X$ will denote a Banach space and $X^*$ its dual. For any $x \in X$ and $x^* \in X^*$ we denote by $\langle x^*, x\rangle$ the value of $x^*$ at $x$. For $x \in X$ and $\varepsilon > 0$ we denote by $B(x, \varepsilon)$ the closed ball centered at $x$ with radius $\varepsilon > 0$, while for $x, y \in X$ we denote by $[x, y]$ the closed segment $\{tx + (1-t)y : t \in [0, 1]\}$. For any closed segment $[x, y]$ in $X$ and any $\varepsilon > 0$ we denote by $B([x, y], \varepsilon)$ the $\varepsilon$-neighbourhood of the segment $[x, y]$, that is,

$$B([x, y], \varepsilon) := \{w \in X : \exists z \in [x, y] \text{ with } \|z - w\| \le \varepsilon\}.$$

Given a function $f : X \to \mathbb{R} \cup \{+\infty\}$, we denote by $\mathrm{dom} f := \{x \in X : f(x) \in \mathbb{R}\}$ its domain. We say that $f$ is *continuous* (respectively *lsc*), if it is continuous (respectively lsc) at every point $x \in X$, where $\mathbb{R} \cup \{+\infty\}$ is equipped with the topology generated by the family $\Im_{\mathbb{R}} \cup \{]x, +\infty], x \in \mathbb{R}\}$ ($\Im_{\mathbb{R}}$ being the usual topology of $\mathbb{R}$). Note that such functions may take infinite values, as for instance the function $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ with $f(x) = 1/x$ if $x > 0$ and $+\infty$ if $x \le 0$.

Concurrently, a function $f$ is said to have a *continuous restriction to a subset $S$ of its domain*, if $f|_S$ is a continuous (real-valued) function, see also [7, page 82].

Throughout this article we shall deal with *proper* (that is, not identically equal to $\{+\infty\}$) *lsc convex* functions. Let us remark that the class of lsc convex functions with a continuous restriction in their domain is much larger than the one of convex continuous functions. It contains in particular all indicator functions of closed convex sets, as well as many other non-continuous functions, see [6, Example 3.8 (a)].

We recall from [6] that the subdifferential $\partial f$ of the function $f$ at a point $x \in \operatorname{dom} f$ is defined as follows

$$\partial f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x\rangle, \ \forall y \in X\}. \tag{1}$$

Finally, $T : X \rightrightarrows X^*$ will denote a multivalued operator defined on $X$ and taking as values subsets of $X^*$. We denote by $\operatorname{dom} T := \{x \in X : T(x) \neq \emptyset\}$ its domain. We recall that $T$ is *cyclically monotone* if for any $n \in \mathbb{N}$, for any $x_0, x_1, \ldots, x_n$ in $X$ and for any $x_0^* \in T(x_0), x_1^* \in T(x_1), \ldots, x_n^* \in T(x_n)$ we have

$$\sum_{i=0}^{n} \langle x_i^*, x_{i+1} - x_i\rangle \leq 0, \tag{2}$$

where $x_{n+1} := x_0$. Moreover, if $T$ is not strictly contained (in the graph sense) in any other cyclically monotone operator, then it is called *maximal cyclically monotone*. Typical (and in fact exclusive) examples of maximal cyclically monotone operators are the subdifferentials $\partial f$ of convex lsc functions (see [6], for example).

## 3. Representation of convex functions

Rockafellar has proved in [8] that if $T$ is cyclically monotone, then there exists a lsc convex function $f$ such that $T \subseteq \partial f$. The proof of this result involves a typical construction based on $T$. In particular, starting from any point $x_0$ of the domain of $T$ (which is supposed to be nonempty), he defined the following lsc convex function $f_T$

$$f_T(x) = \sup \left\{ \langle x_n^*, x - x_n\rangle + \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i\rangle \right\} + c, \tag{3}$$

where $c$ is an arbitrary constant and the supremum is taken over all $n \geq 1$, all $x_1, x_2, \ldots, x_n$ in $\operatorname{dom} T$ and all $x_0^* \in T(x_0), x_1^* \in T(x_1), \ldots, x_n^* \in T(x_n)$.

Let us note that $x_0$ appears in all sums at the right hand side of (3) and that cyclic monotonicity property guarantees that $f_T(x_0) = c$, hence, in particular, $\operatorname{dom} f_T \neq \emptyset$.

It is proved in [8] that the function $f_T$ is unique up to a constant, whenever the operator $T$ is maximal cyclically monotone, in which case $T = \partial f_T$.

Applying this result to the maximal cyclically monotone operator $\partial f$, (3) yields (for $c = f(x_0)$) the following representation for the lsc convex function $f$

$$f(x) = \sup \left\{ \langle x_n^*, x - x_n\rangle + \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i\rangle \right\} + f(x_0), \tag{4}$$

where the supremum is taken over all $n \geq 1$, all $x_1, x_2, \dots, x_n$ in $\operatorname{dom} \partial f$ and all $x_0^* \in \partial f(x_0), x_1^* \in \partial f(x_1), \dots, x_n^* \in \partial f(x_n)$.

This representation is global, in the sense that there is no limitation for the choice of the points $x_1, x_2, \dots, x_n$ in the above supremum. Let us now produce a localized version of the formula (4) which will be useful in the sequel.

PROPOSITION 1. *Let $x_0 \in \operatorname{dom} \partial f$. For any $x \in X$ and $\varepsilon > 0$ we have*

$$f(x) = f(x_0) + \sup \left\{ \langle x_n^*, x - x_n \rangle + \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle \right\}, \tag{5}$$

*where the supremum is taken over all $n \geq 1$, all*

$$x_1, x_2, \dots, x_n \quad in \quad \operatorname{dom} \partial f \cap B([x_0, x], \varepsilon) \tag{6}$$

*and all $x_0^* \in \partial f(x_0), x_1^* \in \partial f(x_1), \dots, x_n^* \in \partial f(x_n)$.*

Note that comparing with (4), the choice of the points $x_1, x_2, \dots, x_n$ is constrained into the $\varepsilon$-neighbourhood of $[x_0, x]$.

PROOF. Let us consider the indicator function $h$ of the closed set $B([x_0, x], \varepsilon)$, given by

$$h(y) = \begin{cases} 0 & \text{if } y \in B([x_0, x], \varepsilon); \\ +\infty & \text{if } y \notin B([x_0, x], \varepsilon). \end{cases}$$

Since $B([x_0, x], \varepsilon)$ is convex, for every $y \in B([x_0, x], \varepsilon)$ and $y^* \in \partial h(y)$ we have

$$\langle y^*, x' - y \rangle \leq 0, \forall x' \in B([x_0, x], \varepsilon). \tag{7}$$

Let us now consider the lsc convex function $g(y) = f(y) + h(y)$ and let us remark that $\operatorname{dom} \partial g \subset \operatorname{dom} g \subset B([x_0, x], \varepsilon)$. Since $\operatorname{dom} f \cap \operatorname{int} \operatorname{dom} h \neq \emptyset$, it follows [6, Proposition 3.15] that

$$\partial g(y) = \partial f(y) + \partial h(y), \tag{8}$$

for all $y \in B([x_0, x], \varepsilon)$.

Applying formula (4) for the lsc convex function $g$ at the point $x$, and using the fact that $g(y') = f(y')$ for all $y' \in B([x_0, x], \varepsilon)$, we get

$$f(x) = f(x_0) + \sup \left\{ \langle x_n^*, x - x_n \rangle + \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle \right\}, \tag{9}$$

where the supremum is taken over all $n \geq 1$, all $x_1, x_2, \ldots, x_n$ in $\operatorname{dom} \partial g$ and all $x_0^* \in \partial g(x_0), x_1^* \in \partial g(x_1), \ldots, x_n^* \in \partial g(x_n)$.

In particular, for any $M < f(x) - f(x_0)$, there exist $x_1, x_2, \ldots, x_n$ in $B([x_0, x], \varepsilon)$ and $x_0^* \in \partial g(x_0), x_1^* \in \partial g(x_1), \ldots, x_n^* \in \partial g(x_n)$, such that

$$M < \langle x_n^*, x - x_n \rangle + \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle.$$

Using (8), for $i = 0, 1, \ldots, n$, we can write $x_i^* = z_i^* + y_i^*$, where $z_i^* \in \partial f(x_i)$ and $y_i^* \in \partial h(x_i)$. It now follows from (7) that

$$M < \langle z_n^*, x - x_n \rangle + \sum_{i=0}^{n-1} \langle z_i^*, x_{i+1} - x_i \rangle.$$

Since $M$ is arbitrarily chosen, the proof is complete. □

## 4. Main results

In this section we establish dual characterizations for the class of proper lsc convex functions $f$ such that $f|_{\operatorname{dom} \partial f}$ (respectively $f|_{\operatorname{dom} f}$) is continuous. These characterizations involve properties stronger than (but reminiscent of) cyclic monotonicity for the subdifferential $\partial f$.

Let us first state the following interesting result.

PROPOSITION 2. *Let $x_0 \in \operatorname{dom} \partial f$. Then the following are equivalent:*

(i) *$f|_{\operatorname{dom} f}$ is continuous at $x_0$;*
(ii) *$f|_{\operatorname{dom} \partial f}$ is continuous at $x_0$.*

PROOF. We obviously have (i) implies (ii). For the inverse implication, suppose that $f|_{\operatorname{dom} f}$ is not continuous at $x_0$. There then exists $\varepsilon > 0$ such that for all $i \geq 1$, there exists $x_i \in \operatorname{dom} f \cap B(x_0, 1/i)$ with

$$|f(x_i) - f(x_0)| > \varepsilon.$$

Now for each $i \geq 1$, we may find $y_i \in \operatorname{dom} \partial f \cap B(x_0, 2/i)$ with

$$|f(x_i) - f(y_i)| < \frac{\varepsilon}{2}$$

(this is possible because $\operatorname{dom} \partial f$ is graphically dense in $\operatorname{dom} f$). Then we have $\{y_i\} \to x_0$ and $|f(y_i) - f(x_0)| > \varepsilon/2$ for all $i \geq 1$, which contradicts (ii). □

**4.1. Relative continuity on dom $\partial f$**

We recall from [4] the following definition.

DEFINITION 3. (i) Let $x_0 \in X$. An operator $T$ is called *$\sigma$-cyclically monotone* at $x_0$, if

$$\limsup_{n\to\infty} \sum_{i=0}^{n} \langle x_i^*, x_{i+1} - x_i \rangle \le 0 \tag{10}$$

for all $x_0^* \in T(x_0)$, for all $\{x_i\}_{i\ge1}$ in dom $T$ satisfying $\lim_{i\to\infty} x_i = x_0$ and all $\{x_i^*\}_{i\ge1}$ in $X^*$ satisfying $x_i^* \in T(x_i)$ for all $i \ge 1$.

(ii) The operator $T$ is called *$\sigma$-cyclically monotone*, if it is $\sigma$-cyclically monotone at every $x_0 \in X$.

Let us note that Definition 3 implies (in a trivial way) that $T$ is $\sigma$-cyclically monotone at any $x \notin \operatorname{dom} T$.

It is also easily seen that every $\sigma$-cyclically monotone operator is cyclically monotone. Indeed, given points $x_0, x_1, \ldots, x_n$ in dom $T$, it suffices to consider the sequence $\{x_i\}_{i\ge1}$ in dom $T$, where $x_i = x_0$, for $i \ge n+1$. Then relation (10) clearly yields (2).

The following theorem gives a positive answer to a conjecture raised in [4].

THEOREM 4. *Let $x_0 \in \operatorname{dom} \partial f$. The following are equivalent:*

(i) *$f|_{\operatorname{dom}\partial f}$ is continuous at $x_0$;*

(ii) *$\partial f$ is $\sigma$-cyclically monotone at $x_0$.*

PROOF. (i) implies (ii). Let $\{x_i\}_{i\ge1}$ be any sequence in dom $\partial f$ such that $\lim_{i\to\infty} x_i = x_0$. For any $x_i^* \in \partial f(x_i)$, relation (1) implies

$$f(x_{i+1}) - f(x_i) \ge \langle x_i^*, x_{i+1} - x_i \rangle. \tag{11}$$

Adding (11) from $i = 0$ to an arbitrary integer $n$ we obtain

$$f(x_{n+1}) - f(x_0) \ge \sum_{i=0}^{n} \langle x_i^*, x_{i+1} - x_i \rangle.$$

As $n \to +\infty$, the continuity of $f|_{\operatorname{dom}\partial f}$ at $x_0$ yields (10).

(ii) implies (i). Suppose that $f|_{\operatorname{dom}\partial f}$ is not continuous at $x_0$. Since $f$ is lsc at $x_0$, we deduce the existence of a sequence $\{x_i\}_{i\ge1}$ in dom $\partial f$, such that $x_i \to x_0$ and $\liminf_{i\to\infty} f(x_i) > f(x_0)$. Take any $\varepsilon > 0$ such that

$$\liminf_{i\to+\infty} f(x_i) \ge f(x_0) + 2\varepsilon.$$

Let us fix $i \in \mathbb{N}$. Using Proposition 1 for the points $x_i \in \operatorname{dom}\partial f$ and $x_{i+1} \in X$ and for the number $1/(i+1) > 0$, we deduce the existence of a finite sequence

$y_{i,1}, y_{i,2}, \dots, y_{i,k_i}$ in $\operatorname{dom} \partial f \cap B([x_i, x_{i+1}], 1/(i+1))$ and of $y^*_{i,0} \in \partial f(y_{i,0}), y^*_{i,1} \in \partial f(y_{i,1}), \dots, y^*_{i,k_i} \in \partial f(y_{i,k_i})$ such that

$$\sum_{j=0}^{k_i} \langle y^*_{i,j}, y_{i,j+1} - y_{i,j} \rangle \geq f(x_{i+1}) - f(x_i) - \frac{\varepsilon}{2^{i+1}}, \tag{12}$$

where by convention $y_{i,0} := x_i$, $y_{i,k_i+1} := x_{i+1}$. Summing (12) from $i = 0$ to an arbitrary integer $n$, we obtain

$$\sum_{i=0}^{n} \sum_{j=0}^{k_i} \langle y^*_{i,j}, y_{i,j+1} - y_{i,j} \rangle \geq f(x_{n+1}) - f(x_0) - \sum_{i=0}^{n} \frac{\varepsilon}{2^{i+1}}.$$

Taking the upper limit as $n \to \infty$, the last inequality yields

$$\limsup_{n \to +\infty} \sum_{i=0}^{n} \sum_{j=0}^{k_i} \langle y^*_{i,j}, y_{i,j+1} - y_{i,j} \rangle \geq \varepsilon.$$

Since the sequence $\{y_{0,0}, \dots, y_{0,k_0}, y_{1,0}, \dots, y_{1,k_1}, \dots\}$ is norm converging to $x_0$, we conclude that $\partial f$ is not $\sigma$-cyclically monotone. □

We can easily deduce the following corollary.

COROLLARY 5. *The following statements are equivalent*:

(i) $f|_{\operatorname{dom} \partial f}$ *is continuous*;

(ii) $\partial f$ *is* $\sigma$*-cyclically monotone.*

We state below some typical examples of lsc convex functions such that $f|_{\operatorname{dom} \partial f}$ is discontinuous.

EXAMPLE 1 ([7, page 83]). Let the function $f : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ be defined by

$$f(x_1, x_2) = \begin{cases} x_2^2/x_1 & \text{if } x_1 > 0; \\ 0 & \text{if } x_1 = x_2 = 0; \\ +\infty & \text{elsewhere.} \end{cases}$$

The above function is lsc and convex (the latter can be verified by calculating the Hessian). Considering the sequence $(1/n^3, 1/n)$ we conclude that $f|_{\operatorname{dom} \partial f}$ is not continuous at $(0, 0)$.

EXAMPLE 2. Let the function $f : \ell^2(\mathbb{N}) \to \mathbb{R} \cup \{+\infty\}$ be defined by

$$f(x) = \|x\|_1 := \sum_{i=0}^{+\infty} |x_i|$$

for every $x = \{x_i\} \in \ell^2(\mathbb{N})$. Since $f$ is the pointwise supremum of the convex continuous functions $f_n : \ell^2(\mathbb{N}) \to \mathbb{R} \cup \{+\infty\}$ defined for all $x = \{x_i\} \in \ell^2(\mathbb{N})$ by $f_n(x) = \sum_{i=0}^{n} |x_i|$, it is obviously convex and lsc. However, $f\,|_{\mathrm{dom}\,\partial f}$ is discontinuous at any point of its domain. (Note that in this example the domain of the function $f$ is dense).

### 4.2. Relative continuity on dom $f$

Let us first give the following definition.

DEFINITION 6. Let $x_0 \in X$. An operator $T$ is called strongly cyclically monotone at $x_0$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x_1 \in \mathrm{dom}\, T \cap B(x_0, \delta)$, for every sequence $\{x_i\}_{i \geq 2}$ in dom $T$ satisfying $\lim_{i\to\infty} x_i = x_0$, and for every sequence $\{x_i^*\}_{i\geq 1}$ in $X^*$ satisfying $x_i^* \in T(x_i)$ for all $i \geq 1$, we have

$$\limsup_{n\to\infty} \sum_{i=1}^{n} \langle x_i^*, x_{i+1} - x_i \rangle \leq \varepsilon. \tag{13}$$

Let us note that $T$ is strongly cyclically monotone (in a trivial way) at every point in the complement of $\overline{\mathrm{dom}\, T}$.

PROPOSITION 7. *Let $x_0 \in X$. If $T$ is strongly cyclically monotone at $x_0$, then $T$ is also $\sigma$-cyclically monotone at $x_0$. The converse is true whenever $x_0 \in \mathrm{dom}\, T$.*

PROOF. Suppose that $T$ is strongly cyclically monotone at $x_0$. It suffices to consider only the case $x_0 \in \mathrm{dom}\, T$. If $\{x_i\}_{i\geq 1}$ is any sequence in dom $T$ such that $\lim_{i\to\infty} x_i = x_0$, then for any $\varepsilon > 0$ we can apply (13) for the sequence $\{y_i\}_{i\geq 1}$ defined by $y_1 := x_0$ and $y_i := x_{i-1}$ for all $i \geq 2$ (note that $y_1 \in \mathrm{dom}\, T \cap B(x_0, \delta)$ for all $\delta$). Since $\varepsilon$ is arbitrary, we easily conclude that (10) is verified, hence $T$ is $\sigma$-cyclically monotone.

Conversely, suppose that $x_0 \in \mathrm{dom}\, T$ and that $T$ is $\sigma$-cyclically monotone at $x_0$. We shall show that $T$ is strongly cyclically monotone at $x_0$. Let $\varepsilon > 0$. Then pick any $x_0^*$ in $T(x_0)$ and set $\delta = \varepsilon/\|x_0^*\|$ (if $x_0^* = 0$, then take $\delta = 1$). Then for every $x_1 \in \mathrm{dom}\, T \cap B(x_0, \delta)$, we have

$$|\langle x_0^*, x_1 - x_0 \rangle| \leq \varepsilon. \tag{14}$$

Since $T$ is $\sigma$-cyclically monotone at $x_0$, it follows that for every sequence $\{x_i\}_{i\geq 2}$ in dom $T$ satisfying $\lim_{i\to\infty} x_i = x_0$, and for every sequence $\{x_i^*\}_{i\geq 1}$ in $X^*$ satisfying $x_i^* \in T(x_i)$ for all $i \geq 1$, we have

$$\limsup_{n\to\infty} \sum_{i=0}^{n} \langle x_i^*, x_{i+1} - x_i \rangle \leq 0. \tag{15}$$

Combining (14) and (15) we conclude that $T$ is strongly cyclically monotone at $x_0$. □

Consider now the case where $T = \partial f$. Since $\overline{\operatorname{dom} f} = \overline{\operatorname{dom} \partial f}$, we conclude that $\partial f$ is $\sigma$-cyclically monotone at every $x_0 \in X \setminus \operatorname{dom} \partial f$ and strongly cyclically monotone at every $x_0 \in X \setminus \overline{\operatorname{dom} f}$. It now follows from Proposition 7 that for any $x_0 \in \operatorname{dom} \partial f$, $\partial f$ is strongly cyclically monotone at $x_0$ if and only if $\partial f$ is $\sigma$-cyclically monotone at $x_0$ (if and only if $f\,|_{\operatorname{dom} \partial f}$ is continuous at $x_0$). The following theorem (analogue to Theorem 4) deals with the case $x_0 \in \operatorname{dom} f$.

THEOREM 8. *Let $x_0 \in \operatorname{dom} f$. The following are equivalent:*
(i) *$f\,|_{\operatorname{dom} f}$ is continuous at $x_0$;*
(ii) *$\partial f$ is strongly cyclically monotone at $x_0$.*

PROOF. (i) implies (ii). Let $\varepsilon > 0$. According to (*i*), there exists $\delta > 0$ such that for all $x \in \operatorname{dom} \partial f \cap B(x_0, \delta)$

$$|f(x) - f(x_0)| < \varepsilon. \tag{16}$$

Fix any $x_1$ in $\operatorname{dom} \partial f \cap B(x_0, \delta)$. Consider now any sequence $\{x_i\}_{i \ge 1}$ in $\operatorname{dom} \partial f$ satisfying $\lim_{i \to \infty} x_i = x_0$. Then for all $i \ge 1$ and all $x_i^* \in \partial f(x_i)$, we have

$$f(x_{i+1}) - f(x_i) \ge \langle x_i^*, x_{i+1} - x_i \rangle. \tag{17}$$

Adding (17) from $i = 1$ to an arbitrary integer $n$, we obtain

$$f(x_{n+1}) - f(x_1) \ge \sum_{i=1}^{n} \langle x_i^*, x_{i+1} - x_i \rangle.$$

As $n \to +\infty$, the continuity of $f\,|_{\operatorname{dom} f}$ together with (16) yields (13).

(ii) implies (i). Suppose that $f\,|_{\operatorname{dom} f}$ is not continuous at $x_0$. Then since $f$ is lsc, there exists $\varepsilon > 0$ and $\{x_i\}_{i \ge 1}$ in $\operatorname{dom} f$ satisfying $\lim_{i \to \infty} x_i = x_0$ such that

$$\liminf_{i \to \infty} f(x_i) > f(x_0) + 3\varepsilon. \tag{18}$$

Since $\operatorname{dom} \partial f$ is graphically dense in $\operatorname{dom} f$ ([6, Theorem 3.17]), without loss of generality we assume that $\{x_i\}_{i \ge 1}$ is in $\operatorname{dom} \partial f$. Moreover, for any $\delta > 0$, we may choose $x_{1,\delta}$ in $\operatorname{dom} \partial f \cap B(x_0, \delta)$ such that

$$|f(x_{1,\delta}) - f(x_0)| < \varepsilon.$$

It follows that

$$\liminf_{i \to \infty} f(x_i) > f(x_{1,\delta}) + 2\varepsilon. \tag{19}$$

Applying Proposition 1 successively for the points $\{x_{1,\delta}, x_1\}$ and $\{x_i, x_{i+1}\}$ for $i \ge 1$, and repeating the arguments of the final part of the proof of Theorem 4 ((ii) implies (i)) we deduce the existence of a sequence $\{z_i\}_i$ in $X$ satisfying $z_0 := x_{1,\delta}$ and $\lim_{i\to\infty} z_i = x_0$, and of a sequence $\{z_i^*\}_{i\geq 1}$ satisfying $z_i^* \in \partial f(z_i)$ for all $i$, such that for every $n$ we have

$$\sum_{i=0}^{\varphi(n)} \langle z_i^*, z_{i+1} - z_i \rangle \geq f(x_{n+1}) - f(x_{1,\delta}) - \varepsilon,$$

for some increasing function $\varphi : \mathbb{N} \to \mathbb{N}$. As $n \to \infty$, thanks to (19), we obtain

$$\limsup_{n\to\infty} \sum_{i=0}^{n} \langle z_i^*, z_{i+1} - z_i \rangle > \varepsilon.$$

Thus $\partial f$ is not strongly cyclically monotone at $x_0$. □

We shall say that $\partial f$ is strongly cyclically monotone on $\operatorname{dom} f$ if it is strongly cyclically monotone at every point of $\operatorname{dom} f$. The following corollary is analogue to Corollary 5.

COROLLARY 9. *The following are equivalent*:

(i) $f|_{\operatorname{dom} f}$ *is continuous*;
(ii) $\partial f$ *is strongly cyclically monotone on* $\operatorname{dom} f$.

REMARK. Combining Proposition 2 with Theorem 4 and Theorem 8 we obtain an indirect way to establish Proposition 7 for the special case $T = \partial f$.

Finally, the following proposition shows that $\partial f$ is not strongly monotone at any $x_0 \in \overline{\operatorname{dom} f} \setminus \operatorname{dom} f$.

PROPOSITION 10. *Let* $x_0 \in \overline{\operatorname{dom} f} \setminus \operatorname{dom} f$. *Then* $\partial f$ *is not strongly cyclically monotone at* $x_0$.

PROOF. Since $f$ is lsc and $x_0 \notin \operatorname{dom} f$, it follows that for any sequence $\{x_i\}_{i\geq 1}$ in $\operatorname{dom} \partial f$ satisfying $\lim_{i\to\infty} x_i = x_0$ we have $\lim_{i\to\infty} f(x_i) = +\infty$. Using the same arguments as in the proof of Theorem 8 ($(ii)$ implies $(i)$), for every $M > 0$ we obtain the existence of a sequence $\{z_i\}_{i\geq 1}$ in $X$ satisfying $z_1 := x_1$ and $\lim_{i\to\infty} z_i = x_0$, and of a sequence $\{z_i^*\}_{i\geq 1}$ satisfying $z_i^* \in \partial f(z_i)$ for all $i \geq 1$, such that for $n$ large enough

$$\sum_{i=1}^{n} \langle z_i^*, z_{i+1} - z_i \rangle \geq M.$$

Thus $\partial f$ cannot be strongly cyclically monotone at $x_0$. □

Let us now show that Theorem 4 and Theorem 8 characterize different classes of functions. This is illustrated in the following example.

EXAMPLE. Consider the following proper lsc convex function $f : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f(x_1, x_2) = \begin{cases} x_2^2/x_1 - \sqrt{x_1} & \text{if } x_1 > 0; \\ 0 & \text{if } x_1 = x_2 = 0; \\ +\infty & \text{elsewhere.} \end{cases}$$

It is easily seen that

$$\operatorname{dom} \partial f = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\},$$

while

$$\operatorname{dom} f = \operatorname{dom} \partial f \cup \{(0, 0)\}.$$

Clearly, the function $f\,|_{\operatorname{dom} \partial f}$ is continuous. On the other hand, the function $f\,|_{\operatorname{dom} f}$ is discontinuous at $(0, 0)$, as can be shown by considering the sequence $(1/n^3, 1/n)$ which converges to $(0, 0)$.

REMARK. The above example exhibits in particular the difference between Definition 3 and Definition 6 for $x_0 \in \overline{\operatorname{dom} T} \setminus \operatorname{dom} T$ (compare also with Proposition 2).

### 4.3. Classification of the various concepts of cyclic monotonicity

We first give the following definitions.

DEFINITION 11. We say that
(i) $\partial f$ is locally bounded on $\operatorname{dom} f$, if for every $x_0 \in \operatorname{dom} f$ there exist $M > 0$ and $\delta > 0$ such that

$$(20) \qquad \forall z \in \operatorname{dom} \partial f \cap B(x_0, \delta),\ \forall z^* \in \partial f(z),\ \|z^*\| \leq M.$$

(ii) $\partial f$ has a locally bounded selection on $\operatorname{dom} f$, if for every $x_0 \in \operatorname{dom} f$ there exist $M > 0$ and $\delta > 0$ such that

$$(21) \qquad \forall z \in \operatorname{dom} \partial f \cap B(x_0, \delta), \quad \exists z^* \in \partial f(z), \quad \|z^*\| \leq M.$$

Let us observe that if $f$ is the indicator function of any closed convex subset $K$ of $X$, then the operator $\partial f$ has a locally bounded selection on $\operatorname{dom} f$, without being locally bounded (unless $K = X$).

The following result is well known (see [6], for example).

THEOREM 12. *$f$ is continuous if and only if $\partial f$ is locally bounded on* $\operatorname{dom} f$.

In the above case, it follows that $\operatorname{dom} f$ is open and $\operatorname{dom} f = \operatorname{dom} \partial f$. Let us remark that it is possible to have $\operatorname{dom} f \neq X$. It suffices to consider the function $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ with

$$f(x) = \begin{cases} +\infty & \text{if } x \leq 0; \\ 1/x & \text{if } x > 0. \end{cases}$$

We now state the following sufficient condition for strong cyclic monotonicity.

PROPOSITION 13. *If* $\partial f$ *has a locally bounded selection on* $\operatorname{dom} f$, *then* $\partial f$ *is strongly cyclically monotone on* $\operatorname{dom} f$.

PROOF. Let $x_0 \in \operatorname{dom} f$. In view of Theorem 8 it suffices to show that the function $f|_{\operatorname{dom} f}$ is continuous at $x_0$. Since $\partial f$ has a locally bounded selection on $\operatorname{dom} f$ and since $\operatorname{dom} \partial f$ is dense in $\operatorname{dom} f$, there exists $\delta > 0$ such that for every $x \in \operatorname{dom} f \cap B(x_0, \delta)$ we can find a sequence $\{x_i\}_{i \geq 1}$ in $\operatorname{dom} \partial f$ satisfying $\lim_{i \to \infty} x_i = x$ and a sequence $\{x_i^*\}_{i \geq 1}$ in $X^*$ satisfying $x_i^* \in \partial f(x_i)$ and $\|x_i^*\| \leq M$ for all $i \geq 1$. Then (1) yields that

$$f(x_i) \leq f(x_0) + \langle x_i^*, x_i - x_0 \rangle.$$

Since $f$ is lsc, we conclude as $i \to +\infty$ that

$$f(x) \leq f(x_0) + M\|x_0 - x\|. \tag{22}$$

Since (22) holds for all $x \in \operatorname{dom} f \cap B(x_0, \delta)$, it follows that

$$\limsup_{x \to x_0} f|_{\operatorname{dom} f}(x) \leq f(x_0).$$

Since $f$ is lsc we conclude that $f|_{\operatorname{dom} f}$ is continuous at $x_0$. □

The converse of Proposition 13 is not true as it is shown by the following example.

EXAMPLE. Consider the function $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ with

$$f(x) = \begin{cases} -\sqrt{-x} & \text{if } x \leq 0; \\ +\infty & \text{if } x > 0. \end{cases}$$

Then the restriction $f|_{\operatorname{dom} f}$ is obviously continuous. On the other hand, $\partial f$ does not have a locally bounded selection (take $x_0 = 0$).

Thus, between the various concepts we have considered, the following implications hold, and none other:

$$
\begin{array}{ccc}
\partial f \text{ locally bounded on } \operatorname{dom} f & \Longleftrightarrow & f \text{ continuous} \\
\Downarrow & & \\
\partial f \text{ locally bounded selection on } \operatorname{dom} f & & \Downarrow \\
\Downarrow & & \\
\partial f \text{ strongly cyclically monotone on } \operatorname{dom} f & \Longleftrightarrow & f\,|_{\operatorname{dom} f} \text{ continuous} \\
\Downarrow & & \Downarrow \\
\partial f \ \sigma\text{-cyclically monotone} & \Longleftrightarrow & f\,|_{\operatorname{dom} \partial f} \text{ continuous}
\end{array}
$$

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# A DUAL CHARACTERISATION OF THE RADON-NIKODYM PROPERTY

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We prove that a Banach space $X$ has the Radon-Nikodym property if, and only if, every weak*-lower semicontinuous convex continuous function $f$ of $X^*$ is Gâteaux differentiable at some point of its domain with derivative in the predual space $X$.

## 1. INTRODUCTION

Collier [5] has shown that a Banach space $X$ has the Radon-Nikodym property if, and only if, all weak*-lower semicontinuous convex continuous functions on the dual space $X^*$ are generically Fréchet differentiable. (Such a dual space was called in [5] weak*-Asplund.) In this article we give the following characterisation of the Radon-Nikodym property in terms of Gâteaux derivatives.

**THEOREM 1.** *A Banach space $X$ has the Radon-Nikodym property if, and only if, every weak*-lower semicontinuous convex continuous function on $X^*$ is Gâteaux differentiable at some point of its domain with derivative in the predual space $X$.*

Since Fréchet derivatives of weak*-lower semicontinuous convex continuous functions of $X^*$ are always elements of $X$ (see [7], for example), the improvement upon the aforementioned result of Collier consists on replacing the Fréchet derivative by Gâteaux and on passing from a dense differentiability assumption to the existence of the derivative at one point.

If $X$ does not have the Radon-Nikodym property, then it is possible to have nowhere Fréchet differentiable weak*-lower semicontinuous convex continuous functions on $X^*$ for which the set of points where the Gâteaux derivative exists and belongs to the predual space is dense (see Proposition 8). Concurrently, it is also possible to have weak*-lower semicontinuous convex continuous functions on $X^*$ that are generically Gâteaux differentiable with all derivatives in $X^{**} \setminus X$. Indeed, consider the Banach space $X = c_0(\mathbb{N})$, its dual space $X^* = \ell^1(\mathbb{N})$ and the function $g(x) = \|x\|_1$, see [10, Example 1.4 (b)] for details.

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Let us finally note that characterisations of the Radon-Nikodym property for dual Banach spaces in terms of the Gâteaux derivative are recently established by Giles in [**8**, Theorem 2].

The proof of Theorem 1 is given in Section 3, while in the following section we fix our notation and we recall relevant definitions.

## 2. Preliminaries

In the sequel, $(X, \|.\|)$ will be a Banach space and $(X^*, \|.\|)$ will be its dual. We denote by $B_X$ the closed unit ball of $X$ and by $\mathbb{R}$ (respectively, $\mathbb{N}$) the set of all real (respectively, positive integer) numbers. For any $x \in X$ and any $p \in X^*$ we denote by $\langle p, x\rangle$ the value of the functional $p$ at the point $x$. Similarly, for any $z^{**}$ in $X^{**}$ we denote by $\langle p, z^{**}\rangle$ the value of $z^{**}$ at $p$. We also denote by $\overline{\text{co}}\, F$ the closed convex hull of the set $F$. For any non-empty closed bounded subset $F$ of $X$ we denote by $\psi_F$ the indicator function of $F$ ($\psi_F(x) := 0$, if $x \in F$ and $+\infty$, if $x \notin F$) and by $\psi_F^*$ its Fenchel conjugate, that is, for all $p \in X^*$

$$\psi_F^*(p) = \sup_{x\in F} \langle p, x\rangle. \tag{1}$$

It is known that $\psi_F^*$ is a weak*-lower semicontinuous convex continuous function. (The latter follows from the fact that the boundedness of $F$ yields dom $\psi_F^* = X^*$.) We also recall that every weak*-lower semicontinuous convex continuous function $g : X^* \to \mathbb{R} \cup \{+\infty\}$ coincides with the first conjugate $f^*$ of a lower semicontinuous convex function $f$ defined on $X$ (take $f := g^*$). We denote by $\operatorname{dom} g := \{p \in X^* : g(p) < +\infty\}$ the domain of the function $g$. Then, the Fenchel-Moreau subdifferential $\partial g$ of $g$ at any $p_0 \in$ dom $g$ is defined as follows:

$$\partial g(p_0) = \{z^{**} \in X^{**} : g(p) - g(p_0) \geqslant \langle p - p_0, z^{**}\rangle,\ \forall p \in X^*\} \tag{2}$$

If $p_0 \in X^* \setminus$ dom $g$, then we set $\partial g(p_0) = \emptyset$.

Given a closed subset $F$ of $X$ and a point $x_0$ of $F$ we say that $x_0$ is a *strongly exposed* point of $F$, if there exists $p_0 \in X^*$ such that any sequence $\{x_n\}_{n\geqslant 1}$ in $F$ satisfying $\lim_{n\to+\infty} \langle p_0, x_n\rangle = \sup_{x\in F} \langle p_0, x\rangle$ converges to $x_0$ in the norm topology. In such case we say that the functional $p_0$ strongly exposes $x_0$ in $F$. We denote by se$(F)$ the set of strongly exposed points of $F$.

We now introduce the notion of a *weakly exposed* point, which will be useful in the sequel, see Lemma 5.

**Definition 2.** *Let $F$ be a closed subset of $X$. A point $x_0 \in F$ is called a weakly exposed point in $F$, if there exists $p_0 \in X^*$ such that any sequence $\{x_n\}_{n\geqslant 1}$ in $F$ with $\lim_{n\to+\infty} \langle p_0, x_n\rangle = \sup_{x\in F} \langle p_0, x\rangle$ weakly converges to $x_0$.*

In the case of the above definition we say that the functional $p_0$ weakly exposes $x_0$ in $F$. It follows easily that $p_0$ attains its unique maximum on $F$ at $x_0$, hence in particular $x_0$ is an extreme point of $F$. We denote by $\mathrm{we}(F)$ the set of weakly exposed points of $F$. Furthermore, a point $x_0$ is called a *point of continuity* of $F$, if the identity mapping $\mathrm{id} : (F, \Im_w) \to (F, \Im_{\|\cdot\|})$ is continuous, where $\Im_w$ (respectively, $\Im_{\|\cdot\|}$) denotes the relative weak (respectively, norm) topology of $F$. It follows directly that $x_0$ is a strongly exposed point of $F$ if, and only if, it is both weakly exposed and a point of continuity of $F$. Finally, a point $x_0$ is called *weakly denting* (or *strongly extreme*, according to the terminology in [**4**, p.67]), if for any relatively weakly open subset $W$ in $F$ containing $x_0$ there exist $p \in X^*$ and $\alpha > 0$ such that the set $\{x \in F : \langle p, x\rangle > \langle p, x_0\rangle - \alpha\}$ is included in $W$.

## 3. Proof of the main result

The proof of Theorem 1 is based on the following result of Bourgain [**3**, Chapter 1; Theorem 4]. (For a proof in English, see [**4**, Corollary 3.7.6].)

**Theorem 3.** *A Banach space $X$ has the Radon-Nikodym property if, and only if, every nonempty closed convex bounded subset $F$ of $X$ has at least one weakly denting point.*

We can easily deduce the following corollary. The analogous result for dual Banach spaces is given in [**8**, Theorem 4].

**Corollary 4.** *For a Banach space $X$, the following are equivalent:*

(i) *$X$ has the Radon-Nikodym property*

(ii) *Every closed convex bounded subset of $X$ is the closed convex hull of its weakly exposed points.*

(iii) *Every nonempty closed convex bounded subset of $X$ has at least one weakly exposed point.*

Proof: It is known ([**4**, Corollary 3.5.7], [**10**, Theorem 5.21], for example) that a Banach space $X$ has the Radon-Nikodym property if, and only if, every closed convex bounded subset of $X$ is the closed convex hull of its strongly exposed points. This shows that (i)$\Longrightarrow$(ii). Implication (ii)$\Longrightarrow$(iii) is trivial, while (iii)$\Longrightarrow$(i) follows from Theorem 3 and the observation that every weakly exposed point of $F$ is weakly denting. $\square$

Remark 1. A weakly denting point is not in general weakly exposed, even in finite dimensions. Indeed, let $X = \mathbb{R}^2$, $F = \{(x_1, x_2) : f(x_1) \leqslant x_2 \leqslant g(x_1)\}$, where $f(x) = \max\{0, x^3\}$ and $g(x) = x + 1$, and $\overline{x} = (0, 0)$. Then $\overline{x}$ is a weakly denting point of the compact convex set $F$, without being weakly exposed.

Remark 2. A Banach space $X$ has the Radon-Nikodym property if, and only if, for every nonempty closed convex bounded subset $F$ of $X$ we have $\mathrm{se}(F) \neq \emptyset$. However, if $X$ does not have the Radon-Nikodym property, then the fact that $\mathrm{we}(F) \neq \emptyset$ (or even that $\overline{\text{co}}\left(\text{we}\,(F)\right) = F$) for some closed convex bounded subset $F$ of $X$ does not necessarily imply that $\text{se}(F) \neq \emptyset$. (Consider the subset $F$ of $c_0(\mathbb{N})$ defined by (12) in Proposition 8 and Claims 1 and 2 therein.)

We shall finally need the following lemma.

**Lemma 5.** *Let $X$ be a Banach space and $F$ be any non-empty closed convex bounded subset $X$. Then the following are equivalent:*

(i) *The function $\psi_F^*$ is Gâteaux differentiable at $p_0$ with derivative $x_0 \in X$.*

(ii) *$x_0 \in F$ and the functional $p_0$ is weakly exposing $x_0$ in $F$.*

Proof: (i)$\Longrightarrow$(ii): Assume that (i) holds. Since $x_0 = \nabla^G \psi_F^*(p_0)$ (where $\nabla^G \psi_F^*$ denotes the Gâteaux derivative of $\psi_F^*$), we obviously have $x_0 \in \partial\psi_F^*(p_0)$, that is for all $p \in X^*$

$$\psi_F^*(p) - \psi_F^*(p_0) \geqslant \langle p - p_0, x_0\rangle.$$

For $p = 0$ we obtain

$$\psi_F^*(p_0) := \sup_{x\in F} \langle p_0, x\rangle = \langle p_0, x_0\rangle. \tag{3}$$

Let now $\{x_n\}_{n\geqslant 1}$ be a sequence in $F$ such that

$$\lim_{n\to+\infty} \langle p_0, x_n\rangle = \sup_{x\in F} \langle p_0, x\rangle. \tag{4}$$

It suffices to show that $\{x_n\}_{n\geqslant 1}$ weakly converges to $x_0$. (Then, since the weak and the norm closure of the convex set $F$ coincide, it will also follow that $x_0 \in F$.)

Let us assume, towards a contradiction, that there exists a subsequence $\{x_{n_k}\}_{k\geqslant 1}$ of $\{x_n\}_{n\geqslant 1}$, $h \in X^*$ and $\alpha > 0$ such that for all $k \geqslant 1$

$$\langle h, x_{n_k}\rangle - \langle h, x_0\rangle > \alpha. \tag{5}$$

Thanks to (3) and (4), we can consider $\varepsilon_n \searrow 0^+$ in a way that

$$\langle p_0, x_n\rangle \geqslant \langle p_0, x_0\rangle - \varepsilon_n. \tag{6}$$

Since $\psi_F^*(p) \geqslant \langle p, x_n\rangle$, using (3) we get

$$\psi_F^*(p) \geqslant \psi_F^*(p_0) + \langle p, x_n\rangle - \langle p_0, x_0\rangle,$$

which in view of (6) yields

$$\psi_F^*(p) \geqslant \psi_F^*(p_0) + \langle p - p_0, x_n\rangle - \varepsilon_n. \tag{7}$$

Set $t_n = 2\varepsilon_n/\alpha$. Then for $p = p_0 + t_n h$ relation (7) yields

$$(\psi_F^*)(p_0 + t_n h) - (\psi_F^*)(p_0) \geqslant \langle t_n h, x_n\rangle - \varepsilon_n$$

for all $n \geqslant 1$. In view of (5) this implies

$$\frac{(\psi_F^*)(p_0 + t_{n_k}h) - (\psi_F^*)(p_0)}{t_{n_k}} - \langle h, x_0 \rangle \geqslant \frac{\alpha}{2} > 0$$

for all $k \geqslant 1$. It follows that $x_0$ is not the Gâteaux derivative of $\psi_F^*$ at $p_0$, hence a contradiction.

(ii)→(i): Suppose that $p_0$ is weakly exposing $x_0$ in $F$, hence in particular $\langle p_0, x_0 \rangle = \sup_{x \in F} \langle p_0, x \rangle$. It follows easily from (2) that $x_0 \in \partial\psi_F^*(p_0)$. Let us now suppose that (i) does not hold. Then there exist $\varepsilon > 0$, $h \in X^*$ with $\|h\| \leqslant 1$ and $t_n \searrow 0^+$ such that

$$(\psi_F^*)(p_0 + t_n h) - (\psi_F^*)(p_0) > \langle t_n h, x_0 \rangle + \varepsilon t_n. \tag{8}$$

For every $n \geqslant 1$, choose $x_n$ in $F$ such that

$$\langle p_0 + t_n h, x_n \rangle > (\psi_F^*)(p_0 + t_n h) - \frac{t_n}{n}. \tag{9}$$

Since $(\psi_F^*)(p_0) \geqslant \langle p_0, x_n \rangle$, the above inequality yields

$$\langle p_0 + t_n h, x_n \rangle - \langle p_0, x_n \rangle > (\psi_F^*)(p_0 + t_n h) - (\psi_F^*)(p_0) - \frac{t_n}{n}.$$

Hence

$$\langle t_n h, x_n \rangle > (\psi_F^*)(p_0 + t_n h) - (\psi_F^*)(p_0) - \frac{t_n}{n}. \tag{10}$$

Combining (8) and (10) we conclude

$$\langle h, x_n - x_0 \rangle > \varepsilon - \frac{1}{n},$$

which shows that $\{x_n\}_{n \geqslant 1}$ does not weakly converge to $x_0$. However, since the sequence $\{x_n\}_{n \geqslant 1}$ is bounded and the function $\psi_F^*$ is continuous, relation (9) yields $\lim_{n \to +\infty} \langle p_0, x_n \rangle = (\psi_F^*)(p_0)$, obtaining thus a contradiction to Definition 2. □

REMARK. The above proof was inspired from techniques developed in [**2**] where a connection between well-posed problems and differentiability was established. Results in the same spirit are also established in [**6**, Section 1], via a different approach. We are grateful to C. Zălinescu for bringing the aforementioned reference to our attention.

PROOF OF THEOREM 1: The "only if" part follows from the result of Collier [**5**] and the fact that the Fréchet derivatives of weak*-lower semicontinuous convex continuous functions on $X^*$ always belong to the predual space $X$ (see [**7**], for example).

For the "if" part, let $F$ be any closed convex bounded subset of $X$. Then the function $\psi_F^*$ of $X^*$ (given in (1)) is weak*-lower semicontinuous convex and continuous. From our hypothesis and Lemma 5 we conclude that we $(F) \neq \emptyset$. Since $F$ is arbitrary, Corollary 4 asserts that $X$ has the Radon-Nikodym property. □

Let us recall that a Banach space $X$ is called *weakly sequentially complete*, if every weakly Cauchy sequence of $X$ is weakly converging in $X$. A typical example of a non-reflexive weakly sequentially complete Banach space is the space $L^1(\mu)$, where $\mu$ is a $\sigma$-finite positive measure. The following remark is due to Godefroy.

**COROLLARY 6.** *Let $X$ be a weakly sequentially complete Banach space. Then $X$ has the Radon-Nikodym property if, and only if, every weak*-lower semicontinuous convex continuous function on $X^*$ is Gâteaux differentiable at some point of its domain.*

PROOF: The "only if" part is a direct consequence of Theorem 1. The "if" part follows from the following observation: if $F$ is a nonempty closed convex bounded subset of $X$, and if $\nabla^G\psi_F^*(p)$ is the Gâteaux derivative of the function $\psi_F^*$ at $p \in X^*$, then there exists $\{x_n\}_{n\geqslant 1}$ in $F$ that weakly*-converges to $\nabla^G\psi_F^*(p)$ (see the proof of Lemma 5 (i)$\Longrightarrow$(ii)). It follows that $\{x_n\}_{n\geqslant 1}$ is a weakly Cauchy sequence, hence in view of our hypothesis $\nabla^G\psi_F^*(p) \in X$. (For similar considerations, see also [**9**].) We conclude by Lemma 5 (i)$\Longrightarrow$(ii) and Corollary 4 (iii)$\Longrightarrow$(i). ∎

Lemma 5 has also the following consequence. (The proof below is similar to [**10**, Theorem 5.20].)

**COROLLARY 7.** *Let $F$ be a closed convex bounded subset of $X$. If $\psi_F^*$ is Gâteaux differentiable in a dense subset of $X^*$ with derivatives in $X$, then $F = \overline{\mathrm{co}}\left(\mathrm{we}\,(F)\right)$.*

PROOF: Since $F$ is bounded, we have $\mathrm{dom}\,(\psi_F^*) = X^*$. (In particular the function $\psi_F^*$ is convex and Lipschitz.) Since $F$ is closed and convex, we have $\overline{\mathrm{co}}\left(\mathrm{we}\,(F)\right) \subseteq F$. Let us suppose, towards a contradiction, that there exists some $x_0$ in $F \setminus \overline{\mathrm{co}}\left(\mathrm{we}\,(F)\right)$. Then by applying the Hahn-Banach theorem, we can find $p \in X^*$ ($p \neq 0$) and $\alpha \in \mathbb{R}$ such that

$$\sup\Big\{\langle p, x\rangle : x \in \overline{\mathrm{co}}\left(\mathrm{we}\,(F)\right)\Big\} < \alpha < \langle p, x_0\rangle.$$

Set $D = \{q \in X^* : \exists\nabla^G(\psi_F^*)(q) \in X\}$. Since $D$ is dense in $X^*$, we can find $q \in D$ close to $p$ such that $z := \nabla^G(\psi_F^*)(q) \in X$ and

$$\sup\Big\{\langle q, x\rangle : x \in \overline{\mathrm{co}}\left(\mathrm{we}\,(F)\right)\Big\} < \alpha < \langle q, x_0\rangle. \tag{11}$$

By Lemma 5 we conclude that $z \in \mathrm{we}\,(F)$ and that the functional $q$ weakly exposes $z$. This clearly contradicts (11). ∎

The space $c_0(\mathbb{N})$ is a typical example of a Banach space without the Radon-Nikodym property. In this case, as already mentioned in Section 1, the norm $\|\cdot\|_1$ provides an example of a weak*-lower semicontinuous convex continuous function of $\ell^1(\mathbb{N})$, which is generically Gâteaux differentiable with all derivatives in $X^{**}\setminus X$. In the following proposition we give an example of a (nowhere Fréchet differentiable) weak*-lower semicontinuous convex continuous function of $\ell^1(\mathbb{N})$, which is Gâteaux differentiable with derivatives in the predual space in a dense set.

**PROPOSITION 8.** *Let $X = c_0(\mathbb{N})$. Then there exists a weak*-lower semicontinuous convex continuous function $f : X^* \to \mathbb{R}$ such that:*

(i) *there exists a dense subset $D$ of $X^*$ such that $f$ is Gâteaux differentiable at every point of $D$ with derivative in the predual space ;*

(ii) *$f$ is nowhere Fréchet differentiable.*

PROOF: Set $X = c_0(\mathbb{N})$ and consider the set

$$F = B_X \cap c_0^+(\mathbb{N}) := \{x = (x^i)_i : \|x\|_\infty \leqslant 1 \text{ and } x^i \geqslant 0 \ (\forall i \in \mathbb{N})\}. \tag{12}$$

It is easily seen that $F$ is closed convex bounded and that

$$\text{ext}\,(F) = \{x \in F : \ x^i \in \{0,1\} \text{ for all } i\}$$

where ext $(F)$ denotes the set of the extreme points of $F$.

CLAIM 1. Let $\overline{x} \in \text{ext}\,(F)$ and consider the finite set

$$I_{\overline{x}} = \{i \in \mathbb{N} : \overline{x}^i = 1\}. \tag{13}$$

Then any functional $p = (p^i)_i$ of $X^* := \ell^1(\mathbb{N})$ satisfying

$$\begin{aligned} &p^i > 0, \quad \text{if } i \in I_{\overline{x}} \\ &p^i < 0, \quad \text{if } i \in \mathbb{N}\backslash I_{\overline{x}} \end{aligned} \tag{14}$$

weakly exposes the point $\overline{x}$. In particular ext $(F) =$ we $(F)$ (and hence we $(F) \neq \emptyset$).

PROOF OF CLAIM 1: Let $\overline{x} \in \text{ext}\,(F)$, $I_{\overline{x}} = \{i \in \mathbb{N} : \overline{x}^i = 1\}$ and consider any $p$ in $\ell^1(N)$ satisfying (14). We first note that for all $x \in F$ and all $i \in \mathbb{N}$ we have

$$p^i x^i \leqslant p^i \overline{x}^i. \tag{15}$$

It follows that $\langle p, x\rangle \leqslant \langle p, \overline{x}\rangle$, for all $x \in F$, that is, $\langle p, \overline{x}\rangle = \sup_{x\in F} \langle p, x\rangle$. Take now any sequence $\{x_n\}_{n\geqslant 1}$ in $F$ such that $\lim_{n\to+\infty} \langle p, x_n\rangle = \langle p, \overline{x}\rangle$. We show that

$$\lim_{n\to+\infty} x_n^i = \overline{x}^i \tag{16}$$

for all $i \geqslant 0$. Indeed, assume that for some $i_0$ (16) does not hold. Then there exist a subsequence $\{x_{n_k}^{i_0}\}_{k\geqslant 1}$ of $\{x_n^{i_0}\}_{n\geqslant 1}$, $\varepsilon > 0$ and $k_0 \in \mathbb{N}$ such that for all $k \geqslant k_0$

$$|x_{n_k}^{i_0} - \overline{x}^{i_0}| > \frac{\varepsilon}{|p^{i_0}|}.$$

Using (15) we infer that

$$p^{i_0}\, x_{n_k}^{i_0} \ < \ p^{i_0}\, \overline{x}^{i_0} - \varepsilon.$$

Combining with (15) we get $\langle p, x_{n_k}\rangle < \langle p, \overline{x}\rangle - \varepsilon$, for all $k \geqslant k_0$. This contradicts the fact that $\langle p, x_n\rangle \to \langle p, \overline{x}\rangle$. It follows that (16) holds for all $i \geqslant 0$. Since the sequence $\{x_n\}_{n\geqslant 1}$ is bounded, we conclude from (16) that $x_n \xrightarrow{w} \overline{x}$. Hence the functional $p$ is weakly exposing $\overline{x}$ in $F$. Since every weakly exposed point is obviously extreme, the proof of the claim is complete. □

CLAIM 2: $\operatorname{se}(F) = \emptyset$.

PROOF OF CLAIM 2: It clearly suffices to show that any point $\overline{x}$ in $\operatorname{we}(F)$ is not a point of continuity for $F$. To this end, take any $\overline{x} \in \operatorname{we}(F)$ and consider the sequence $\{x_n\}_{n\geqslant 1}$ in $F$ with

$$x_n^i = \begin{cases} 1 & \text{if } i \in I_{\overline{x}} \cup \{n\} \\ 0 & \text{elsewhere} \end{cases}$$

where $I_{\overline{x}}$ is given by (13). Then it follows easily that $x_n \xrightarrow{w} \overline{x}$. On the other hand, for $n$ sufficiently large, we have $\|x_n - \overline{x}\|_\infty = 1$. □

Consider now the weak*-lower semicontinuous convex continuous function $\psi_F^* : \ell^1(\mathbb{N}) \to \mathbb{R}$ defined by

$$\psi_F^*(p) := \sup_{x\in F} \langle p, x\rangle = \|p_+\|_1 \tag{17}$$

where $\|\cdot\|_1$ is the usual norm of $\ell^1(\mathbb{N})$ and

$$p_+^i = \begin{cases} p^i & \text{if } p^i > 0 \\ 0 & \text{if } p^i \leqslant 0. \end{cases}$$

Let us denote by $D$ the set of all functionals $p = (p^i)_i$ in $\ell^1(\mathbb{N})$ satisfying (14) for some finite (possibly empty) subset $I$ of $\mathbb{N}$. For every such functional $p$, consider the point $\overline{x} = (\overline{x}^i)_i$ of $c_0(\mathbb{N})$ defined by

$$\overline{x}^i = \begin{cases} 1 & \text{if } i \in I \\ 0 & \text{if } i \in \mathbb{N}\backslash I. \end{cases}$$

Then $\overline{x} \in F$ and $I = I_{\overline{x}}$ (where $I_{\overline{x}}$ is given in (13)). It follows by Claim 1 that the functional $p$ weakly exposes $\overline{x}$. Applying Lemma 5 (ii)$\to$(i) we conclude that $\overline{x}$ is the Gâteaux derivative of $\psi_F^*$ at $p$.

We now show that $D$ is dense in $\ell^1(\mathbb{N})$. Indeed, take any $q = (q^i)_i$ in $\ell^1(\mathbb{N})$ and any $\varepsilon > 0$. Then for some $n_0 \in \mathbb{N}$ we have:

$$\|q\|_1 \leqslant \sum_{i=0}^{n_0} |q^i| + \frac{\varepsilon}{2}.$$

Consider now the functional $p = (p^i)_i$ defined by

$$p^i = \begin{cases} q^i & \text{if } i \leqslant n_0 \text{ and } q^i \neq 0 \\ -\dfrac{\varepsilon}{2^{i+2}} & \text{elsewhere.} \end{cases}$$

It is easily seen that $p \in D$. Moreover,

$$\|q - p\|_1 = \sum_{i=0}^{+\infty} |q^i - p^i| \leqslant \sum_{i>n_0}^{+\infty} |q^i| + \sum_{i=0}^{+\infty} \frac{\varepsilon}{2^{i+2}} \leqslant \varepsilon.$$

We have shown that the function $\psi_F^*$ is densely Gâteaux differentiable with derivatives in the predual space $X$. On the other hand, since by Claim 2 the set $F$ has no strongly exposed points, it follows from [**1**, p. 450] that $\psi_F^*$ is nowhere Fréchet differentiable.

Let us finally note that the function $\psi_F^*$ is in fact generically Gâteaux differentiable. Indeed, it is easily seen that for every $p = (p^i)_i$ with $p^i \neq 0$ for all $i$, we have $\nabla^G \psi_F^*(p) = z^{**}$ where $z^{**} \in \ell^\infty(\mathbb{N})$ is given by

$$(z^{**})^i = \begin{cases} 1 & \text{if } p^i > 0 \\ 0 & \text{if } p^i < 0 \end{cases}$$

□

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# INTEGRATION OF FENCHEL SUBDIFFERENTIALS OF EPI-POINTED FUNCTIONS*

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**Abstract.** It is shown that in finite dimensions Rockafellar's technique of integrating cyclically monotone operators, applied to the Fenchel subdifferential of an epi-pointed function, yields the closed convex hull of that function.

**Key words.** Fenchel subdifferential, cyclically monotone operator, integration, epi-pointed function

**AMS subject classifications.** Primary, 52A41, 47H05; Secondary, 26E25

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**1. Introduction.** By the term *integration of a multivalued operator* $T : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$, we mean the problem of finding a lower semicontinuous (lsc) function $f$ such that $T \subseteq \partial f$, where $\partial f$ corresponds to some notion of subdifferential for the function $f$. This problem has recently attracted researchers' interest; see, for instance, [3], [5], [6], [9], and references therein.

If we impose the further restriction that $\partial f$ is the Fenchel subdifferential (defined below), then a complete answer (even in infinite dimensions) to the aforementioned problem has been established by Rockafellar [7], with the introduction of the class of cyclically monotone operators. Indeed, as shown in [7] (see also [4]), every such operator $T$ is included in the subdifferential $\partial f$ of an lsc convex function $f$. In particular, $T$ coincides with $\partial f$ if and only if it is maximal, and in such a case $f$ is unique up to a constant.

In dealing with the above problem, Rockafellar used a technique consisting of a formal construction of an lsc convex function $f_T$ started from a given cyclically monotone operator $T$. The function $f_T$ is further called the *convex integral* of $T$. Let us recall that Fenchel subdifferentials are particular cases of cyclically monotone operators. Consequently, for every lsc function $f$ with $\operatorname{dom} \partial f \neq \emptyset$, the convex integral $f_{\partial f}$ (also denoted $\widehat{f}$ in this paper) of its subdifferential $\partial f$ naturally defines an lsc convex function minorizing $f$. If in particular $f$ is convex, then the convex integral $\widehat{f}$ is equal to $f$ up to a constant [7]. In the general case, a natural question arises:

($\mathcal{Q}$) Given an lsc function $f$, is $\widehat{f}$ equal to the closed convex hull $\overline{\text{co}}\, f$ of $f$?

This question was first considered in [1, Proposition 2.6], where the authors provided a positive answer (in finite dimensions) for the class of *strongly coercive functions*, that is, functions satisfying

$$\lim_{\|x\| \to \infty} \frac{f(x)}{\|x\|} = +\infty. \tag{1.1}$$

In this paper we improve the above result by establishing the same conclusion for the larger class of *epi-pointed functions* introduced in [2] (see definition below).

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Moreover, we shall give an easy example of a non-epi-pointed function for which $(\mathcal{Q})$ is no longer valid. However, for the one-dimensional case ($d = 1$), we shall show that $(\mathcal{Q})$ holds true for every lsc function defined on $\mathbb{R}$.

The paper is organized as follows. In the next section, we fix our notation and give some preliminaries concerning Fenchel duality and convex integration of the (Fenchel) subdifferential of a nonconvex function. The result of [1] for the class of strongly coercive functions is recalled, and an example where the convex integration does not yield the closed convex hull of the function is illustrated. Finally, in section 3 we state and prove the main result of this article, concerning the class of epi-pointed functions.

**2. Convex integration.** Throughout this paper we consider the Euclidean space $\mathbb{R}^d$ equipped with the usual scalar product $\langle \cdot, \cdot \rangle$. In what follows, we denote by $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ a lsc function which is proper, that is $\operatorname{dom} f := \{x \in \mathbb{R}^d : f(x) \in \mathbb{R}\}$ is nonempty. We also denote by $\operatorname{epi} f$ the epigraph of $f$, that is the set $\{(x,t) \in \mathbb{R}^d \times \mathbb{R} : f(x) \le t\}$. We recall that the second conjugate $\overline{\operatorname{co}}\, f$ (also denoted by $f^{**}$) of $f$ is given by

$$\overline{\operatorname{co}}\, f(x) = \sup_{x^* \in \mathbb{R}^d} \{\langle x^*, x\rangle - f^*(x^*)\}, \tag{2.1}$$

where

$$f^*(x^*) = \sup_{x \in \mathbb{R}^d} \{\langle x^*, x\rangle - f(x)\}. \tag{2.2}$$

It is known that $\overline{\operatorname{co}}\, f$ is the greatest lsc convex function majorized by $f$, and that its epigraph coincides with the closed convex hull of the epigraph of $f$. By the term subdifferential we shall always mean the Fenchel subdifferential $\partial f$, defined for every $x \in \operatorname{dom} f$ as follows

$$\partial f(x) = \{x^* \in \mathbb{R}^d : f(y) \ge f(x) + \langle x^*, y - x\rangle \;\; \forall y \in \mathbb{R}^d\}. \tag{2.3}$$

If $x \in \mathbb{R}^d \setminus \operatorname{dom} f$, we set $\partial f(x) = \emptyset$. Throughout this paper, the set

$$\operatorname{dom} \partial f := \{x \in \mathbb{R}^d : \partial f(x) \neq \emptyset\}$$

is assumed to be nonempty. Further, let $x_0$ denote an arbitrary point of $\operatorname{dom} \partial f$. We call convex integral of $\partial f$ the lsc convex function $\widehat{f} : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ defined for all $x \in \mathbb{R}^d$ by the formula

$$\widehat{f}(x) := f(x_0) + \sup \left\{ \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i\rangle + \langle x_n^*, x - x_n\rangle \right\}, \tag{2.4}$$

where the supremum is taken for all $n \ge 1$, all $x_1, x_2, \dots, x_n$ in $\operatorname{dom} \partial f$, and all $x_0^* \in \partial f(x_0), x_1^* \in \partial f(x_1), \dots, x_n^* \in \partial f(x_n)$. According to (2.3), we can easily check that $\widehat{f} \le f$, and consequently $f$ is proper and

$$\widehat{f} \le \overline{\operatorname{co}}\, f. \tag{2.5}$$

Rockafellar [8] has shown that if $f$ is in particular convex, then the convex integral $\widehat{f}$ of $\partial f$ is equal to $f$, that is

$$\widehat{f} = f. \tag{2.6}$$

In [1, Proposition 2.6] the authors generalized (2.6) to the nonconvex case by showing that if $f$ is strongly coercive (that is $f$ satisfies (1.1)), then (2.5) becomes

$$\widehat{f} = \overline{\text{co}}\, f.$$

However, the exact relation between $\widehat{f}$ and $\overline{\text{co}}\, f$ for a function not satisfying (1.1) remains to be discovered. In particular, while in one-dimensional spaces we always have $\widehat{f} = \overline{\text{co}}\, f$ (see Corollary 3.7), the following simple counterexample shows that this is not the case in general.

*Example* 2.1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as follows:

$$f(a,b) = \begin{cases} \exp(-a^2) + \frac{1}{2}b^2 & \text{if } (a,b) \neq (0,0), \\ 0 & \text{if } (a,b) = (0,0). \end{cases}$$

We can easily check that

$$f^*(a,b) = \begin{cases} \frac{1}{2}b^2 & \text{if } a = 0, \\ +\infty & \text{if } a \neq 0 \end{cases}$$

and that

$$\overline{\text{co}}\, f(a,b) = \frac{1}{2}b^2.$$

On the other hand, since

$$\partial f(a,b) = \begin{cases} \{0\} & \text{if } (a,b) = (0,0), \\ \emptyset & \text{if } (a,b) \neq (0,0), \end{cases}$$

formula (2.4) yields (for $x_0 = (0,0)$) that $\widehat{f}(x) = 0$ for all $x \in \mathbb{R}^2$. Hence $\widehat{f} \neq \overline{\text{co}}\, f$.

*Remark*. Appropriately modifying the function $f$ around the origin, we can obtain a continuous function $g : \mathbb{R}^2 \to \mathbb{R}$ such that $\widehat{g} \neq \overline{\text{co}}\, g$.

Let us also remark that in the previous example we have

$$\text{int}\,(\text{dom}\, f^*) = \emptyset. \tag{2.7}$$

It will follow from the main theorem of section 3 that (2.7) is in fact a necessary condition for obtaining such examples.

**3. Epi-pointed functions.** The aim of this section is to establish the equality between the convex integral $\widehat{f}$ of $\partial f$ and the closed convex hull $\overline{\text{co}} f$ of $f$ for the class of proper, lsc, and epi-pointed functions defined in $\mathbb{R}^d$.

Let us recall the following definition [2].

DEFINITION 3.1. *The function $f$ is called epi-pointed if* $\text{int}\,(\text{dom}\, f^*) \neq \emptyset$.

It follows easily (see [2, Proposition 4.5 (iv)]) that every strongly coercive function is epi-pointed. Note also that for every $\overline{x}^* \in \text{int}\,(\text{dom}\, f^*)$ we can always find $\overline{x} \in \mathbb{R}^d$ such that $f^*(\overline{x}^*) = \langle \overline{x}^*, \overline{x} \rangle - f(\overline{x})$ (that is the "sup" in (2.2) is attained). This obviously yields that $\overline{x}^* \in \partial f(x) \cap \text{int}\,(\text{dom}\, f^*)$. In particular, if $f$ is epi-pointed the set $\text{dom}\, \partial f$ is nonempty. If now $x_0$ is any point of $\text{dom}\, \partial f$, we can consider the lsc convex function $\tilde{f}$ defined for all $x \in \mathbb{R}^d$ by

$$\tilde{f}(x) = f(x_0) + \sup \left\{ \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle \; + \; \langle x_n^*, x - x_n \rangle \right\}, \tag{3.1}$$

where the supremum is taken for all $n \geq 1$, all $x_1, x_2, \ldots, x_n$ in $\mathbb{R}^d$, all $x_0^* \in \partial f(x_0)$, and all

$$x_i^* \in \partial f(x_i) \cap \operatorname{int}(\operatorname{dom} f^*),$$

where $i \in \{1, \ldots, n\}$. Note that whenever $f$ is epi-pointed, the set

$$\{x \in \mathbb{R}^d : \partial f(x) \cap \operatorname{int}(\operatorname{dom} f^*) \neq \emptyset\}$$

is nonempty, so that $\tilde{f}$ is proper. Comparing formulas (2.4) and (3.1) we immediately conclude that

$$\tilde{f} \leq \widehat{f}.$$

We shall show that if the function $f$ is convex and epi-pointed, then $f$ is equal to $\tilde{f}$ and so, in view of (2.6), the previous inequality becomes an equality. This is the context of Proposition 3.3 below.

We shall first need the following lemma.

LEMMA 3.2. *Suppose that $f$ is lsc convex and epi-pointed. Then we have the inclusion*

$$\partial f^*(x^*) \subseteq \partial \tilde{f}^*(x^*) \quad \textit{on } \operatorname{int}(\operatorname{dom} f^*).$$

*Proof.* A classic result (see [8]) states that for the lsc convex function $f$ and all $x, x^* \in \mathbb{R}^d$ we have

$$x \in \partial f^*(x^*) \quad \text{if and only if} \quad x^* \in \partial f(x).$$

Similarly, for the lsc convex function $\tilde{f}$,

$$x \in \partial \tilde{f}^*(x^*) \quad \text{if and only if} \quad x^* \in \partial \tilde{f}(x).$$

Let $x^* \in \operatorname{int}(\operatorname{dom} f^*)$ and $x \in \partial f^*(x^*)$. We shall show that $x \in \partial \tilde{f}^*(x^*)$. It follows that

$$x^* \in \partial f(x) \cap \operatorname{int}(\operatorname{dom} f^*). \tag{3.2}$$

For any $t < \tilde{f}(x)$, using formula (3.1), we may choose $x_1, \ldots, x_n$ in $\mathbb{R}^d$, $x_0^* \in \partial f(x_0)$, and $x_1^* \in \partial f(x_1) \cap \operatorname{int}(\operatorname{dom} f^*), \ldots, x_n^* \in \partial f(x_n) \cap \operatorname{int}(\operatorname{dom} f^*)$ such that

$$t < f(x_0) + \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle + \langle x_n^*, x - x_n \rangle. \tag{3.3}$$

For any $y \in \mathbb{R}^d$, adding to both sides of (3.3) the quantity $\langle x^*, y - x \rangle$, we obtain

$$t + \langle x^*, y - x \rangle < f(x_0) + \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle + \langle x_n^*, x - x_n \rangle + \langle x^*, y - x \rangle. \tag{3.4}$$

In view of (3.1), the right part of (3.4) is always less than or equal to $\tilde{f}(y)$. Letting $t \to \tilde{f}(x)$, we infer

$$\tilde{f}(x) + \langle x^*, y - x \rangle \leq \tilde{f}(y),$$

which yields $x^* \in \partial \tilde{f}(x)$, or, equivalently, $x \in \partial \tilde{f}^*(x^*)$. ◻

Proposition 3.3. *If $f$ is lsc convex and epi-pointed, then $\tilde{f} = f$.*

*Proof.* Since the functions $f^*$ and $\tilde{f}^*$ are proper, lsc, and convex, we deduce from [8] and Lemma 3.2 that

$$f^* = \tilde{f}^* + k \quad \text{on } \operatorname{int}(\operatorname{dom} f^*) \tag{3.5}$$

for some constant $k \in \mathbb{R}$.

Let us now prove that the equality in (3.5) can be extended to all $\mathbb{R}^d$. According to [7, Corollary 7.3.4], it suffices to prove that the relative interiors of the convex sets $\operatorname{dom} f^*$ and $\operatorname{dom} \tilde{f}^*$ are equal or, equivalently (since $\operatorname{int}(\operatorname{dom} f^*)$ is nonempty), that

$$\operatorname{int}(\operatorname{dom} f^*) = \operatorname{int}(\operatorname{dom} \tilde{f}^*). \tag{3.6}$$

Let us now prove this last equality. Taking conjugates in both sides of the inequality $\tilde{f} \leq f$ we obtain $f^* \leq \tilde{f}^*$; hence in particular

$$\operatorname{dom} \tilde{f}^* \subseteq \operatorname{dom} f^*,$$

and so

$$\operatorname{int}(\operatorname{dom} \tilde{f}^*) \subseteq \operatorname{int}(\operatorname{dom} f^*). \tag{3.7}$$

Conversely, let $x^* \in \operatorname{int}(\operatorname{dom} f^*)$. Since $f^*$ is convex, we have $\partial f^*(x^*) \neq \emptyset$. By Lemma 3.2 we get $\partial \tilde{f}^*(x^*) \neq \emptyset$, yielding that $x^* \in \operatorname{dom} \partial \tilde{f}^*$. It follows that

$$\operatorname{int}(\operatorname{dom} f^*) \subseteq \operatorname{dom} \tilde{f}^*. \tag{3.8}$$

Combining (3.7) with (3.8), we conclude that equality (3.6) holds as desired. Hence we obtain

$$f^* = \tilde{f}^* + k.$$

Taking conjugates, this last equality yields $f = \tilde{f} - k$. Since $f(x_0) = \tilde{f}(x_0)$, we conclude that $k = 0$ and thus $f = \tilde{f}$. ◻

We shall finally need the following lemma.

Lemma 3.4. *Suppose that $f$ is lsc and epi-pointed, and set $g = \overline{\operatorname{co}}\, f$. Then for any $x \in \operatorname{dom} \partial f$ and $x^* \in \partial g(x) \cap \operatorname{int}(\operatorname{dom} f^*)$ there exist $y_1, \ldots, y_p$ in $\mathbb{R}^d$ such that $x \in \operatorname{co}\{y_1, y_2, \ldots, y_p\}$ and*

$$x^* \in \bigcap_{i=1}^{p} \partial f(y_i).$$

*Proof.* From [2, Theorem 4.6] we conclude that for any $x^* \in \partial g(x)$ there exist $y_1, \ldots, y_p$ in $\mathbb{R}^d$ and $w_1, \ldots, w_q$ in $\mathbb{R}^d \setminus \{0\}$ such that

$$x - \sum_{j=1}^{q} w_j \;\in\; \operatorname{co}\{y_1, y_2, \ldots, y_p\}$$

and

$$x^* \in \left[\bigcap_{i=1}^{p} \partial f(y_i)\right] \cap \left[\bigcap_{j=1}^{q} \partial f_\infty(w_j)\right], \tag{3.9}$$

where $f_\infty$ is defined via the relation $\operatorname{epi}(f_\infty) = (\operatorname{epi} f)_\infty$, where

$$(\operatorname{epi} f)_\infty := \left\{ d \in X : \exists \{x_n\}_{n\geq 1} \text{ in } \operatorname{epi} f, \exists \{t_n\} \searrow 0^+ \text{ with } d = \lim_{n\to+\infty} t_n x_n \right\}.$$

It suffices to show that for $x^* \in \operatorname{int}(\operatorname{dom} f^*)$, (3.9) yields $q = 0$. In order to find a contradiction, suppose that $q \neq 0$. Since the function $f_\infty$ is sublinear positively homogeneous and $f_\infty(0) = 0$ (e.g., [2]), it follows easily that for any $w_j \neq 0$ and any $x^* \in \partial f_\infty(w_j)$ we have $\langle x^*, w_j \rangle = f_\infty(w_j)$. Since $x^* \in \operatorname{int}(\operatorname{dom} f^*)$, we may find some $z^* \in \mathbb{R}^d$ (near $x^*$) such that $z^* \in \operatorname{int}(\operatorname{dom} f^*)$ and $\langle z^*, w_j \rangle > f_\infty(w_j)$. The latter yields easily that

$$z^* \notin \partial f_\infty(0). \tag{3.10}$$

On the other hand, since $z^* \in \operatorname{int}(\operatorname{dom} f^*) \subseteq \operatorname{dom} \partial f^*$, we conclude the existence of $x$ in $\mathbb{R}^d$ such that $x \in \partial f^*(z^*)$, or, equivalently,

$$z^* \in \partial g(x). \tag{3.11}$$

Since $\partial g(x) \subseteq \partial f_\infty(0)$ [2, Theorem 4.6], relations (3.10) and (3.11) give the contradiction. ∎

We are now ready to establish the main result of this section.

THEOREM 3.5. *If $f$ is lsc and epi-pointed, then $\widehat{f} = \overline{\operatorname{co}}\, f$.*

*Proof.* Set $g = \overline{\operatorname{co}}(f)$. Then $g$ is lsc convex and $\operatorname{int}(\operatorname{dom} g^*) = \operatorname{int}(\operatorname{dom} f^*)$. In particular, $g$ is epi-pointed. Using Proposition 3.3 we conclude that

$$g(x) = g(x_0) + \sup \left\{ \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle \; + \; \langle x_n^*, x - x_n \rangle \right\},$$

where the supremum is taken over all $n \geq 1$, all $x_1, \dots, x_n$ in $\mathbb{R}^d$, all $x_0^* \in \partial g(x_0)$, and all

$$x_i^* \in \partial g(x_i) \cap \operatorname{int}(\operatorname{dom} f^*),$$

where $i \in \{1, \dots, n\}$. Take any $x \in \mathbb{R}^d$ and any $t < g(x)$. Then there exist $x_1, \dots, x_n$ in $\mathbb{R}^d$, $x_0^* \in \partial g(x_0)$, and $x_i^* \in \partial g(x_i) \cap \operatorname{int}(\operatorname{dom} f^*)$ (for $i = 1$ to $n$) such that

$$t < g(x_0) + \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle \; + \; \langle x_n^*, x - x_n \rangle. \tag{3.12}$$

Recalling that $x_0 \in \operatorname{dom} \partial f$, we easily check that $g(x_0) = f(x_0)$ and $\partial g(x_0) = \partial f(x_0)$. On the other hand, for all $i \in \{1, \dots, n\}$ Lemma 3.4 guarantees the existence of points $y_i^1, \dots, y_i^{p_i}$ in $\mathbb{R}^d$ such that $x_i \in \operatorname{co}\{y_i^1, y_i^2, \dots, y_i^{p_i}\}$ and

$$x_i^* \in \bigcap_{j=1}^{p_i} \partial f\left(y_i^j\right).$$

We claim that, for $i = 1$, there exists an index $j_1$ in $\{1, 2, \dots, p_1\}$ such that

$$\langle x_0^*, x_1 - x_0 \rangle + \langle x_1^*, x_2 - x_1 \rangle \leq \langle x_0^*, y_1^{j_1} - x_0 \rangle + \langle x_1^*, x_2 - y_1^{j_1} \rangle.$$

Indeed, if this were not the case, then for every $j$ we would have

$$\langle x_0^*, x_1 - x_0\rangle + \langle x_1^*, x_2 - x_1\rangle > \langle x_0^*, y_1^j - x_0\rangle + \langle x_1^*, x_2 - y_1^j\rangle.$$

This yields a contradiction, since $x_1 \in \mathrm{co}\,\{y_1^1, \dots, y_1^{p_1}\}$.

Proceeding like this for $i \geq 1$, we inductively replace all $x_i$'s in (3.12) by $y_i^{j_i}$'s in a way that $x_i^* \in \partial f(y_i^{j_i})$, thus obtaining the formula

$$t < f(x_0) + \langle x_0^*, y_1^{j_1} - x_0\rangle + \langle x_1^*, y_2^{j_2} - y_1^{j_1}\rangle + \cdots + \langle x_n^*, x - y_n^{j_n}\rangle.$$

Comparing with (2.4), we obtain $t < \widehat{f}(x)$. Letting $t \to g(x)$ we infer $g(x) = \overline{\mathrm{co}}\, f(x) \leq \widehat{f}(x)$, which finishes the proof in view of (2.5). □

COROLLARY 3.6. *Suppose that $f$, $h$ are proper lsc and epi-pointed functions. If $\partial f = \partial h$, then $\overline{\mathrm{co}}\, f$ and $\overline{\mathrm{co}}\, h$ are equal up to a constant.*

*Proof.* For $x_0 \in \mathrm{dom}\,\partial f$ and $c = g(x_0) - f(x_0)$ we obviously have $\widehat{f} = \widehat{h} + c$, which, in view of Theorem 3.5, yields $\overline{\mathrm{co}}\, f = \overline{\mathrm{co}}\, h + c$. □

The class of proper, lsc, and epi-pointed functions is not minimal, in order to ensure the conclusion of Theorem 3.5. For example, every constant function $f$ satisfies $\widehat{f} = \overline{\mathrm{co}}\, f = f$, and obviously $\mathrm{dom}\, f^* = \{0\}$. (In fact, one can consider any lsc convex function $f$ which is not epi-pointed.) Furthermore, the example of the function $f(x) = \min\{\|x\|, 1\}$ shows that the conclusion $\widehat{f} = \overline{\mathrm{co}}\, f$ might be true even in cases where $f$ is nonconvex and non-epi-pointed at the same time. In particular, in one-dimensional spaces the following result is true.

COROLLARY 3.7. *If $d = 1$ (that is $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$) and $\mathrm{dom}\,\partial f \neq \emptyset$, then $\widehat{f} = \overline{\mathrm{co}}\, f$.*

*Proof.* In view of Theorem 3.5, it suffices to consider only the case $\mathrm{int}\,(\mathrm{dom}\, f^*) = \emptyset$. Since $f^*$ is convex (and $\mathrm{dom}\,\partial f \neq \emptyset$) it follows that $\mathrm{dom}\, f^* = \{\alpha\}$ for some $\alpha \in \mathbb{R}$. We easily conclude from (2.1) that

$$\overline{\mathrm{co}}\, f(x) = \alpha x - f^*(\alpha) \tag{3.13}$$

for all $x \in \mathbb{R}$. On the other hand, for any $x_0 \in \mathrm{dom}\,\partial f$ we have $\partial f(x_0) = \{\alpha\}$, which yields, in view of (2.2) and (2.3), that

$$f^*(\alpha) = \alpha x_0 - f(x_0). \tag{3.14}$$

Finally, it follows easily from relation (2.4) that

$$\widehat{f}(x) = f(x_0) + \alpha(x - x_0). \tag{3.15}$$

Relations (3.13), (3.14), and (3.15) directly yield $\widehat{f} = \overline{\mathrm{co}}\, f$. □

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# Coincidence Theorems for Convex Functions

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The paper aims at creating a new insight into our perception of convexity by focusing on two fundamental problems: the coincidence of two functions (at least one being convex) upon an information on a dense set and the clarification of the relation between convexity and Fenchel subdifferential. Various results are established into these directions. Several examples are also illustrated showing that some rather unexpected situations can often occur.

*Keywords:* Convex function, coincidence, subdifferential

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## 1. Introduction

If $X$ is a Banach space, would you say that two lower semicontinuous (in short lsc) convex functions $g^1, g^2 : X \to \mathbb{R} \cup \{+\infty\}$, equal on a dense subset of their domain, necessarily coincide? In other words, can we determine a lsc convex function if we know its values on a dense set? In Proposition 3.4, we show that in infinite dimensions the answer is negative even if it is assumed that $g^1 \leq g^2$ and that both functions are positively homogeneous. Motivated by these considerations, we introduce the class $\mathcal{G}_1(X)$ of lsc convex functions $g$ that do not admit any non trivial lsc majorant $f$ coinciding with $g$ on a dense subset of $\operatorname{dom} g$. We show that $\mathcal{G}_1(\mathbb{R}^d)$ coincides with the set of lsc convex functions (see Corollary 3.7). This is not the case in infinite dimensions, since - as we prove in Theorem 3.8 - a lsc convex function with a dense domain belongs to the class $\mathcal{G}_1(X)$ if, and only if, its domain is equal to $X$.

We also investigate the relation between Fenchel subdifferential and convexity by focusing on the following question: given a lsc function $f : X \to \mathbb{R} \cup \{+\infty\}$, can we conclude that $f$ is necessarily convex whenever the domain of the Fenchel subdifferential of $f$ is dense in $X$? This assertion is true in finite dimensions but fails impressively as soon as we consider infinite dimensional spaces (see Proposition 4.3). An interesting relevant question is the following. Given a lsc convex function $g$ and a lsc function $f$ such that the domain of the Fenchel subdifferential of $f$ is dense in the domain of $g$ and such that the closed convex

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envelope of $f$ is equal to $g$, can we conclude that $f = g$? If $\mathcal{G}_2(X)$ denotes the class of lsc convex functions $g$ for which the latter conclusion is true for all $f$, we show in Proposition 4.5 that $\mathcal{G}_2(X)$ strictly contains $\mathcal{G}_1(X)$. Finally we obtain the analogue of Theorem 3.8 for the class $\mathcal{G}_2(X)$. More precisely, we show in Theorem 4.8 that a lsc convex and positively homogeneous function with a dense domain belongs to the class $\mathcal{G}_2(X)$ if, and only if, its domain is equal to $X$.

## 2. Preliminaries

In the sequel, let $\mathbb{N}$ denote the set of strictly positive integers, $X$ a Banach space and $X^*$ its dual. For any $x \in X$ and $p \in X^*$ we denote by $\langle p, x\rangle$ the value of $p$ at $x$. Given a function $f : X \to \mathbb{R} \cup \{+\infty\}$, we denote by $\operatorname{dom} f = \{x \in X : f(x) \in \mathbb{R}\}$ its domain. Throughout this article, we shall deal with proper (i.e. $\operatorname{dom} f$ is nonempty) and lsc functions. Let us recall that the Fenchel subdifferential $\partial f$ of any such function $f$ at the point $x \in \operatorname{dom} f$ is defined as follows

$$\partial f(x) = \{p \in X^* : f(y) - f(x) \geq \langle p, y - x\rangle, \ \forall y \in X\}. \tag{1}$$

If $x \notin \operatorname{dom} f$, we set $\partial f(x) = \emptyset$. We denote by $\operatorname{dom} \partial f = \{x \in X : \partial f(x) \neq \emptyset\}$ the domain of the subdifferential of $f$. We recall that whenever $f$ is convex, $\operatorname{dom} \partial f$ is dense in $\operatorname{dom} f$. In fact, the following result holds (see for example [5, Theorem 3.17]).

**Proposition 2.1.** *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc convex function. Then $\operatorname{dom} \partial f$ is $f$-graphically dense in $\operatorname{dom} f$, that is, for every $x \in \operatorname{dom} f$, there exists a sequence $\{x^n\}$ in $\operatorname{dom} \partial f$ converging to $x$ such that the sequence $\{f(x^n)\}$ converges to $f(x)$.*

Let us further recall that the closed convex hull of the function $f$, denoted $\overline{co}\, f$, is defined as the greatest lsc convex function majorized by $f$. It is well known that its epigraph coincides with the closed convex hull of the epigraph of $f$. Let us remark that $\overline{co}\, f$ takes its values in $\mathbb{R} \cup \{+\infty\}$ if, and only if, there exists an affine continuous function minimizing $f$ on $X$. It is known (see [4] for example) that for a proper lsc function $f : X \to \mathbb{R} \cup \{+\infty\}$ one has $f = \overline{co}\, f$ on $\operatorname{dom} \partial f$. This fact can be seen in the following simple way. Fix $x \in \operatorname{dom} \partial f$, take any $p \in \partial f(x)$ and consider the function $g = p + (f(x) - \langle p, x\rangle)$. It follows from (1) that $f \geq g$. Since $g$ is lsc convex, we get that $f \geq \overline{co}\, f \geq g$. The result follows from the observation that $f(x) = g(x)$.

## 3. Uniquely determined majorants of convex functions

In this section we are interested in the question of determination of a convex function, by means of an information for its values on a dense set.

A relevant, but more specific question is the following. Let $g : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc convex function. If $f$ is lsc and $D$ is a dense subset of $\operatorname{dom} g$ then

$$\left.\begin{array}{r} f \geq g \\ f = g \text{ on } D \end{array}\right\} \implies f = g \ ? \tag{2}$$

Let us first tackle (2) for the special case where $D = \operatorname{dom} \partial g$.

**Lemma 3.1.** *Let $g : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc convex function. Then for every lsc function $f$ satisfying $f \geq g$ we have*

$$f\,|_{\mathrm{dom}\,\partial g} = g\,|_{\mathrm{dom}\,\partial g} \Longrightarrow f = g.$$

**Proof.** Since $f \geq g$, it clearly suffices to show that $f$ coincides with $g$ on $\mathrm{dom}\, g$. By Proposition 2.1, for every $x \in \mathrm{dom}\, g$, there exists a sequence $\{x^n\}$ in $\mathrm{dom}\,\partial g$ converging to $x$ such that the sequence $\{g(x^n)\}$ converges to $g(x)$. By our hypothesis, $f(x^n) = g(x^n)$ for all $n$. Since $f$ is lsc at $x$, letting $n \to +\infty$ this last equality yields $f(x) \leq g(x)$. Recalling that $f \geq g$, we conclude that $f(x) = g(x)$. □

We now have the following proposition.

**Proposition 3.2.** *Let $g : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc convex function and let $D$ be a subset of $\mathrm{dom}\, g$ which is $g$-graphically dense in $\mathrm{dom}\,\partial g$ (that is, any point in $\mathrm{dom}\,\partial g$ is the $g$-graphical limit of a sequence in $D$). Then for every lsc function $f$ satisfying $f \geq g$ we have*

$$f\,|_D = g\,|_D \Longrightarrow f = g.$$

**Proof.** In view of Lemma 3.1, it suffices to show that $f$ coincides with $g$ on $\mathrm{dom}\,\partial g$. Let any $x \in \mathrm{dom}\,\partial g$. Then there exists a sequence $\{x^n\}$ in $D$ converging to $x$ with $\{g(x^n)\}$ converging to $g(x)$. Then $f(x^n) = g(x^n)$ for all $n \geq 1$, and letting $n \to +\infty$ we obtain $f(x) \leq g(x)$, since $f$ is lsc at $x$. Recalling that $f \geq g$, we conclude that $f(x) = g(x)$. □

The following corollary is a direct consequence of Proposition 3.2.

**Corollary 3.3.** *Let $g : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc convex function such that $g\,|_{\mathrm{dom}\,\partial g}$ is continuous and let $D$ be a dense subset of $\mathrm{dom}\,\partial g$. Then for every lsc function $f$ satisfying $f \geq g$ we have*

$$f\,|_D = g\,|_D \Longrightarrow f = g.$$

**Remark.**

1. As observed by the referee, the assumptions of Lemma 3.1 (as well as those of Proposition 3.2 and Corollary 3.3) yield $\partial g(x) \subset \partial f(x)$, for all $x \in X$. Thanks to the convexity of $g$, this inclusion still holds even if $\partial$ is remplaced by any abstract subdifferential in the sense of [8, page 35], which relates directly to results concerning integration of subdifferentials (see [6], [8], [9], [3] and references therein, as well as [7, Theorem 24.9] for the convex case).

2. The assumption "$g\,|_{\mathrm{dom}\,\partial g}$ is continuous" adapted in Corollary 3.3 is strictly weaker than the relative continuity of $g$ on $\mathrm{dom}\, g$. We refer to [2] for further details and a dual characterization of this property.

The above results establish positive answers for the assertion (2) provided that the dense set $D$ satisfies certain conditions. The following proposition shows that without these conditions, assertion (2) may fail even if both functions $g^1$ and $g^2$ are convex and positively homogeneous. As usual, $\ell^2(\mathbb{N})$ denotes the Hilbert space of square summable sequences and $\{e^i\}$ the canonical basis.

**Proposition 3.4.** *There exist two distinct positively homogeneous lsc convex functions $g^1, g^2 : \ell^2(\mathbb{N}) \to \mathbb{R} \cup \{+\infty\}$ such that $g^1 \leq g^2$ and $g^1 = g^2 < +\infty$ on an open dense subset of $\ell^2(\mathbb{N})$.*

**Proof.** Consider the functions $g^1, g^2 : \ell^2(\mathbb{N}) \to \mathbb{R}\cup\{+\infty\}$ defined for any $x = (x_i) \in \ell^2(\mathbb{N})$ as follows:

$$g^1(x) = \sup_{i\geq 1} \{ i\,|x_i| \} \quad \text{and} \quad g^2(x) = \max\ \{ 2|x_1|\,,\, g^1(x)\ \}.$$

It follows directly that both functions are lsc convex and positively homogeneous and that $g^1 \leq g^2$. Let us also note that $g^1(e^1) \neq g^2(e^1)$ and consequently $g^1 \neq g^2$. Moreover, it is easily seen that $g^1 = g^2 < +\infty$ on the subset

$$D = \{(x_i) \in \ell^2(\mathbb{N}) :\ \exists i \in \mathbb{N}, |x_1| < \frac{i}{2}\,|x_i|\}.$$

Since $D$ is an open dense subset of $\ell^2(\mathbb{N})$, the assertion is established. □

In the sequel we consider the question (2) globally, in the sense that we are interested to lsc convex functions $g$ for which the assertion holds true simultaneously for all dense subsets $D$ of $\operatorname{dom} g$. This class is introduced in the following definition.

**Definition 3.5.** We say that a proper lsc convex function $g : X \to \mathbb{R}\cup\{+\infty\}$ belongs to the class $\mathcal{G}_1(X)$, if $g$ does not admit any non trivial lsc majorant that coincides with $g$ on a dense set of $\operatorname{dom} g$.

Proposition 3.2 guarantees that $\mathcal{G}_1(X)$ contains all convex continuous functions on $X$. The following result shows that, more generally, $\mathcal{G}_1(X)$ contains also all lsc convex functions $g$ with $\operatorname{int}\operatorname{dom} g \neq \emptyset$.

**Proposition 3.6.** *If $g : X \to \mathbb{R}\cup\{+\infty\}$ is a proper lsc convex function satisfying* $\operatorname{int}\operatorname{dom} g \neq \emptyset$, *then $g \in \mathcal{G}_1(X)$.*

**Proof.** Let $f$ be a lsc majorant of $g$ such that $f = g$ on a dense subset $D$ of $\operatorname{dom} g$. Let $D_1 = D \cap \operatorname{int}\operatorname{dom}\partial g$. Thanks to the continuity of $g$ on $\operatorname{int}\operatorname{dom} g$ it is easily seen that $D_1$ is $g$-graphically dense in $\operatorname{dom}\partial g$, whence $f = g$, in view of Proposition 3.2. □

Taking the relative interior, one gets the following corollary.

**Corollary 3.7.** $\mathcal{G}_1(\mathbb{R}^d)$ *coincides with the class of all proper lsc convex functions.*

The following theorem gives information on $\mathcal{G}_1(X)$ in the infinite dimensional case.

**Theorem 3.8.** *Let $g : X \to \mathbb{R}\cup\{+\infty\}$ be a proper lsc convex function with a dense domain. Then $g \in \mathcal{G}_1(X)$ if, and only if, $\operatorname{dom} g = X$.*

**Proof.** Suppose that $\operatorname{dom} g \neq X$. Since $\operatorname{dom} g$ is convex and dense in $X$, it follows easily that $\operatorname{int}\operatorname{dom} g = \emptyset$. Hence $g$ takes at least one infinite value in every neighborhood of each point of $X$. Since the function $g$ is lsc at every $x$ in $X$, we conclude that for every integer $n$ the set

$$D_n = \{x \in \operatorname{dom} g : g(x) \geq n\}$$

is dense in $X$. Let us now take $n > \inf g$ and consider the lsc function $f : X \to \mathbb{R}\cup\{+\infty\}$ defined by

$$f(x) = \max\{g(x), n\}.$$

It is easily seen that $f$ violates (2), i.e. $f \geq g$, $f = g$ on the dense set $D_n$ and $f \neq g$. Hence $g \notin \mathcal{G}_1(X)$, which proves the "necessity" part. The "sufficiency" part is a direct consequence of Proposition 3.6, since $\operatorname{dom} g = X$ implies that $\operatorname{dom} \partial g = X$. □

## 4. Fenchel subdifferential and convexity

In this section we investigate the relation between Fenchel subdifferential and convexity. The central question in this section is whether the non-emptiness of the Fenchel subdifferential of a lsc function $f$ on a dense set guarantees the convexity of $f$.

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc function. If we assume that $f$ is convex, then Proposition 2.1 asserts that $\operatorname{dom} \partial f$ is dense in $\operatorname{dom} f$. Let us observe that the converse assertion is not true even if $X = \mathbb{R}$ as shows the example below:

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or } 1; \\ +\infty & \text{elsewhere.} \end{cases}$$

Indeed, the function $f$ is obviously lsc, non-convex and $\operatorname{dom} f = \operatorname{dom} \partial f = \{0, 1\}$.

However, assuming that $\operatorname{dom} f$ is convex, the following proposition ensures the converse in finite dimensions.

**Proposition 4.1.** *Let $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc function with a convex domain. Then $f$ is convex if, and only if,* $\operatorname{dom} \partial f$ *is dense in* $\operatorname{dom} f$.

**Proof.** The "necessity" part follows from Proposition 2.1. For the "sufficiency" part, assume that $f$ is lsc and that $D = \operatorname{dom} \partial f$ is dense in $\operatorname{dom} f$. Setting $g = \overline{co} f$ we obviously have that $g$ is lsc convex, $f \geq g$ and $f\,|_D = g\,|_D$. Since $\operatorname{dom} f$ is convex, it follows that $\operatorname{dom} f$ (and subsequently also $\operatorname{dom} \partial f$) is dense in $\operatorname{dom} g$. This finishes the proof in view of Corollary 3.7. □

In case where the lsc function $f$ has a dense domain in $X$, the converse assertion becomes:

$$\text{if } \operatorname{dom} \partial f \text{ is dense in } X, \text{ is } f \text{ convex ?} \tag{3}$$

This question was first considered in [1] where the following positive result was established (see [1, Section 3]).

**Proposition 4.2.** *Let $f$ be a lsc function such that* $\operatorname{dom} \partial f$ *is dense in $X$. Suppose that at least one of the conditions (a), (b) or (c) is satisfied:*

*(a)* $X = \mathbb{R}^d$;

*(b)* $\operatorname{dom} f = X$;

*(c)* $\partial f$ *has a locally bounded selection on* $\operatorname{dom} \partial f$, *i.e., for every $x \in X$ there exist $M > 0$ and $r > 0$ such that for all $y \in \operatorname{dom} \partial f$,*

$$\|y - x\| \leq r \implies \exists p \in \partial f(y) : \|p\| \leq M.$$

*Then $f$ is a convex continuous function.*

The following proposition completes the above results by exhibiting an example showing that, in infinite dimensions, (3) is not true without additional assumptions. Let us note that in the forthcoming example the domain of the constructed nonconvex function $f$ is convex as was the case in Proposition 4.1.

**Proposition 4.3.** *In the Banach space $X = \ell^2(\mathbb{N})$, there exists a proper lsc non-convex function with a dense domain of Fenchel subdifferential.*

**Proof.** Let us consider the functions $g, f : \ell^2(\mathbb{N}) \to \mathbb{R}\cup\{+\infty\}$ defined for $x = (x_i) \in \ell^2(\mathbb{N})$ by

$$g(x) = \sup_{i\neq 1} \{\, i\,|x_i|\,\} \quad \text{and} \quad f(x) = \max\left\{\chi_{(-1,1)}(x_1),\, g(x)\right\},$$

where $\chi_{(-1,1)}(x_1) = 0$ if $x_1 \in (-1,1)$, and $\chi_{(-1,1)}(x_1) = 1$ if not. The functions $f, g$ are clearly lsc and $g$ is convex. Considering the restriction of $f$ to the one-dimensional subspace $\mathbb{R}e^1$ we deduce that $f$ is not convex. Set

$$U = \{(x_i) \in \ell^2(\mathbb{N}) : \exists i \in \mathbb{N}, |x_i| > \frac{1}{i}\}.$$

We obviously have

$$f \geq g \geq 0 \quad \text{and} \quad f\,|_U = g\,|_U\,. \tag{4}$$

Let us remark that the domain of $g$ contains the subspace of almost everywhere null sequences, thus it is dense in $\ell^2(\mathbb{N})$. Since $g$ is a lsc convex function, it follows that $\operatorname{dom}\partial g$ is dense in $\operatorname{dom} g$, and consequently also in $\ell^2(\mathbb{N})$. On the other hand, we conclude from (4) that $\operatorname{dom}\,\partial g \cap U \subset \operatorname{dom}\,\partial f \cap U$. Since $U$ is open and dense, it follows that $\operatorname{dom}\partial f$ is dense in $X$. □

Motivated by the above example, let us consider the following general problem. Given a lsc convex function $g : X \to \mathbb{R}\cup\{+\infty\}$ and a lsc function $f : X \to \mathbb{R}\cup\{+\infty\}$, is it true that

$$\left.\begin{array}{r}\overline{co} f = g \\ \operatorname{dom}\partial f \text{ dense in } \operatorname{dom} g\end{array}\right\} \implies f = g\,? \tag{5}$$

Analogously to Definition 3.5, we introduce the following class of lsc convex functions.

**Definition 4.4.** We say that a lsc convex function $g : X \to \mathbb{R}\cup\{+\infty\}$ belongs to the class $\mathcal{G}_2(X)$, if $g$ does not admit any non trivial lsc $f$ such that $\overline{co}\, f = g$ and $\operatorname{dom}\partial f$ is dense in $\operatorname{dom} g$.

It is easily seen that assertion (5) is related with (2). In particular, if $g \in \mathcal{G}_1(X)$, then obviously (5) holds and $g \in \mathcal{G}_2(X)$. Hence, for any Banach space $X$, we have

$$\mathcal{G}_1(X) \subset \mathcal{G}_2(X). \tag{6}$$

In finite dimensions, in view of Corollary 3.7, $\mathcal{G}_1(X) = \mathcal{G}_2(X)$ and both classes coincide with the class of all proper lsc convex functions. The following proposition shows that inclusion (6) can be strict in infinite dimensions.

**Proposition 4.5.** *Consider the lsc convex function $g : \ell^2(\mathbb{N}) \to \mathbb{R}\cup\{+\infty\}$ defined for every $x = (x_i) \in \ell^2(\mathbb{N})$ by*

$$g(x) = \sum_{i=1}^{+\infty} |x_i|^{3/2}\,.$$

*Then $g \in \mathcal{G}_2(X)$, but $g \notin \mathcal{G}_1(X)$. More precisely, for every lsc function $f : \ell^2(\mathbb{N}) \to \mathbb{R} \cup \{+\infty\}$, the following implication is satisfied*

$$\overline{co} f = g \Longrightarrow f = g.$$

**Proof.** Let us observe that

$$g(x) = \sum_{i=1}^{+\infty} \varphi(x_i), \tag{7}$$

where the function $\varphi : \mathbb{R} \to \mathbb{R}$ defined for all $t \in \mathbb{R}$ by $\varphi(t) = |t|^{3/2}$ is convex, differentiable on $\mathbb{R}$ and twice continuously differentiable on $\mathbb{R} \setminus \{0\}$ with $\varphi''(t) > 0$, for all $t \neq 0$. Clearly the function $g$ is lsc and convex. Since $\varphi(0) = 0$, the domain of $g$ contains the subspace of almost everywhere null sequences, thus it is dense in $\ell^2(\mathbb{N})$. Considering the sequence $x = \left(\sqrt[3]{(\frac{1}{i})^2}\right)_{i \geq 1}$, we conclude that $\operatorname{dom} g \neq \ell^2(\mathbb{N})$. Hence according to Theorem 3.8 we have $g \notin \mathcal{G}_1(X)$.

Let us now prove that $g \in \mathcal{G}_2(X)$. To this end, let $f : \ell^2(\mathbb{N}) \to \mathbb{R} \cup \{+\infty\}$ be a lsc function satisfying $\overline{co} f = g$ (the other assumption in (5) that dom $\partial f$ is dense in dom $g$ will be superfluous in the sequel). By Lemma 3.1, in order to show that $f = g$ it suffices to ensure that $f|_{\operatorname{dom} \partial g} = g|_{\operatorname{dom} \partial g}$.

Let us show that for every $x \in \operatorname{dom} \partial g$ and $p = (p_i) \in \partial g(x)$, there exists $m > 0$ such that for all $u \in X$ with $\|u\| \leq 1$ we have

$$g(x+u) \geq g(x) + \langle p, u \rangle + m\,\|u\|^2. \tag{8}$$

Indeed, let us fix $x \in \operatorname{dom} \partial g$ and $p = (p_i) \in \partial g(x)$. Applying (1) for $y = x + te^i$ (where $t \in \mathbb{R}$ and $\{e^i\}$ is the canonical basis), we have $\varphi(x_i + t) \geq \varphi(x_i) + p_i t$. This inequality shows that $p_i \in \partial \varphi(x_i)$. Since $\varphi$ is convex and differentiable we conclude that

$$p_i = \varphi'(x_i). \tag{9}$$

Choose $m = \frac{1}{2} \min \{\varphi''(t) : \ |t| \ \leq \|x\| + 1,\ t \neq 0\}$; obviously $m > 0$. Consider now any $u = (u_i) \in \ell^2(\mathbb{N})$ with $\|u\| \leq 1$. Then for all $i \geq 1$ a direct calculation gives

$$\varphi(x_i + u_i) = \varphi(x_i) + u_i \varphi'(x_i) + \int_{x_i}^{x_i+u_i} (x_i + u_i - t)\varphi''(t)\, dt, \tag{10}$$

where the above integral is defined as the sum of the generalized integrals $\lim_{\delta \to 0^+} \int_{x_i}^{-\delta} (x_i + u_i - t)$ $\varphi''(t)\, dt$ and $\lim_{\delta \to 0^+} \int_{\delta}^{x_i+u_i} (x_i + u_i - t)\ \varphi''(t)\, dt$ in case that $x_i \leq 0 \leq x_i + u_i$ (and analogously, if $x_i + u_i \leq 0 \leq x_i$). (Note that if $x_i(x_i + u_i) > 0$, then (10) is nothing but the Taylor's integration formula for the twice differentiable function $\varphi$ on the segment $[x_i, x_i + u_i]$ (respectively on $[x_i + u_i, x_i]$, if $u_i < 0$)).

Let us remark that for all $i \geq 1$

$$\int_{x_i}^{x_i+u_i} (x_i + u_i - t)\varphi''(t)\, dt \geq 2m \int_{x_i}^{x_i+u_i} (x_i + u_i - t)\, dt = m\, u_i^2.$$

Combining with (10) we obtain

$$\varphi(x_i + u_i) \geq \varphi(x_i) + u_i\varphi'(x_i) + m\, u_i^2. \tag{11}$$

Adding the above inequalities for all $i$ and recalling (7) and (9), we obtain

$$g(x+u) \geq g(x) + \langle p, u\rangle + m\, \|u\|^2.$$

Hence (8) holds and the proof finishes in view of the following lemma.

**Lemma 4.6.** *Let $g$ be a proper lsc convex function on $X$ and $x \in X$. Suppose that there exist $p \in X^*$, $r > 0$ and $m > 0$ such that for all $u \in X$ with $\|u\| \leq 1$ we have*

$$g(x+u) \geq g(x) + \langle p, u\rangle + m\, \|u\|^r. \tag{12}$$

*Then, for every lsc function $f$ such that $f \geq g$, the following implication is satisfied:*

$$f(x) > g(x) \implies \overline{co} f \neq g. \tag{13}$$

**Proof.** Let us set

$$G(u) = g(x+u) - g(x) - \langle p, u\rangle \quad \text{and} \quad F(u) = f(x+u) - g(x) - \langle p, u\rangle.$$

Thus $G$ is a lsc convex function with $G(0) = 0$, $F$ is a lsc function with $F(0) > 0$ and $F \geq G$. Moreover, relation (12) becomes:

$$m\, \|u\|^r \leq G(u), \tag{14}$$

for all $u \in X$ with $\|u\| \leq 1$. To prove the assertion of the lemma, it suffices to show that $\overline{co} F \neq G$. Let us remark that if $x \in X$ with $\|x\| > 1$, we have

$$G\left(\frac{x}{\|x\|}\right) \leq \frac{1}{\|x\|}G(x) + (1 - \frac{1}{\|x\|})G(0) = \frac{1}{\|x\|}G(x),$$

since $G$ is convex. This shows, in view of (14), that for all $\|x\| > 1$

$$m\|x\| \leq G(x). \tag{15}$$

Let us now prove that there exists $a > 0$ such that $F \geq a$. Indeed, if this were not the case, there would exist a sequence $\{x^n\}$ in $X$ such that for all $n \geq 1$

$$F(x^n) < \frac{1}{n}. \tag{16}$$

Since $G(x^n) \leq F(x^n)$ we conclude thanks to (15) that for $n$ large enough $\|x^n\| \leq 1$. Thus (14) yields $m\|x^n\|^r < \frac{1}{n}$. It follows that the sequence $\{x^n\}$ converges to 0. Letting $n \to \infty$ in (16), and using the fact that $F$ is lsc we conclude that $F(0) \leq 0$. This contradiction shows that what asserted is true.

Thus $F \geq a$, which implies that $\overline{co} F \geq a$. Consequently $\overline{co} F$ and $G$ cannot coincide at 0. This finishes the proof. □

**Remark 4.7.** It is easily seen that, more generally, the lsc convex function $g : \ell^2(\mathbb{N}) \to \mathbb{R} \cup \{+\infty\}$ defined for all $x = (x_i) \in \ell^2(\mathbb{N})$ by $g(x) = \sum_{i=1}^{+\infty} |x_i|^p$ $(1 < p < 2)$ satisfies the conclusion of Proposition 4.5.

The following result, analogous to Theorem 3.8, deals with the case of lsc positively homogeneous convex functions $g$ with dense domain.

**Theorem 4.8.** *Let $g : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc positively homogeneous convex function with a dense domain. Then $g \in \mathcal{G}_2(X)$ if, and only if,* $\operatorname{dom} g = X$.

**Proof.** Suppose that $\operatorname{dom} g \neq X$ and let us prove that $g \notin \mathcal{G}_2(X)$. For that, let us consider the function $f : X \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f(x) = \begin{cases} \max\,\{g(x), 1\} & \text{if } g(x) > 0; \\ g(x) & \text{if } g(x) \leq 0. \end{cases}$$

It is directly seen that $f \geq g$, which implies $\overline{co} f \geq g$ since $g$ is convex, and that for all $x \in X$

$$f(x) = g(x) \Longleftrightarrow x \in D,$$

where $D = \{\, x \in X : g(x) \notin \ (0,1)\,\}$.

Let us show that $\overline{co} f = g$. This equality holds clearly on $D$. Let us now consider any $x \notin D$ and let us set $x^1 = \frac{1}{g(x)}x$, $x^2 = 0$ and $\lambda = g(x) \in (0,1)$. Since $g$ is positively homogeneous, $g(x^1) = 1$ and $g(x^2) = 0$ which implies that both points $x^1$ and $x^2$ belong to $D$. Thus $\overline{co} f(x^i) = g(x^i)$ for $i = 1,2$. Since the functions $g$ and $\overline{co} f$ are convex, and since $x = \lambda x^1 + (1-\lambda)x^2$ we deduce

$$\overline{co} f(x) \leq \lambda \overline{co} f(x^1) + (1-\lambda)\overline{co} f(x^2) = \lambda g(x^1) + (1-\lambda) g(x^2) = g(x).$$

It follows that $\overline{co} f(x) = g(x)$.

Let us show that $f$ is lsc. Since $g$ is lsc, $f \geq g$ and $f\,|_D = g\,|_D$, the function $f$ is obviously lsc at every point of $D$. Let now any $x \notin D$ and let $\{x^n\}$ be a sequence in $X$ converging to $x$. Since $g$ is lsc at $x$ and $g(x) > 0$, we have $g(x^n) > 0$ for $n$ large enough, and consequently $f(x^n) = \max\{g(x^n), 1\} \geq 1 = f(x)$. This shows that $f$ is lsc at $x$.

Let us show that $\operatorname{dom} \partial f$ is dense in $\operatorname{dom} g$. To this end, let us consider any $x \in \operatorname{dom} g$. Since $g$ takes arbitrarily large finite values around $x$ (see the proof of Theorem 3.8), there exists a sequence $\{x^n\}$ in $\operatorname{dom} g$ such that $g(x^n) > 2$ and $\|x^n - x\| < \frac{1}{n}$ for all $n \geq 1$. Using Proposition 2.1, we obtain a sequence $\{y^n\}$ in $\operatorname{dom} \partial g$ such that $g(y^n) > 2$ and $\|x^n - y^n\| < \frac{1}{n}$ for all $n$. Since $f \geq g$ and $f(y^n) = g(y^n)$ for all $n$, by definition of the Fenchel subdifferential we have $\partial g(y^n) \subset \partial f(y^n)$ for all $n$. It follows that the sequence $\{y^n\}$ is included in $\operatorname{dom} \partial f$. Since the sequence $\{y^n\}$ converges to $x$, it follows that $\operatorname{dom} \partial f$ is dense in $\operatorname{dom} g$.

Since $g$ is positively homogeneous and takes finite arbitrarily large (hence in particular positive) values, it follows that $D \neq X$, whence $f \neq g$. Thus $g \notin \mathcal{G}_2(X)$, which proves the "necessity" part.

The "sufficiency" part is a direct consequence of Theorem 3.8 and inclusion (6). □

**Remark 4.9.** The example of the function given in Proposition 4.5 shows that the assumption "$g$ is positively homogeneous" is indispensable in Theorem 4.8.

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# PARTIE II

# Analyse quasi-convexe

*Approche via l'analyse non-lisse*

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**Article [6]**

"Characterization of nonsmooth semistrictly quasiconvex and strictly quasiconvex functions"
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# Characterization of Nonsmooth Semistrictly Quasiconvex and Strictly Quasiconvex Functions[1]

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**Abstract.** New concepts of semistrict quasimonotonicity and strict quasimonotonicity for multivalued maps are introduced. It is shown that a locally Lipschitz map is (semi)strictly quasiconvex if and only if its Clarke subdifferential is (semi)strictly quasimonotone. Finally, an existence result for the corresponding variational inequality problem is obtained.

**Key Words.** Subdifferentials, semistrictly quasiconvex functions, strictly quasiconvex functions.

## 1. Introduction

Among various notions used in generalized convexity, semistrict quasiconvexity is one of the oldest; see for instance Ref. 1, where it was called "strict quasiconvexity." Recently, it regained attention because of its applications to the multicriteria optimization problem; see Refs. 2 and 3 and references therein.

After the work of Karamardian and Schaible in generalized monotonicity (Refs. 4 and 5) and the developments in the area of nonsmooth analysis, there has been an effort to characterize the generalized convexity of functions in terms of the generalized monotonicity of their subdifferential; see for instance Refs. 6–9. In particular, it was shown that a lower semicontinuous function $f$ is quasiconvex if and only if its Clarke–Rockafellar subdifferential is quasimonotone; under the further assumption that the function $f$ is radially continuous, $f$ is pseudoconvex if and only if its Clarke–Rockafellar subdifferential is pseudomonotone. However, for the classes of semistrictly

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quasiconvex and strictly quasiconvex functions, few results are available: in Refs. 10 and 11, a characterization was given for differentiable functions only; in Ref. 9, it was shown that, if the Clarke subdifferential of a locally Lipschitz function $f$ is pseudomonotone (i.e., $f$ is pseudoconvex), then $f$ is semistrictly quasiconvex.

In this paper, we introduce the notions of semistrict quasimonotonicity and strict quasimonotonicity for multivalued operators and show that a locally Lipschitz function $f$ is semistrictly (strictly) quasiconvex if and only if $\partial f$ is semistrictly (strictly) quasimonotone. Various related results and a mixed characterization are also established. In the last section, we give an application to variational inequalities. In particular, we show that, for a semistrictly quasimonotone operator defined on a weakly compact convex set $K$, the dual variational inequality problem has a solution. Since no additional assumptions are used, we thus strengthen some recent relevant results (Ref. 12).

## 2. Notation and Preliminary Results

Let $X$ be a Banach space and let $X^*$ be its topological dual. For any $x \in X$ and $\epsilon > 0$, we denote by $B_\epsilon(x)$ the ball $\{x' \in X: \|x' - x\| \leq \epsilon\}$. The value of the functional $u^* \in X^*$ at the point $u \in X$ will be denoted by $(u^*, u)$.

Given $x, y \in X$, we define the closed line segment

$$[x, y] = \{tx + (1-t)y: 0 \leq t \leq 1\}.$$

The segments $(x, y]$, $[x, y)$, and $(x, y)$ are defined analogously. For any $A \subset X$, we denote by $\mathrm{co}(A)$ the convex hull of $A$.

We shall consider always functions $f: X \to \mathbb{R} \cup \{+\infty\}$ with domain

$$\mathrm{dom}(f) := \{x \in X: f(x) \in \mathbb{R}\} \neq \varnothing.$$

Functions defined on a subset of $X$ will be considered as taking the value $+\infty$ outside this subset. Let $f$ be a lower semicontinuous function. The Clarke–Rockafellar generalized derivative of $f$ at $x_0 \in \mathrm{dom}(f)$ in the direction $d \in X$ is given by (see Ref. 13)

$$f^\uparrow(x_0; d) = \sup_{\epsilon > 0} \limsup_{\substack{x \to_f x_0 \\ t \searrow 0}} \inf_{d' \in B_\epsilon(d)} [f(x + td') - f(x)]/t.$$

Here, $t \searrow 0$ indicates the fact that $t > 0$ and $t \to 0$; $x \to_f x_0$ means that both $x \to x_0$ and $f(x) \to f(x_0)$.

The (Clarke–Rockafellar) subdifferential of $f$ at $x_0 \in \mathrm{dom}(f)$ is defined by

$$\partial f(x_0) = \{x^* \in X^*: (x^*, d) \leq f^\uparrow(x_0, d), \forall d \in X\};$$

if $x_0 \in X \setminus \mathrm{dom}(f)$, then

$$\partial f(x_0) = \varnothing.$$

In this paper, we shall consider mainly functions $f$ which are locally Lipschitz, in which case $f^\dagger$ coincides with the Clarke generalized derivative

$$f^0(x_0; d) = \limsup_{\substack{x \to x_0 \\ t \searrow 0}} [f(x+td) - f(x)]/t.$$

In the latter case, we have $\partial f(x_0) \neq \varnothing$, whenever $x_0 \in \mathrm{dom}(f)$.

A function $f$ is called:

(i) quasiconvex if, for all $x, y \in \mathrm{dom}(f)$, one has

$$f(z) \leq \max\{f(x), f(y)\}, \qquad \forall z \in [x, y];$$

(ii) semistrictly quasiconvex if $\mathrm{dom}(f)$ is convex and, for all $x, y \in \mathrm{dom}(f)$, the following implication holds:

$$f(x) < f(y) \Rightarrow f(z) < f(y), \qquad \forall z \in (x, y);$$

(iii) strictly quasiconvex if, for all $x, y \in \mathrm{dom}(f)$, one has

$$f(z) < \max\{f(x), f(y)\}, \qquad \forall z \in (x, y).$$

A function $f$ is strictly quasiconvex if and only if it is quasiconvex and is not constant on any line segment $[x, y]$ of its domain. A lower semicontinuous, semistrictly quasiconvex function is quasiconvex. Also, any local minimum $x_0 \in \mathrm{dom}(f)$ of a semistrictly quasiconvex function $f$ is a global minimum. For these and other properties of strictly or semistrictly quasiconvex functions, we refer the reader to Ref. 14.

We shall consider also multivalued operators $T: X \to 2^{X^*}$ with nonempty domain

$$D(T) := \{x \in X: T(x) \neq \varnothing\}.$$

A multivalued operator $T$ is called quasimonotone if, for all $x, y \in X$, the following implication holds:

$$\exists x^* \in T(x): (x^*, y-x) > 0 \Rightarrow \forall y^* \in T(y): (y^*, y-x) \geq 0.$$

We recall from Ref. 15 the following characterizations of quasiconvexity for the class of lower semicontinuous functions.

**Theorem 2.1.** *For a lower semicontinuous function $f$, the following statements are equivalent:*

(a) $f$ is quasiconvex;

(b) $\partial f$ is quasimonotone;

(c) for all $x, y \in \mathrm{dom}(f)$,

$$\exists x^* \in \partial f(x): (x^*, y-x) > 0 \Rightarrow f(z) \leq f(y), \qquad \text{for all } z \in [x, y].$$

The following lemma, which reveals an interesting property of lower semicontinuous quasiconvex functions, will be used frequently in the sequel.

**Lemma 2.1.** Let $f: X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous, quasiconvex function. Suppose that, for some $x, y \in \mathrm{dom}(f)$, the function $f$ is constant on the segment $[x, y]$ and that there exists $x^* \in \partial f(x)$ such that $(x^*, y-x) > 0$. Then, the following statements are true:

(i) every $z \in (x, y]$ is a local minimum of the function $f$;
(ii) the point $x$ is not a local minimum;
(iii) for all $z \in (x, y)$ and all $z^* \in \partial f(z)$, we have $(z^*, y-x) = 0$.

**Proof.**

(i) By Theorem 2.1, $\partial f$ is quasimonotone. Let $z \in (x, y]$. Since $(x^*, z-x) > 0$, there exists $\epsilon > 0$ such that $(x^*, z'-x) > 0$, for all $z' \in B_\epsilon(z)$. Applying Theorem 2.1, we infer that

$$f(z') \geq f(x) = f(z).$$

Accordingly, $z$ is a local minimum of $f$.

(ii) Since $(x^*, y-x) > 0$, we have

$$f^{\uparrow}(x; y-x) > 0.$$

From the definition of the Clarke–Rockafellar derivative, it follows that there exist $\epsilon > 0$ and sequences $x_n \to x$ and $t_n \searrow 0$ such that, for all $n \in N$,

$$\inf_{d' \in B_\epsilon(y-x)} [f(x_n + t_n d') - f(x_n)]/t_n > 0. \tag{1}$$

Choose $n \in N$ sufficiently large so that $y - x_n \in B_\epsilon(y-x)$. Then, relation (1) implies that

$$f(x_n + t_n(y - x_n)) > f(x_n).$$

Since $x_n + t_n(y-x_n)$ belongs to the segment $(x_n, y)$ and $f$ is quasiconvex, we infer that

$$f(y) > f(x_n),$$

hence

$$f(x) > f(x_n),$$

and $x$ is not a local minimum.

(iii) Suppose that, for some $z \in (x, y)$ and some $z^* \in \partial f(z)$, we have $(z^*, y-x) \neq 0$. From the assumption $(x^*, y-x) > 0$, it follows that $(x^*, z-x) > 0$, and quasimonotonicity implies that $(z^*, z-x) \geq 0$. Hence, $(z^*, y-x) > 0$ and accordingly $(z^*, y-z) > 0$. From part (ii), we deduce that $z$ is not a local minimum of $f$. This clearly contradicts (i) □

## 3. Semistrictly Quasimonotone Operators

Given a differentiable function $f: C \to \mathbb{R}$, where $C$ is an open convex subset of $\mathbb{R}^n$, it is known (Refs. 10 and 11) that $f$ is semistrictly quasiconvex if and only if the single-valued map $F = \nabla f$ is quasimonotone and, for any distinct $x, y \in C$, one has the following implication:

$$(F(x), y-x) > 0 \Rightarrow \exists z \in ((x+y)/2, y): (F(z), y-x) > 0. \tag{2}$$

Quasimonotone operators satisfying (2) were called in Ref. 10 semistrictly quasimonotone. We now generalize this notion to the multivalued case, considering also the general framework of a Banach space $X$.

**Definition 3.1.** A multivalued operator $T: X \to 2^{X^*}$ is called semistrictly quasimonotone, if it is quasimonotone and, for any distinct $x, y \in D(T)$, one has the following implication:

$$\exists x^* \in T(x): (x^*, y-x) > 0 \Rightarrow$$
$$\exists z \in ((x+y)/2, y), \exists z^* \in T(z): (z^*, y-x) > 0. \tag{3}$$

The above definition will be justified later by Theorem 3.2. Relation (3) has also an equivalent formulation.

**Proposition 3.1.** Relation (3) is equivalent to the following: If $(x^*, y-x) > 0$ for some $x^* \in T(x)$, then the set

$$\{z \in (x, y): (z^*, y-x) > 0, \qquad \text{for some } z^* \in T(z)\}$$

is dense in $[x, y]$.

**Proof.** Suppose that (3) holds and that

$$(x^*, y-x) > 0, \qquad \text{for some } x^* \in T(x).$$

Let $w \in (x, y]$. We define inductively a sequence $(z_n)_n \subseteq (x, w)$ such that

$$\|z_n - w\| \leq (1/2^{n-1}) \|x - w\|$$

and

$$(z_n^*, y-x)>0, \qquad \text{for some } z_n^* \in T(z_n),$$

as follows.

Set $z_1 = x$. If $z_n$ is defined, then we have

$$(z_n^*, w-z_n)>0, \qquad \text{for some } z_n^* \in T(z_n).$$

Applying relation (3), we choose $z_{n+1} \in ((z_n+w)/2, w)$, such that $(z_{n+1}^*, w-z_n)>0$, for some $z_{n+1}^* \in T(z_{n+1})$. Then obviously,

$$(z_{n+1}^*, y-x)>0$$

and

$$\|z_{n+1}-w\| \leq (1/2)\|z_n-w\| \leq (1/2^n)\|x-w\|.$$

Hence, $z_n \to w$ and the proposition follows. □

We shall need the following mixed characterization of semistrictly quasiconvex functions, which is analogous to the characterization of quasiconvex functions, given in Theorem 2.1(c).

**Theorem 3.1.** A locally Lipschitz function $f: X \to \mathbb{R} \cup \{+\infty\}$ is semistrictly quasiconvex if and only if, for any $x, y \in \mathrm{dom}(f)$, the following implication holds:

$$\exists x^* \in \partial f(x): (x^*, y-x)>0 \Rightarrow \forall z \in [x, y): f(z)<f(y). \tag{4}$$

**Proof.** Suppose that (4) holds for all $x, y \in X$. Then, by Theorem 2.1, $f$ is quasiconvex. If $f$ is not semistrictly quasiconvex, then there exists $x, y \in \mathrm{dom}(f)$ and $z \in (x, y)$ such that $f(x)<f(z)=f(y)$. Applying the Lebourg mean-value theorem (see Theorem 4.5 in Ref. 16) to the segment $[x, z]$, we obtain $w \in (x, z)$ and $w^* \in \partial f(w)$, such that

$$(w^*, z-x)=f(z)-f(x)>0.$$

It follows that

$$(w^*, y-w)>0.$$

Since $z \in (w, y)$, relation (4) would then imply $f(z)<f(y)$, a contradiction. Hence, $f$ is semistrictly quasiconvex.

Conversely, suppose that $f$ is semistrictly quasiconvex. To show (4), it is sufficient to show that, if $(x^*, y-x)>0$ for some $x^* \in \partial f(x)$, then $f(x)<f(y)$. Suppose that $f(x) \geq f(y)$. For any $z \in (x, y]$, we have $(x^*, z-x)>0$; since $f$

is quasiconvex, Theorem 2.1 implies that $f(x) \leq f(z)$. In particular, we have $f(x) = f(y)$. It follows easily that $f$ is constant on $[x, y]$.

Since $(x^*, y-x) > 0$, applying Lemma 2.1(i), we get that $y$ is a local minimum; since $f$ is semistrictly quasiconvex, we conclude that $y$ is also a global minimum. This contradicts Lemma 2.1(ii) and the fact that $f(x) = f(y)$. □

**Remark 3.1.** It is clear from the above proof that relation (4) holds also for lower semicontinuous and semistrictly quasiconvex functions.

**Corollary 3.1.** Let $f: X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous and semistrictly quasiconvex function. For two distinct points $x, y \in \text{dom}(f)$, let the function $f$ be constant on the segment $[x, y]$. Then, for any $z \in (x, y)$ and $z^* \in \partial f(z)$, one has $(z^*, y-x) = 0$.

**Proof.** Since $f(x) = f(z)$, applying the implication (4) to the segment $[z, x]$ we conclude that

$$(z^*, x-z) \leq 0, \qquad \text{for all } z^* \in \partial f(z).$$

It follows that $(z^*, x-y) \leq 0$. Similarly, applying (4) to the segment $[z, y]$, we conclude that

$$(z^*, y-x) \leq 0, \qquad \text{for all } z^* \in \partial f(z).$$

Combining the previous inequalities we get the desired result. □

We now give a characterization of semistrictly quasiconvex functions by means of their subdifferential.

**Theorem 3.2.** Let $f: X \to \mathbb{R} \cup \{+\infty\}$ be a locally Lipschitz function. Then, $f$ is semistrictly quasiconvex if and only if $\partial f$ is semistrictly quasimonotone.

**Proof.**

(i) Let first $f$ be semistrictly quasiconvex. Then, $f$ is quasiconvex, hence $\partial f$ is quasimonotone. If

$$(x^*, y-x) > 0, \qquad \text{for some } x, y \in X \text{ and } x^* \in \partial f(x),$$

then Theorem 3.1 implies that $f((x+y)/2) < f(y)$. Applying the Lebourg mean-value theorem, we obtain some $w \in ((x+y)/2, y)$ and $w^* \in \partial f(w)$ such that

$$(1/2)(w^*, y-x) = (w^*, y-(x+y)/2) \\ = f(y) - f((x+y)/2) > 0,$$

as desired.

(ii) Let $\partial f$ be semistrictly quasimonotone. Then, $\partial f$ is quasimonotone, hence $f$ is quasiconvex. Suppose that $f$ is not semistrictly quasiconvex. Then, there exist $x, y \in \mathrm{dom}(f)$ and $z \in (x, y)$ such that $f(x) < f(z) = f(y)$. Since $f$ is quasiconvex, it must be constant on $[z, y]$. From the Lebourg mean-value theorem, we get $x_1 \in (x, y)$ and $x_1^* \in \partial f(x_1)$ such that $(x_1^*, y-x) > 0$. Since $\partial f$ is semistrictly quasimonotone, by Proposition 3.1 there exists $z_1 \in (z, y)$ and $z_1^* \in \partial f(z_1)$ such that $(z_1^*, y-x) > 0$; hence $(z_1^*, y-z_1) > 0$. Lemma 2.1(iii) implies that, for all $w \in (z_1, y)$ and $w^* \in \partial f(w)$, one has $(w^*, y-x) = 0$. This clearly contradicts Proposition 3.1. □

## 4. Strictly Quasimonotone Operators

Let $f: C \to \mathbb{R}$ be a differentiable function, where $C \subseteq \mathbb{R}^n$ is open and convex. It is known (Ref. 10) that $f$ is strictly quasiconvex if and only if the single-valued map $\nabla f$ is quasimonotone and, for any distinct $x, y \in C$, there exists $z \in (x, y)$ such that $(\nabla f(z), y-x) \neq \varnothing$. This leads to the following definition for the multivalued case in the infinite-dimensional setting.

**Definition 4.1.** A multivalued operator $T: X \to 2^{X^*}$ is called strictly quasimonotone, if it is quasimonotone and, for any distinct $x, y \in D(T)$, there exists $z \in (x, y)$ and $z^* \in T(z)$ such that $(z^*, y-x) \neq 0$.

We have the following easy connection to semistrict quasimonotonicity.

**Proposition 4.1.** If the operator $T$ is strictly quasimonotone, then it is semistrictly quasimonotone.

**Proof.** Suppose that $x, y \in D(T)$ and $(x^*, y-x) > 0$ for some $x^* \in T(x)$. Since $T$ is quasimonotone, for all $z \in (x, y)$ and all $z^* \in T(z)$, it follows easily that $(z^*, y-x) \geq 0$. In addition, from Definition 4.1 there exists $w \in ((x+y)/2, y)$, $w^* \in T(w)$ such that $(w^*, y-(x+y)/2) \neq 0$, i.e., $(w^*, y-x) \neq 0$. It follows that $(w^*, y-x) > 0$, hence $T$ is semistrictly quasimonotone. □

Definition 4.1 is justified by the following theorem.

**Theorem 4.1.** Let $f: X \to \mathbb{R} \cup \{+\infty\}$ be a locally Lipschitz function. Then, $f$ is strictly quasiconvex if and only if $\partial f$ is strictly quasimonotone.

**Proof.** If $f$ is strictly quasiconvex, then it is quasiconvex, hence $\partial f$ is quasimonotone. In addition, for any $x, y \in \mathrm{dom}(f)$, the function $f$ cannot be constant on the segment $[x, y]$. Hence, there exists $w \in (x, y]$ such that $f(w) \neq f(x)$. Applying the Lebourg mean-value theorem, we get $z \in (x, w)$ and $z^* \in \partial f(z)$ such that $(z^*, w-x) \neq 0$, i.e., $(z^*, y-x) \neq 0$.

Conversely, suppose that $\partial f$ is strictly quasimonotone. Then, it is semistrictly quasimonotone, hence $f$ is semistrictly quasiconvex. Corollary 3.1 shows that $f$ cannot be finite and constant on any segment $[x, y]$. Hence, $f$ is strictly quasiconvex. □

## 5. Application to Variational Inequalities

Let $K$ be a nonempty, closed and convex subset of $X$, and let $T: K \to 2^{X^*}$ be a multivalued operator. We recall that the variational inequality problem (VIP) is the following:

Find $x_0 \in K$ such that,

$$\forall x \in K,\ \exists x_0^* \in T(x_0): (x_0^*, x - x_0) \geq 0. \tag{5}$$

It is known (see for instance Refs. 12, 17, and 18) that the previous problem is closely related to the following one:

Find $x_0 \in K$ such that,

$$\forall x \in K,\ \forall x^* \in T(x): (x^*, x - x_0) \geq 0. \tag{6}$$

Following Ref. 12, we shall call problem (6) the dual variational inequality problem (DVIP).

It is well known that a solution of DVIP is always a solution of VIP, provided that the operator $T$ is, say, upper hemicontinuous. That is why we shall restrict our attention to DVIP.

In Ref. 12, Theorems 4.1 and 4.2, it was shown that the DVIP (6) has a solution under the following assumptions:

(a) the operator $T$ is quasimonotone and, for every $x, y \in K$, the following implication holds:

$$\exists x^* \in T(x); (x^*, y-x) > 0 \Rightarrow$$
$$\exists z \in ((x+y)/2, y): \forall z^* \in T(z), (z^*, y-x) > 0; \tag{7}$$

(b) $T$ is upper hemicontinuous and its values are $w^*$-compact;
(c) the set $K$ has inner points (see Ref. 19 for the relevant definition);
(d) $K$ is weakly compact, or alternatively there exists a weakly compact subset $W$ of $K$ and a point $x_0 \in W$ such that the following condition holds:

$$\forall x \in K \setminus W, \exists x_0^* \in T(x_0) \text{ such that } (x_0^*, x_0 - x) < 0. \tag{8}$$

We intend to show that assumptions (b) and (c) can be omitted, and that assumption (a) can also be weakened considerably. We recall first the definition of a properly quasimonotone operator, introduced in Ref. 20.

**Definition 5.1.** An operator $T: X \to 2^{X^*}$ is called properly quasimonotone if, for every $x_1, x_2, \ldots, x_n \in X$ and every $y \in \text{co}\{x_1, x_2, \ldots, x_n\}$, there exists $i$ such that

$$\forall x_i^* \in T(x_i): (x_i^*, y - x_i) \leq 0. \tag{9}$$

Choosing $y = (x_1 + x_2/2)$, we see that a properly quasimonotone operator is quasimonotone. Now, we show the following proposition.

**Proposition 5.1.** Any semistrictly quasimonotone operator $T$ is properly quasimonotone.

**Proof.** If $T$ is not properly quasimonotone, then there would exist $x_1, x_2, \ldots, x_n \in K$ and $y = \sum_{i=1}^n \lambda_i x_i$, with $\sum_{i=1}^n \lambda_i = 1$ and $\lambda_i \geq 0$, such that, for each $i = 1, 2, \ldots, n$, there exists $x_i^* \in T(x_i)$ with $(x_i^*, y - x_i) > 0$. It follows that there exists $\epsilon > 0$ such that, for all $y' \in B_\epsilon(y)$, one has

$$(x_i^*, y' - x_i) > 0, \qquad \forall i = 1, 2, \ldots, n. \tag{10}$$

Suppose that $T$ is semistrictly quasimonotone. Then, $(x_1^*, y - x_1) > 0$ implies that there exists $z \in (x_1, y) \cap B_\epsilon(y)$ and $z^* \in T(z)$ such that $(z^*, y - x_1) > 0$; so, in particular, $(z^*, y - z) > 0$. It follows that

$$\sum_{j=1}^n \lambda_j (z^*, x_j - z) = (z^*, y - z) > 0;$$

hence, for some $j = 1, 2, \ldots, n$, we must have $(z^*, x_j - z) > 0$. Since $T$ is quasimonotone, we deduce that

$$(x_j^*, x_j - z) \geq 0, \qquad \text{for all } x_j^* \in T(x_j),$$

thus contradicting (10). □

We note that the converse of Proposition 5.1 does not hold since, for instance, a subdifferential of a locally Lipschitz, quasiconvex function $f$ is properly quasimonotone (in Ref. 20, this was shown even for lower semicontinuous functions), while it is not semistrictly quasimonotone, unless $f$ is semistrictly quasiconvex.

The argument used in the proof of the next theorem is well known; we reproduce it here for the sake of completeness.

**Theorem 5.1.** Let $T: X \to 2^{X^*}$ be a properly quasimonotone operator whose domain contains the closed and convex set $K$. If assumption (d) holds, then the DVIP has a solution.

**Proof.** Define the multivalued mapping $G: K \to 2^{X^*} \backslash \{\varnothing\}$ by

$$G(x) = \{y \in K: \langle x^*, y - x \rangle \leq 0, \text{ for all } x^* \in T(x)\}. \tag{11}$$

For every $x_1, x_2, \ldots, x_n \in K$ and $y \in \text{co}\{x_1, x_2, \ldots, x_n\}$, proper quasimonotonicity implies that $y \in \bigcup_{i=1}^n G(x_i)$. In addition, for each $x \in K$, $G(x)$ is weakly closed; thus, if $K$ is weakly compact, then for each $x \in K$, $G(x)$ is also weakly compact. Otherwise, the coercivity condition (8) gives $G(x_0) \subseteq W$, hence $G(x_0)$ is weakly compact. In both cases, the well-known Ky Fan lemma (Ref. 21) implies that $\bigcap_{x \in K} G(x) \neq \varnothing$. It is clear that any $x_0 \in \bigcap_{x \in K} G(x)$ is a solution of DVIP. □

It is obvious that the above theorem, together with Proposition 5.1, strengthen Theorems 4.1 and 4.2 of Ref. 12, since condition (a) implies that $T$ is semistrictly quasimonotone, hence properly quasimonotone, while conditions (b) and (c) are not used.

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**Article [7]**

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# On the Subdifferentials of Quasiconvex and Pseudoconvex Functions and Cyclic Monotonicity[1]

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**ABSTRACT.** The notions of cyclic quasimonotonicity and cyclic pseudomonotonicity are introduced. A classical result of convex analysis concerning the cyclic monotonicity of the (Fenchel-Moreau) subdifferential of a convex function is extended to corresponding results for the Clarke-Rockafellar subdifferential of quasiconvex and pseudoconvex functions.
The notion of proper quasimonotonicity is also introduced. It is shown that this new notion retains the characteristic property of quasimonotonicity (i.e. a lower semicontinuous function is quasiconvex if and only if its Clarke-Rockafellar subdifferential is properly quasimonotone), while it is also related to the KKM property of multivalued maps; this makes it more useful in applications to variational inequalities.

## 1. Introduction

Let $X$ be a Banach space and $f : X \to R \bigcup \{+\infty\}$ a lower semicontinuous (lsc) function. According to a relatively recent result of Correa, Joffre and Thibault (see [7] for reflexive and [8] for arbitrary Banach spaces), the function $f$ is convex if and only if its Clarke-Rockafellar subdifferential $\partial f$ is monotone. In the same line of research, much work has been done to characterize the generalized convexity of lsc functions by a corresponding generalized monotonicity of the subdifferential. Thus Luc [15] and, independently, Aussel, Corvellec and Lassonde [2], showed that $f$ is quasiconvex if and only if $\partial f$ is quasimonotone. Similarly, Penot and Quang [16] showed that if the function $f$ is also radially continuous, then $f$ is pseudoconvex if and only if $\partial f$ is pseudomonotone (in the sense of Karamardian and Schaible [14], as generalized for multivalued operators by Yao [20]). In section

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2, we review these results, together with some notation and definitions, and show that in most cases the radial continuity assumption is not necessary.

However, since the Clarke-Rockafellar subdifferential of a convex function coincides with the classical Fenchel-Moreau subdifferential [19], it is not only monotone, but also cyclically monotone [17]. In section 3 of this work, we define analogous notions of cyclic quasimonotonicity and cyclic pseudomonotonicity and show that the subdifferential of quasimonotone and pseudomonotone functions have these properties respectively. Cyclic generalized monotonicity is not just a stronger property than the corresponding generalized monotonicity, but it expresses a behavior of a specific kind; In particular, an operator can even be strongly monotone without being cyclically quasimonotone.

Cyclic (generalized) monotonicity describes the behavior of an operator around a "cycle" consisting of a finite number of points. In section 4 we consider instead the convex hull of such a cycle. We show that the definitions of monotone and pseudomonotone operators can be equivalently stated in terms of this convex hull. This is not so for quasimonotone operators; this leads to the introduction of the new notion of a properly quasimonotone operator. We show that this new notion, while retaining the important characteristics of quasimonotonicity (in particular, $f$ is quasiconvex if and only if $\partial f$ is properly quasimonotone) is often easier to handle; in particular, it is closely related to the KKM property of multivalued maps. We show this by an application to Variational Inequalities. In addition, quasimonotonicity and proper quasimonotonicity are identical on one dimensional spaces, which is probably the reason why the latter escaped attention.

## 2. Relations between generalized convexity and generalized monotonicity

We denote by $X^*$ the dual of $X$ and by $(x^*, x)$ the value of $x^* \in X^*$ at $x \in X$. For $x, y \in X$ we set $[x, y] = \{tx + (1-t)y : 0 \le t \le 1\}$ and define $(x, y]$, $[x, y)$ and $(x, y)$ analogously. Given a lsc function $f : X \to R \bigcup \{+\infty\}$ with domain $dom(f) := \{x \in X : f(x) < +\infty\} \ne \emptyset$, the Clarke-Rockafellar generalized derivative of $f$ at $x_0 \in dom(f)$ in the direction $d \in X$ is given by (see [19]):

$$f^{\uparrow}(x_0, d) = \sup_{\varepsilon > 0} \limsup_{\substack{x \to_f x_0 \\ t \searrow 0}} \inf_{d' \in B_\varepsilon(d)} \frac{f(x + td') - f(x)}{t} \tag{2.1}$$

where $B_\varepsilon(d) = \{d' \in X : \|d' - d\| < \varepsilon\}$, $t \searrow 0$ indicates the fact that $t > 0$ and $t \to 0$, and $x \to_f x_o$ means that both $x \to x_o$ and $f(x) \to f(x_o)$.

The Clarke-Rockafellar subdifferential of $f$ at $x_0$ is defined by

$$\partial f(x_0) = \{x^* \in X : (x^*, d) \leq f^{\uparrow}(x_0, d), \forall d \in X\}. \tag{2.2}$$

We recall that a function $f$ is called quasiconvex, if for any $x, y \in X$ and $z \in [x, y]$ we have

$$f(z) \leq \max\{f(x), f(y)\} \tag{2.3}$$

A lsc function $f$ is called pseudoconvex [16], if for every $x, y \in X$, the following implication holds:

$$\exists x^* \in \partial f(x) : (x^*, y - x) \geq 0 \Longrightarrow f(x) \leq f(y) \tag{2.4}$$

It is known [16] that a lsc pseudoconvex function which is also radially continuous (i.e. its restriction to line segments is continuous), is quasiconvex. Both quasiconvexity and pseudoconvexity of functions are often used in Optimization and other areas of applied mathematics when a convexity assumption would be too restrictive [5].

Let $T : X \to 2^{X^*}$ be a multivalued operator with domain $D(T) = \{x \in X : T(x) \neq \emptyset\}$. The operator $T$ is called

(i) cyclically monotone, if for every $x_1, x_2, ..., x_n \in X$ and every $x_1^* \in T(x_1), x_2^* \in T(x_2), ..., x_n^* \in T(x_n)$ we have

$$\sum_{i=1}^{n} (x_i^*, x_{i+1} - x_i) \leq 0 \tag{2.5}$$

(where $x_{n+1} := x_1$).

(ii) monotone, if for any $x, y \in X$, $x^* \in T(x)$ and $y^* \in T(y)$ we have

$$(y^* - x^*, y - x) \geq 0 \tag{2.6}$$

(iii) pseudomonotone, if for any $x, y \in X$, $x^* \in T(x)$ and $y^* \in T(y)$ the following implication holds:

$$(x^*, y - x) \geq 0 \Longrightarrow (y^*, y - x) \geq 0 \tag{2.7}$$

or equivalently,

$$(x^*, y - x) > 0 \Longrightarrow (y^*, y - x) > 0 \tag{2.8}$$

(iv) quasimonotone, if for any $x, y \in X$, $x^* \in T(x)$ and $y^* \in T(y)$ the following implication holds:

$$(x^*, y - x) > 0 \Longrightarrow (y^*, y - x) \geq 0 \tag{2.9}$$

The above properties were listed from the strongest to the weakest. We recall the hitherto known results connecting generalized convexity with generalized monotonicity:

**Theorem 2.1.** *Let $f : X \to R\bigcup\{+\infty\}$ be a lower semicontinuous function. Then*
*(i) $f$ is convex if and only if $\partial f$ is monotone [8]. In this case $\partial f$ is also cyclically monotone (see for instance [17]).*
*(ii) $f$ is quasiconvex if and only if $\partial f$ is quasimonotone (see [2] or [15]).*
*(iii) Let $f$ be also radially continuous. Then $f$ is pseudoconvex if and only if $\partial f$ is pseudomonotone (see [4] or [16]).*

We now show that pseudoconvexity of a function $f$ implies quasiconvexity of $f$ and pseudomonotonicity of $\partial f$, even without the radial continuity assumption:

**Proposition 2.2.** *Let $f : X \to R\bigcup\{\infty\}$ be a lsc, pseudoconvex function with convex domain. Then:*
*(i) $f$ is quasiconvex*
*(ii) $\partial f$ is pseudomonotone.*

**Proof:** (i) Suppose that for some $x_1, x_2 \in dom(f)$ and some $y \in (x_1, x_2)$ we have $f(y) > \max\{f(x_1), f(x_2)\}$. Set $m = \max\{f(x_1), f(x_2)\}$. Since $f$ is lower semicontinuous, there exists some $\varepsilon > 0$ such that $f(y') > m$, for all $y' \in B_\varepsilon(y)$. From (2.4) it follows (see also [4]) that the sets of local and global minimizers of the function $f$ coincide; hence the point $y$ cannot be a local minimizer, so there exists $w \in B_\varepsilon(y)$ such that $f(w) < f(y)$. Applying Zagrodny's Mean Value Theorem [21, Theorem 4.3] to the segment $[w, y]$, we obtain $u \in [w, y)$, a sequence $u_n \to u$ and $u_n^* \in \partial f(u_n)$, such that $(u_n^*, y - u_n) > 0$. Since $y \in co\{x_1, x_2\}$ it follows that $(u_n^*, x_i - u_n) > 0$, for some $i \in \{1, 2\}$. Using relation (2.4) we get

$m \geq f(x_i) \geq f(u_n)$ and, since $f$ is lower semicontinuous, $m \geq f(u)$. This clearly contradicts the fact that $u \in B_\varepsilon(y)$.

(ii) Let $x^* \in \partial f(x)$ be such that $(x^*, y-x) > 0$. By part (i), $f$ is quasiconvex, so applying Theorem 2.1(ii) we conclude that $\partial f$ is quasimonotone. Hence $(y^*, y-x) \geq 0$, for all $y^* \in \partial f(y)$. Suppose to the contrary that for some $y^* \in \partial f(y)$ we have $(y^*, y-x) = 0$. From relation (2.4) we obtain $f(x) \geq f(y)$.

On the other hand, since $f^\uparrow(x; y-x) > 0$, there exists $\varepsilon_1 > 0$, such that for some $x_n \to x$, $t_n \searrow 0$ and for all $y' \in B_{\varepsilon_1}(y)$, we have $f(x_n + t_n(y'-x_n)) > f(x_n)$. Quasiconvexity of $f$ implies $f(y') > f(x_n)$, for every $y' \in B_{\varepsilon_1}(y)$. In particular $f(y') \geq f(x)$ (since $f$ is lsc), hence $f(y') \geq f(y)$. The latter shows that $y$ is a local minimizer, hence a global one. This is a contradiction, since we have at least $f(y) > f(x_n)$.■

It is still an open question whether pseudomonotonicity of $\partial f$ implies pseudoconvexity of $f$, without the radial continuity assumption. As a partial result, we have the following proposition, which will be of use in the next section.

**Proposition 2.3.** *Let $f$ be a lsc function such that $\partial f$ is pseudomonotone. Then $f$ has the following properties:*

*(i) If $0 \in \partial f(x)$, then $x$ is a global minimizer*

*(ii) $\exists x^* \in \partial f(x) : (x^*, y-x) > 0 \Longrightarrow f(y) > f(x)$*

**Proof:** (i) Suppose that $f(y) < f(x)$. Then using again Zagrodny's Mean Value Theorem, we can find a sequence $z_n \to z \in [y, x)$ and $z_n^* \in \partial f(z_n)$, such that $(z_n^*, x - z_n) > 0$. By pseudomonotonicity, $(x^*, x - z_n) > 0$ for all $x^* \in \partial f(x)$, i.e. $0 \notin \partial f(x)$.

(ii) Let us assume that for some $x^* \in \partial f(x)$ we have $(x^*, y-x) > 0$. We may choose $\varepsilon > 0$ such that $(x^*, y'-x) > 0$, for all $y' \in B_\varepsilon(y)$. Since $\partial f$ is obviously quasimonotone, from Theorem 2.1(ii) we conclude that $f$ is quasiconvex; it then follows that $f(y) \geq f(x)$ (see for instance Theorem 2.1 in [4]). Suppose to the contrary that $f(x) = f(y)$. Then $f(y') \geq f(x) = f(y)$, so $f$ has a local minimum at $y$. It follows that $0 \in \partial f(y)$ (see for instance [21, Th.2.2(c)]). However $\partial f$ is pseudomonotone, hence we should have (see relation (2.8)) $(y^*, y-x) > 0$, for all $y^* \in \partial f(y)$, a contradiction.■

## 3. Generalized cyclic monotonicity

We first introduce cyclic quasimonotonicity.

**Definition 3.1.** *An operator $T : X \to 2^{X^*}$ is called cyclically quasimonotone, if for every $x_1, x_2, ..., x_n \in X$, there exists an $i \in \{1, 2, ..., n\}$ such that*

$$(x_i^*, x_{i+1} - x_i) \leq 0, \forall x_i^* \in T(x_i) \tag{3.1}$$

*(where $x_{n+1} := x_1$).*

It is easy to see that a cyclically monotone operator is cyclically quasimonotone, while a cyclically quasimonotone operator is quasimonotone. Cyclic quasimonotonicity is considerably more restrictive than quasimonotonicity (see Example 3.5 below). However, this property characterizes subdifferentials of quasiconvex functions, as shown by the next theorem.

**Theorem 3.2.** *Let $f : X \to R \bigcup \{+\infty\}$ be a lower semicontinuous function. Then $f$ is quasiconvex if and only if $\partial f$ is cyclically quasimonotone.*

**Proof:** In view of Theorem 2.1(ii), we have only to prove that if $f$ is quasiconvex then $\partial f$ is cyclically quasimonotone.

Assume to the contrary that there exist $x_1, x_2, ..., x_n \in D(\partial f)$ and $x_i^* \in \partial f(x_i)$ such that $(x_i^*, x_{i+1} - x_i) > 0$, for $i = 1, 2, ..., n$ (where as usual $x_{n+1} = x_1$). It follows that $f^{\uparrow}(x_i, x_{i+1} - x_i) > 0$. In particular, for every $i$ there exists $\varepsilon_i > 0$, $\delta_i > 0$ such that

$$\limsup_{\substack{x_i' \to_f x_i \\ t \searrow 0}} \inf_{d \in B_{\varepsilon_i}(x_{i+1} - x_i)} \frac{f(x_i' + td) - f(x_i')}{t} > \delta_i > 0. \tag{3.2}$$

We set $\varepsilon = \min\limits_{i=1,2..n} \varepsilon_i$ and $\delta = \min\limits_{i=1,2..n} \delta_i$. For any $y \in B_{\frac{\varepsilon}{2}}(x_i)$ and $x_{i+1}' \in B_{\frac{\varepsilon}{2}}(x_{i+1})$ we have $y - x_{i+1}' \in B_\varepsilon(x_{i+1} - x_i)$; hence we can choose $\bar{x}_i \in B_{\frac{\varepsilon}{2}}(x_i)$ and $t_i \in (0, 1)$ such that

$$\inf_{x_{i+1}' \in B_{\frac{\varepsilon}{2}}(x_{i+1})} \frac{f(\bar{x}_i + t_i(x_{i+1}' - \bar{x}_i)) - f(\bar{x}_i)}{t_i} > \delta > 0 \tag{3.3}$$

or equivalently

$$f(\bar{x}_i + t_i(x_{i+1}' - \bar{x}_i)) > f(\bar{x}_i) + t_i\delta, \ \forall x_{i+1}' \in B_{\frac{\varepsilon}{2}}(x_{i+1}) \tag{3.4}$$

for $i = 1, 2, .., n$.

Now for every $i$ we choose $x'_{i+1} = \bar{x}_{i+1}$, hence (3.4) becomes

$$f(\bar{x_i} + t_i(\bar{x}_{i+1} - \bar{x_i})) > f(\bar{x_i}) + t_i\delta \tag{3.5}$$

for $i = 1, 2, ..., n$.

Since $f$ is quasiconvex, (3.5) implies that

$$f(\bar{x}_{i+1}) \geq f(\bar{x_i} + t_i(\bar{x}_{i+1} - \bar{x_i})) \tag{3.6}$$

for $i = 1, 2, ..., n$. Combining with (3.5) and adding for $i = 1, 2, ..., n$, we get $0 > \delta(\sum_{i=1}^{n} t_i)$, a contradiction.■

In [18] it was proved that the subdifferential of a convex function is a maximal monotone and maximal cyclically monotone operator. An analogous property does not hold for quasiconvex functions, since for the quasiconvex function $f(x) = sgn(x)\sqrt{\mid x \mid}$, $x \in R$, it is known (see [15]) that $\partial f$ is not maximal quasimonotone. The following proposition shows that it is neither maximal cyclically quasimonotone:

**Proposition 3.3.** *Every quasimonotone operator $T : R \to 2^R$ is cyclically quasimonotone.*

**Proof:** We assume to the contrary that the operator $T$ is quasimonotone and there exist $x_1, x_2, ..., x_n \in R$, $x_i^* \in T(x_i)$, such that

$$(x_i^*, x_{i+1} - x_i) > 0 \tag{3.7}$$

for $i = 1, 2, ..., n$ (where $x_{n+1} = x_1$). Set $x_M = \max_{i=1,2,...,n} x_i$. Then relation (3.7) implies that $x_M^* < 0$. On the other hand, since $x_{M-1} < x_M$, we conclude from (3.7) that $x_{M-1}^* > 0$. Thus $(x_{M-1}^*, x_M - x_{M-1}) > 0$, while $(x_M^*, x_M - x_{M-1}) < 0$, which contradicts the definition of quasimonotonicity.■

We now introduce cyclic pseudomonotonicity:

**Definition 3.4.** *An operator $T : X \to 2^{X^*}$ is called cyclically pseudomonotone, if for every $x_1, x_2, ..., x_n \in X$, the following implication holds:*

$$\begin{aligned} \exists i \in \{1, 2, ..., n\}, \exists x_i^* \in T(x_i) : (x_i^*, x_{i+1} - x_i) > 0 \Longrightarrow \\ \exists j \in \{1, 2, ..., n\}, \forall x_j^* \in T(x_j) : (x_j^*, x_{j+1} - x_j) < 0 \end{aligned} \tag{3.8}$$

*(where $x_{n+1} := x_1$).*

One can easily check that every cyclically monotone operator is cyclically pseudomonotone, while every cyclically pseudomonotone operator is pseudomonotone and cyclically quasimonotone. On the other hand, the following example shows that cyclic generalized monotonicity differs essentially from generalized monotonicity:

**Example 3.5.** *Let $T : R^2 \to R^2$ be defined by $T(a,b) = (\frac{a}{2} - b, a + \frac{b}{2})$. Then the operator $T$ is monotone (and even strongly monotone, i.e. satisfies $(T(x) - T(y), x-y) \geq k \|x - y\|^2$ for all $x, y \in R^2$ where $k$ is a constant). In particular, $T$ is pseudomonotone and quasimonotone. However, it is not cyclically quasimonotone, as one sees by considering the points $x_1 = (1,0), x_2 = (0,1), x_3 = (-1,0)$ and $x_4 = (0,-1)$.*

We now show the following strengthening of Theorem 2.1(iii).

**Theorem 3.6.** *Let $f : X \to R\bigcup\{+\infty\}$ be a lsc function. If $f$ is pseudoconvex, then $\partial f$ is cyclically pseudomonotone. Conversely, if $\partial f$ is pseudomonotone and $f$ is radially continuous, then $f$ is pseudoconvex.*

**Proof:** Again we have only to show that if $f$ is pseudoconvex then $\partial f$ is cyclically pseudomonotone. Assume to the contrary that there exist $x_1, x_2, ..., x_n \in D(\partial f)$ and $x_i^* \in \partial f(x_i)$ such that $(x_i^*, x_{i+1} - x_i) \geq 0$, for $i = 1, 2, ..., n$ (where $x_{n+1} = x_1$), while for some $i_o$ and some $x_{i_o}^* \in \partial f(x_{i_o})$ we have

$$(x_{i_o}^*, x_{i_o+1} - x_{i_o}) > 0 \tag{3.9}$$

By the definition of pseudoconvexity (relation (2.4)) we have $f(x_{i+1}) \geq f(x_i)$, for $i = 1, 2, ..., n$, hence all $f(x_i)$ are equal. In particular, $f(x_{i_o+1}) = f(x_{i_o})$, which contradicts (3.9) in view of Proposition 2.3.■

## 4. Proper Quasimonotonicity

The definitions of monotonicity and pseudomonotonicity have an equivalent formulation, which involves a finite cycle of points and its convex hull:

**Proposition 4.1.** *(i) An operator $T$ is monotone, if and only if for any $x_1, x_2, ..., x_n \in X$ and every $y = \sum_{i=1}^n \lambda_i x_i$, with $\sum_{i=1}^n \lambda_i = 1$ and $\lambda_i > 0$, one has*

$$\sum_{i=1}^n \lambda_i \sup_{x_i^* \in T(x_i)} (x_i^*, y - x_i) \leq 0. \tag{4.1}$$

*(ii) An operator $T$ with convex domain $D(T)$ is pseudomonotone, if and only if for any $x_1, x_2, ..., x_n \in X$ and every $y = \sum_{i=1}^{n} \lambda_i x_i$, with $\sum_{i=1}^{n} \lambda_i = 1$ and $\lambda_i > 0$, the following implication holds:*

$$\exists i \in \{1, 2, ...n\}, \exists x_i^* \in T(x_i) : (x_i^*, y - x_i) > 0 \Longrightarrow \\ \exists j \in \{1, 2, ...n\}, \forall x_j^* \in T(x_j) : (x_j^*, y - x_j) < 0. \tag{4.2}$$

**Proof:** If the operator $T$ satisfies condition (4.1) (respectively (4.2)), then by choosing $y = \frac{x_1+x_2}{2}$, we conclude that it is monotone (respectively pseudomonotone). Hence it remains to show the two opposite directions.

Let us first suppose that $T$ is monotone. Then for any $x_1, x_2, ..., x_n \in X$, any $x_i^* \in T(x_i)$ (for $i = 1, 2, ...n$) and any $y = \sum_{j=1}^{n} \lambda_j x_j$, with $\sum_{j=1}^{n} \lambda_j = 1$ and $\lambda_j > 0$, we have:

$$\sum_{i=1}^{n} \lambda_i (x_i^*, y - x_i) = \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j (x_i^*, x_j - x_i) =$$
$$\sum_{i>j} \lambda_i \lambda_j [(x_i^*, x_j - x_i) + (x_j^*, x_i - x_j)] = \sum_{i>j} \lambda_i \lambda_j (x_i^* - x_j^*, x_j - x_i) \leq 0$$

where the last inequality is a consequence of the monotonicity of $T$. Hence $T$ satisfies relation (4.1).

We now suppose that the operator $T$ is pseudomonotone. If relation (4.2) does not hold, then there exist $x_1, x_2, ..., x_n \in X$, $x_i^* \in T(x_i)$ for $i = 1, 2, ...n$, and some $y = \sum_{j=1}^{n} \lambda_j x_j$ with $\sum_{j=1}^{n} \lambda_j = 1$ and $\lambda_j > 0$, such that

$$(x_i^*, y - x_i) \geq 0 \tag{4.3}$$

while for at least one $i$ (say $i = 1$),

$$(x_1^*, y - x_1) > 0. \tag{4.4}$$

In particular we have $x_1, x_2, ..., x_n \in D(T)$, hence $T(y) \neq \emptyset$. Choose any $y^* \in T(y)$. Relations (2.7) and (4.3) show that

$$(y^*, y - x_i) \geq 0 \tag{4.5}$$

for all $y^* \in T(y)$ and all $i$'s. Since $\sum_i \lambda_i (y^*, y - x_i) = 0$, relations (4.5) show that $(y^*, y - x_i) = 0$ for all $i$'s. On the other hand, relation (4.4) together with (2.8) imply that $(y^*, y - x_1) > 0$, a contradiction. ■

In view of the above Proposition, one could seek an equivalent formulation for the definition of quasimonotonicity, which would involve again the convex hull of a finite cycle. However, in contrast to monotone and pseudomonotone operators, this leads to a different, more restrictive definition:

**Definition 4.2.** *An operator $T : X \to 2^{X^*}$ is called properly quasimonotone, if for every $x_1, x_2, ..., x_n \in X$ and every $y = \sum_{i=1}^{n} \lambda_i x_i$, with $\sum_{i=1}^{n} \lambda_i = 1$ and $\lambda_i > 0$, there exists $i$ such that*

$$\forall x_i^* \in T(x_i) : (x_i^*, y - x_i) \leq 0. \tag{4.6}$$

Choosing $y = \frac{x_1+x_2}{2}$, we see that a properly quasimonotone operator is quasimonotone. As in Proposition 3.3, it is easy to show that the converse is true whenever $X = R$; however, it is not true in general, as the following example shows.

**Example 4.3.** *Let $X = R^2$, $x_1 = (0, 1)$, $x_2 = (0, 0)$, $x_3 = (1, 0)$. We define $T : R^2 \to R^2$ by $T(x_1) = (-1, -1)$, $T(x_2) = (1, 0)$, $T(x_3) = (0, 1)$ and $T(x) = 0$ otherwise. It is easy to check that $T$ is quasimonotone but not properly quasimonotone (it suffices to consider $y = \frac{x_1+x_2+x_3}{3}$).*

The class of properly quasimonotone operators, though strictly smaller than the class of quasimonotone operators, is in a sense not much smaller. This is shown in the next proposition.

**Proposition 4.4.** *(i) Every pseudomonotone operator with convex domain is properly quasimonotone.*
*(ii) Every cyclically quasimonotone operator is properly quasimonotone*

**Proof:** (i) This is an obvious consequence of Proposition 4.1(ii).

(ii) Suppose that the operator $T$ is not properly quasimonotone. Then there would exist $x_1, x_2, ..., x_n \in D(T)$, $x_i^* \in T(x_i)$ and $y = \sum_{i=1}^{n} \lambda_i x_i$ with $\lambda_i > 0$, such that

$$(x_i^*, y - x_i) > 0 \tag{4.7}$$

for $i = 1, 2, ..., n$. Set $x_{i(1)} = x_1$. Relation (4.7) implies that $\sum_j \lambda_j (x_{i(1)}^*, x_j - x_{i(1)}) > 0$. It follows that for some $x_j \neq x_1$ we have $(x_{i(1)}^*, x_j - x_{i(1)}) > 0$. We set $x_{i(2)} = x_j$ and apply relation (4.7) again. Continuing in this way, we define a sequence $x_{i(1)}, x_{i(2)}, ...$ such that

$$(x^*_{i(k)}, x_{i(k+1)} - x_{i(k)}) > 0 \tag{4.8}$$

for all $k \in N$.

Since the set $\{x_1, x_2, ..., x_n\}$ is finite, there exist $m, k \in N$, $m < k$ such that $x_{i(k+1)} = x_{i(m)}$. Thus, for the finite sequence of points $x_{i(m)}, x_{i(m+1)}, ..., x_{i(k)}$ relation (4.8) holds. This means that $T$ is not cyclically quasimonotone. ■

Combining Proposition 4.4(ii) and Theorem 3.2, we get the following corollary.

**Corollary 4.5.** *A lower semicontinuous function $f$ is quasiconvex if and only if $\partial f$ is properly quasimonotone.*

The converse of Proposition 4.4 does not hold. For instance, the operator $T$ defined in Example 3.5 is properly quasimonotone (since it is monotone, hence pseudomonotone), but not cyclically quasimonotone. On the other hand, any subdifferential of a continuous quasiconvex function $f$ is properly quasimonotone, but not pseudomonotone unless $f$ is also pseudoconvex. Thus, between the various generalized monotonicity properties we considered, the following strict implications hold, and none other:

| | | |
|---|---|---|
| cyclically monotone | $\longrightarrow$ | monotone |
| $\downarrow$ | | $\downarrow$ |
| cyclically pseudomonotone | $\longrightarrow$ | pseudomonotone |
| $\downarrow$ | | $\downarrow$ |
| cyclically quasimonotone | $\longrightarrow$ | properly quasimonotone |
| | | $\downarrow$ |
| | | quasimonotone |

Note that the implication (pseudomonotone $\rightarrow$ properly quasimonotone) holds under the assumption that the domain of the operator is convex.

We recall that a multivalued mapping $G : X \rightarrow 2^{X^*}$ is called KKM [11], if for any $x_1, x_2, ..., x_n \in X$ and any $y \in co\{x_1, x_2, ..., x_n\}$ one has $y \in \bigcup_i G(x_i)$. It is easy to see that an operator $T : X \rightarrow 2^{X^*}$ is properly quasimonotone if and only if the multivalued mapping $G : X \rightarrow 2^{X^*}$ defined by

$$G(x) = \{y \in K : \sup_{x^* \in T(x)} (x^*, y - x) \leq 0\} \tag{4.9}$$

is KKM. This suggests an obvious application to Variational Inequalities. All known theorems of existence of solutions for quasimonotone Variational Inequality Problems require extra assumptions on the domain of the operator (see [12]) and, in case of a multivalued operator, on its values (see [9]). As the following theorem shows, existence of solutions for properly quasimonotone operators requires very weak assumptions. We first recall from [1] the following definition.

**Definition 4.6.** *The operator $T : X \to 2^{X^*}$ is called upper hemicontinuous, if its restriction to line segments of its domain is upper semicontinuous, when $X^*$ is equipped to the weak-$*$ topology.*

We now have:

**Theorem 4.7.** *Let $K$ be a nonempty, convex and $w$-compact subset of $X$. If $T$ is a properly quasimonotone, upper hemicontinuous operator with $K \subseteq D(T)$, then there exists an $x_0 \in K$, such that for every $x \in K$, there exists $x^* \in T(x_0)$ such that:*

$$(x^*, x - x_0) \geq 0 \tag{4.10}$$

**Proof:** Since the multivalued map $G$ defined by (4.9) is KKM, and the sets $G(x)$ are obviously weakly closed, by Ky Fan's Lemma [10] one has $\bigcap_{x \in K} G(x) \neq \emptyset$. Take any $x_0 \in \bigcap_{x \in K} G(x)$. We shall show that $x_0$ is actually a solution of (4.10).

We assume to the contrary, that for some $x \in K$ and all $x^* \in T(x_0)$ we have $(x^*, x - x_0) < 0$. The set $V = \{x^* \in X^* : (x^*, x - x_0) < 0\}$ is a w*- neighborhood of $T(x_0)$; hence, if we set $x_t = tx + (1-t)x_0$, by the upper hemicontinuity assumption, we have $T(x_t) \in V$ for all $t$ sufficiently small. Since $x_t - x_0 = t(x - x_0)$, this means that $(x^*, x_t - x_0) < 0$ for all $x^* \in T(x_t)$, i.e. $x_0 \notin G(x_t)$. This contradicts the definition of $x_0$.■

We conclude with a final remark. The notion of a quasimonotone operator was introduced to describe a property that characterizes the subdifferential of a lsc quasiconvex function. Since proper quasimonotonicity does exactly the same thing and is directly related to the KKM property, it is possibly a good candidate to replace quasimonotonicity in most theoretical and practical applications.

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# AN APPROPRIATE SUBDIFFERENTIAL FOR QUASICONVEX FUNCTIONS*

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**Abstract.** In this paper we introduce and study a subdifferential that is related to the quasiconvex functions, much as the Fenchel–Moreau subdifferential is related to the convex ones. It is defined for any lower semicontinuous function, through an appropriate combination of an abstract subdifferential and the normal cone to sublevel sets. We show that this "quasiconvex" subdifferential is always a cyclically quasimonotone operator that coincides with the Fenchel–Moreau subdifferential whenever the function is convex, and that under mild assumptions, the density of its domain in the domain of the function is equivalent to the quasiconvexity of the function. We also show that the "quasiconvex" subdifferential of a lower semicontinuous function contains the derivatives of its differentiable quasiaffine supports. As a consequence, it contains the subdifferential introduced by Martínez-Legaz and Sach in a recent paper [*J. Convex Anal.*, 6 (1999), pp. 1–12]. Several other properties and calculus rules are also established.

**Key words.** subdifferential, quasiconvex function, nonsmooth analysis, quasimonotone operator

**AMS subject classifications.** 26B25, 26E15, 90C26, 49J52

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**1. Introduction.** In the last thirty years, several notions of subdifferentials for quasiconvex functions have been proposed. The oldest ones are the Greenberg–Pierskalla subdifferential [6] and the tangential introduced by Crouzeix [4]. These two subdifferentials have in common that they are convex cones, and are therefore too large to give enough information on the function. The lower subdifferential of Plastria [13] is smaller but still unbounded, as are the related $\alpha$-lower subdifferentials [10]. All of these subdifferentials arise in the context of some quasiconvex conjugation scheme. Of a different nature is the weak lower subdifferential [9], which is more in the spirit of nonsmooth analysis in that its support function partially coincides with the directional derivative; however, this set is not quite satisfactory either, as it is even bigger than the lower subdifferential of Plastria. Trying to remedy this drawback, Martínez-Legaz and Sach [11] recently introduced the Q-subdifferential. Given that it is a subset of the Greenberg–Pierskalla subdifferential, it shares with all other quasiconvex subdifferentials the property that its nonemptiness on the domain of a lower semicontinuous function implies quasiconvexity of the function, which justifies the claim that it is a quasiconvex subdifferential; on the other hand, unlike all other subdifferentials previously introduced in quasiconvex analysis, it can be regarded as

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a small set, as it is contained in the Fréchet subdifferential. But this advantage is, at the same time, the main drawback of this subdifferential, as one has to impose rather strong assumptions on a quasiconvex function to ensure the nonemptiness of its Q-subdifferential on a dense subset of the domain.

In view of all these considerations, one can reasonably say that the problem of defining a sufficiently good subdifferential for quasiconvex functions is still open. To solve it, one has first to set the standards that such a concept should meet. In this sense, we can formulate the general principle that a quasiconvex subdifferential should be related to quasiconvex functions in a way similar to the classical Fenchel–Moreau subdifferential's relation to convex functions. Let us be more precise. The Fenchel–Moreau subdifferential is well defined for an arbitrary function, while, under mild conditions, its nonemptiness on a dense subset of the domain of a lower semicontinuous function is equivalent to convexity of the function. Similarly, a quasiconvex subdifferential should be defined for arbitrary functions, but its nonemptiness on the domain of a lower semicontinuous function should be equivalent (under mild assumptions) to quasiconvexity of the function. Another desirable property of any (quasiconvex) subdifferential is that it should reduce to the Fenchel–Moreau subdifferential in the case of convex functions. As we shall prove below, the quasiconvex subdifferential introduced in this paper satisfies all these requirements. Moreover, it is smaller than all previously defined quasiconvex subdifferentials (except the Q-subdifferential), as it is contained in the upper Dini subdifferential.

The new subdifferential is defined through an appropriate combination of an abstract subdifferential (in the sense of the axiomatic scheme of Aussel–Corvellec–Lassonde [2]) and geometrical considerations based on the notion of the normal cone to sublevel sets, in such a way that it retains important properties from both. For instance, for the class of quasiconvex functions our subdifferential is identical (under mild conditions) to the abstract subdifferential, so that it inherits the same calculus rules; on the other hand, for any continuous function $f$, the existence of a nonzero element of the subdifferential at $x_0$ implies that $f$ is "quasiconvex with respect to $x_0$," in the sense that if $x_0 = \lambda x + (1-\lambda)y$, with $0 \le \lambda \le 1$, then $f(x_0) \le \max\{f(x), f(y)\}$.

The rest of the paper is organized as follows. Section 2 establishes the notation and some preliminaries related to the abstract subdifferentials upon which our quasiconvex subdifferential is built. The central part of the paper is section 3, where the quasiconvex subdifferential is introduced and compared with other subdifferentials, and its main properties are discussed.

**2. Notation and preliminaries.** In what follows, $X \neq \{0\}$ will denote a Banach space and $X^*$ its dual. For any $x \in X$ and $x^* \in X^*$ we denote by $\langle x^*, x\rangle$ the value of $x^*$ at $x$. For $x \in X$ and $\varepsilon > 0$ we denote by $B_\varepsilon(x)$ the closed ball centered at $x$ with radius $\varepsilon > 0$, while for $x$, $y \in X$ we denote by $[x, y]$ the closed segment $\{tx+(1-t)y : t \in [0, 1]\}$. The segments $]x, y]$, $[x, y[$, and $]x, y[$ are defined analogously.

Throughout this article we shall deal with proper functions $f : X \to \mathbb{R} \cup \{+\infty\}$ (i.e., functions for which $dom(f) := \{x \in X : f(x) < +\infty\}$ is nonempty). For any $a \in \mathbb{R}$ the sublevel (resp., strict sublevel) set of $f$ corresponding to $a$ is the set $S_a(f) = \{x \in X : f(x) \le a\}$ (resp., $S_a^<(f) = \{x \in X : f(x) < a\}$). We shall use $S_a$ and $S_a^<$ if there is no risk of confusion.

The Fenchel–Moreau subdifferential $\partial^{FM} f(x)$ of $f$ at any $x \in dom(f)$ is defined by the formula

$$\partial^{FM} f(x) := \{x^* \in X^* : f(y) \ge f(x) + \langle x^*, y - x\rangle \quad \forall y \in X\}. \tag{2.1}$$

(If $x \notin dom(f)$, then we set $\partial^{FM} f(x) = \emptyset$.)

Another useful subdifferential is the Greenberg–Pierskalla subdifferential $\partial^{GP} f$, given by

$$\partial^{GP} f(x) = \{x^* \in X^* : \langle x^*, y - x \rangle \geq 0 \Rightarrow f(y) \geq f(x)\}. \tag{2.2}$$

Given a set $C \subseteq X$ and $x \in X$, the normal cone to $C$ at $x$ is by definition the cone

$$N_C(x) = \{x^* \in X^* : \forall y \in C, \langle x^*, y - x \rangle \leq 0\}.$$

Let $N_f(x) := N_{S_{f(x)}}(x)$ (resp., $N_f^<(x) := N_{S^<_{f(x)}}(x)$) be the normal cone to the sublevel (resp., strict sublevel) set corresponding to the value $f(x)$. The following equivalencies are straightforward:

$$x^* \in N_f(x) \Longleftrightarrow (\forall y \in X, \langle x^*, y - x \rangle > 0 \Rightarrow f(y) > f(x)); \tag{2.3}$$

$$x^* \in N_f^<(x) \Longleftrightarrow (\forall y \in X, \langle x^*, y - x \rangle > 0 \Rightarrow f(y) \geq f(x)). \tag{2.4}$$

Combining the above relations it follows that

$$\partial^{GP} f(x) \subseteq N_f^<(x) \text{ and } N_f(x) \subseteq N_f^<(x).$$

Besides $\partial^{FM}$ and $\partial^{GP}$, one can define other subdifferentials which, unlike the former ones, depend only on the local properties of the function $f$. Such is the Fréchet subdifferential $\partial^F f(x)$, defined by

$$\partial^F f(x) := \{x^* \in X^* : f(y) \geq f(x) + \langle x^*, y - x \rangle + o(y - x) \quad \forall y \in X\},$$

where $o : X \to R$ is some real valued function satisfying

$$\lim_{x \to 0} \frac{o(x)}{\|x\|} = 0.$$

Another "local" subdifferential is the upper Dini subdifferential $\partial^{D^+} f$, defined as follows:

$$\partial^{D^+} f(x) = \begin{cases} \{x^* \in X^* : \langle x^*, d \rangle \leq f^{D+}(x, d), \forall d \in X\} & \text{if } x \in dom(f), \\ \emptyset & \text{if } x \notin dom(f), \end{cases}$$

where

$$f^{D^+}(x, d) = \limsup_{t \searrow 0^+} \frac{1}{t} (f(x + td) - f(x)). \tag{2.5}$$

Both the upper Dini and the Fréchet subdifferential belong to a larger class of subdifferentials defined axiomatically. We recall from [2, Definition 2.1] the relevant definition.

Definition 1. *A subdifferential $\partial$ is an operator that associates to any lower semicontinuous (lsc) function $f : X \to \mathbb{R} \cup \{+\infty\}$ and any $x \in X$ a subset $\partial f(x)$ of $X^*$ so that the following properties are satisfied:*

(P1) $\partial f(x) = \partial^{FM} f(x)$, *whenever $f$ is convex;*

(P2) $0 \in \partial f(x)$, *whenever $f$ has a local minimum at $x$; and*

(P3) $\partial(f + g)(x) \subseteq \partial f(x) + \partial g(x)$

*for all convex continuous functions $g$ for which both $\partial g(x)$ and $\partial(-g)(x)$ are nonempty. (Such functions are called $\partial$-differentiable at $x$.)*

Other subdifferentials satisfying the above properties are the Gâteaux, Hadamard, and Clarke–Rockafellar subdifferentials [2].

*Remark* 2. Let us observe, in relation to Property (P1), that

$$\partial^{FM} f \subseteq \partial f \tag{2.6}$$

for any lsc function $f$. Indeed, take any $x_0 \in X$ and any $x^* \in \partial^{FM} f(x_0)$. Then relation (2.1) guarantees that the function

$$g(x) = f(x) - \langle x^*, x - x_0 \rangle$$

has a minimum at $x_0$, which yields in view of (P2) that $0 \in \partial g(x_0)$. Using Properties (P3) and (P1) we now conclude

$$0 \in \partial f(x_0) + \partial(\langle -x^*, \cdot - x_0 \rangle) = \partial f(x_0) - x^*,$$

i.e., $x^* \in \partial f(x_0)$.

For the purposes of the present paper we shall always use a subdifferential $\partial$ such that $\partial \subseteq \partial^{D^+}$.

We further recall from [2, Definition 2.2] the following definition.

DEFINITION 3. *A norm $\|.\|$ on $X$ is said to be $\partial$-smooth if the functions of the form $x \mapsto \sum_n \mu_n \|x - v_n\|^2$ are $\partial$-differentiable, where the sequence $(v_n)$ converges in $X$, $\mu_n \geq 0$, and the series $\sum_n \mu_n$ is convergent.*

We shall always assume that the space $X$ admits a $\partial$-smooth renorming. (Note that this condition is automatically satisfied if $\partial$ is the Clarke–Rockafellar subdifferential; also, all reflexive Banach spaces admit a $\partial^F$-smooth renorming.) In such a case, the following mean value theorem holds [2, Theorem 4.1].

THEOREM 4. *Let $f$ be lsc and $\partial$ be a subdifferential. If $x, y \in X$ and $f(y) > f(x)$, then there exist $z \in [x, y[$ and sequences $(x_n) \subseteq dom(f)$, $(x_n^*) \subseteq X^*$, such that $x_n \to z$, $x_n^* \in \partial f(x_n)$, and*

$$\langle x_n^*, z + t(y - x) - x_n \rangle > 0 \quad \forall t > 0.$$

*In particular, $dom(\partial f)$ is dense in $dom(f)$.*

Subdifferentials can be used to characterize lsc quasiconvex functions. We recall that a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is called *quasiconvex* if its sublevel sets $S_\alpha$ are convex subsets of $X$ for all $\alpha \in \mathbb{R}$. In [1] it has been shown that a function $f$ is quasiconvex if and only if the following property is true:

$$\text{if } x^* \in \partial f(x) \text{ and } \langle x^*, y - x \rangle > 0, \text{ then } f(z) \leq f(y) \quad \forall z \in [x, y]. \tag{2.7}$$

An easy consequence of (2.7) is the following property of lsc quasiconvex functions (for $\partial f \subseteq \partial^{D^+} f$):

$$\text{if } x^* \in \partial f(x) \text{ and } \langle x^*, y - x \rangle > 0, \text{ then } f(y) > f(x). \tag{2.8}$$

Indeed, $x^* \in \partial f(x)$ and $\langle x^*, y - x \rangle > 0$ yield $f^{D^+}(x, y - x) > 0$; hence for some $t > 0$ (suitably small) we have $f(x) < f(x + t(y - x))$. From (2.7) it follows that $f(x + t(y - x)) \leq f(y)$; hence the result.

Next let $T : X \rightrightarrows X^*$ be a multivalued operator. Following [5] we say that $T$ is *cyclically quasimonotone* if for any $n \geq 1$ and any $x_1, x_2, \ldots, x_n \in X$ there exists $i \in \{1, 2, \ldots, n\}$ such that

$$\langle x_i^*, x_{i+1} - x_i \rangle \leq 0 \quad \forall x_i^* \in T(x_i) \tag{2.9}$$

(where $x_{n+1} := x_1$). If we restrict $n$ in (2.9) to $n = 2$, then $T$ is called *quasimonotone*.

**3. The "quasiconvex" subdifferential $\partial^q$.** In this section we introduce the "quasiconvex" subdifferential $\partial^q$ whose definition depends on both local and global properties of the function. We show that this subdifferential seems completely adapted in quasiconvex analysis (as far as one considers that the Fenchel–Moreau subdifferential $\partial^{FM}$ is apt in convex analysis). In subsection 3.1 we compare the subdifferential $\partial^q$ with the one defined recently in [11], while in subsection 3.2 we present some interesting properties of $\partial^q$.

Given an abstract subdifferential $\partial$ (according to Definition 1) contained in $\partial^{D^+}$, we introduce below the "quasiconvex" subdifferential $\partial^q$.

DEFINITION 5. *The quasiconvex subdifferential $\partial^q f : X \rightrightarrows X^*$ of $f$ is defined for all $x \in dom(f)$ as follows:*

$$\partial^q f(x) = \begin{cases} \partial f(x) \cap N_f(x) & \text{if } N_f^<(x) \neq \{0\}, \\ \emptyset & \text{if } N_f^<(x) = \{0\}. \end{cases}$$

*If $x \notin dom f$, then we set $\partial^q f(x) = \emptyset$.*

We present some fundamental properties of $\partial^q$ in the following propositions.

PROPOSITION 6. *For every proper function $f$, the operator $\partial^q f$ is cyclically quasimonotone.*

*Proof.* It is sufficient to show that the operator $N_f$ (relation (2.3)) is cyclically quasimonotone. The proof follows exactly the same pattern as the proof of quasimonotonicity of $N_f$ in [12]. If $x_i \in X$, $i = 1, 2, \ldots, n$, and $x_i^* \in N_f(x_i)$ are such that $\langle x_i^*, x_{i+1} - x_i \rangle > 0$ for all $i$ (where $x_{n+1} \equiv x_1$), then (2.8) implies that $f(x_{i+1}) > f(x_i)$ for all $i$. By transitivity we conclude $f(x_1) > f(x_1)$; hence we have a contradiction. ∎

PROPOSITION 7. *Let $f$ be a radially continuous function (that is, the restriction of $f$ on line segments is continuous). Then*

(i) *for all $x \in dom(f)$ we have*

$$\partial^q f(x) = \begin{cases} \partial f(x) \cap N_f(x) & \text{if } \partial^{GP} f(x) \neq \emptyset, \\ \emptyset & \text{if } \partial^{GP} f(x) = \emptyset. \end{cases}$$

*In particular for any $x \in X$, if $\partial^q f(x) \neq \emptyset$, then $\partial^{GP} f(x) \neq \emptyset$.*

(ii) $\partial^q f(x) \setminus \{0\} \subseteq \partial^{GP} f(x)$.

*Proof.* (i) If $0 \in \partial^{GP} f(x)$, then $\partial^{GP} f(x) = X^*$. Hence, if $\partial^{GP} f(x) \neq \emptyset$, then $N_f^<(x) \neq \{0\}$. So we have only to prove that if $\partial^{GP} f(x) = \emptyset$, then $N_f^<(x) = \{0\}$. Note that from (2.4) we always have $0 \in N_f^<(x)$. Let us show that $N_f^<(x) \setminus \{0\} \subseteq \partial^{GP} f(x)$. To this end, let $x^* \in N_f^<(x) \setminus \{0\}$ and suppose that $\langle x^*, y - x \rangle \geq 0$. Choose $d \in X$ such that $\langle x^*, d \rangle > 0$. For any $t > 0$ one has $\langle x^*, y + td - x \rangle > 0$; hence $f(y + td) \geq f(x)$. Letting $t \to 0$ and using radial continuity we get $f(y) \geq f(x)$, that is, $x^* \in \partial^{GP} f(x)$.

(ii) The second assertion follows from the following inclusions:

$$\partial^q f(x) \setminus \{0\} \subseteq N_f(x) \setminus \{0\} \subseteq N_f^<(x) \setminus \{0\} \subseteq \partial^{GP} f(x).$$

The proof is complete. ◻

PROPOSITION 8. *Suppose that $f$ is lsc and satisfies one of the following conditions:*

(i) *$f$ is convex;*

(ii) *$f$ is quasiconvex and for all $a > \inf f$ the sublevel sets $S_a(f)$ have nonempty interior.*

*Then*

$$\partial f = \partial^q f.$$

*Proof.* It follows directly from Definition 5 that $\partial^q f \subseteq \partial f$. To show that equality holds, consider any $x^* \in \partial f(x)$. Suppose first that $x^* \neq 0$. Then (2.8) and (2.3) entail that $x^* \in N_f(x)$; hence $x^* \in \partial^q(x)$. If now $x^* = 0$, then obviously $x^* \in \partial f(x) \cap N_f(x)$. According to Definition 5 it suffices to ensure that $N_f^<(x) \neq \{0\}$. Indeed, if $x$ is a global minimum, then $N_f^<(x) = X^*$. If $x$ is not a global minimum, then $f$ cannot be convex; hence assumption (ii) holds. It follows that the convex set $S_{f(x)}^<$ has a nonempty interior. Thus by the Hahn–Banach theorem there exists $y^* \in X^* \setminus \{0\}$ such that $\langle y^*, x\rangle \geq \langle y^*, x'\rangle$ for all $x' \in S_{f(x)}^<$. We now conclude that $y^* \in N_f^<(x)$, i.e., $N_f^<(x) \neq \{0\}$. ◻

*Remark.* The same proof shows that Proposition 8 (ii) holds without any assumption on the sublevel sets, in the case of $X$ finite-dimensional.

Note that if $f$ is lsc, quasiconvex, and radially continuous, then $S_a$ has a nonempty interior for all $a > \inf f$. This is a direct consequence of the following proposition.

PROPOSITION 9. *If $f$ is quasiconvex, lsc, and radially continuous, then it is continuous.*

*Proof.* Since $f$ is lsc, it suffices to show that $S_a^<$ is open. For any $x \in S_a^<$, let $b$ be such that $f(x) < b < a$. Since $f$ is radially continuous, for any $y \in X$ we can find $\varepsilon > 0$ such that $]x - \varepsilon y, x + \varepsilon y[ \subseteq S_b$ . Hence $x \in \operatorname{alg\,int} S_b$. For closed convex sets in Banach spaces the algebraic and the topological interior coincide (e.g., [7, p. 139]). It follows that $x \in \operatorname{int} S_b \subseteq \operatorname{int} S_a^<$. Hence $S_a^<$ is open. ◻

The following lemma is in the same spirit.

LEMMA 10. *Let $K \subseteq X$ be closed. If $\operatorname{alg\,int} K \neq \emptyset$, then $\operatorname{int} K \neq \emptyset$.*

*Proof.* Let $x \in \operatorname{algint} K$. Then obviously

$$\bigcup_{n\in\mathbb{N}} n(K - x) = X.$$

By Baire's lemma, there exists $n_0 \in \mathbb{N}$ such that $\operatorname{int}(n_0(K - x)) \neq \emptyset$. We conclude that $\operatorname{int} K \neq \emptyset$. ◻

We are now ready to state the following result.

PROPOSITION 11. *Let $f$ be lsc, and suppose that either $f$ is radially continuous, or $\operatorname{dom}(f)$ is convex and $S_a$ has nonempty interior for all $a > \inf f$.*

(i) *If the set $\{x \in X : N_f^<(x) \neq \{0\}\}$ is dense in $\operatorname{dom}(f)$, then $f$ is quasiconvex.*

(ii) *$f$ is quasiconvex if and only if the domain of $\partial^q f$ is dense in $\operatorname{dom}(f)$.*

*Proof.* (i) To show that $f$ is quasiconvex, it suffices to show that $S_a$ is convex for all $a$ with $\inf f < a < +\infty$. For this it is sufficient to show that any $x \in X \setminus S_a$ can be strictly separated from $S_a$ by means of a closed hyperplane. By Lemma 10, both assumptions imply that $\operatorname{int} S_a \neq \emptyset$. Choose any $y \in \operatorname{int} S_a$.

*Case* 1. Suppose that $f$ is radially continuous. Then the restriction of $f$ on the line segment $[x, y]$ takes all the values between $f(x)$ and $f(y)$. Hence there exists

$z \in ]x, y[$ such that $a < f(z) < +\infty$. In particular, $z \in dom(f)$, so (by assumption) we can find $c^* \in N_f^<(c) \setminus \{0\}$, where $c$ is as close to $z$ as we wish. Since $f$ is lsc we may assume that $f(c) > a$ and $c \in ]x, y'[$ for some $y' \in \operatorname{int} S_a$. Using (2.4) we now obtain

$$\langle c^*, d \rangle > 0 \Rightarrow f(c+d) \geq f(c).$$

For all $w \in S_a$ we have $\langle c^*, w - c \rangle \leq 0$ (otherwise we would have $f(w) \geq f(c) > a$). In particular, $\langle c^*, w - c \rangle \leq 0$ for all $w \in y' + B_\varepsilon(y')$ for a suitable $\varepsilon > 0$. It follows easily that $\langle c^*, y' - c \rangle < 0$, hence $\langle c^*, x - c \rangle > 0$. Summarizing,

$$\langle c^*, w \rangle \leq \langle c^*, c \rangle < \langle c^*, x \rangle \quad \forall w \in S_a.$$

Consequently, $c^*$ separates strictly $S_a$ and $x$.

*Case* 2. Suppose that $dom(f)$ is convex. If $x \notin \overline{dom}(f)$, then we can strictly separate $x$ and $\overline{dom}(f)$ by means of a closed hyperplane. In particular, the same hyperplane strictly separates $x$ and $S_a$.

If $x \in \overline{dom}(f)$, then $[y, x[ \subseteq \operatorname{int} dom(f)$. Since $S_a$ is closed and $x \notin S_a$, there exists $z \in ]y, x[$ such that $a < f(z) < +\infty$. As in Case 1, it now follows that $x$ and $S_a$ can be strictly separated.

(ii) If $f$ is quasiconvex, then by Proposition 8 we conclude $\partial^q f = \partial f$. Hence (by Theorem 4) $dom(\partial^q f)$ is dense in $dom(f)$. Conversely, if $dom(\partial^q f)$ is dense in $dom(f)$, then the set $\{z \in dom(f) : N_f^<(z) \neq \{0\}\}$ is dense in $dom(f)$; hence by (i) the function $f$ is quasiconvex. □

Combining Proposition 8, Proposition 11, and Theorem 4, we obtain the following corollary.

COROLLARY 12. *Let $f$ be an lsc radially continuous function (respectively, $f$ is an lsc function with convex domain and its sublevel sets have nonempty interior). Then the following are equivalent:*

(i) *$f$ is quasiconvex;*

(ii) $\partial^q f = \partial f$*;*

(iii) $\partial^q f$ *satisfies the conclusion of Theorem* 4 *(mean value theorem);*

(iv) $dom(\partial^q f)$ *is dense in* $dom(f)$*.*

**3.1. Comparison of $\partial^q$ with other subdifferentials.** We start with the following result.

PROPOSITION 13. *For any lsc function $f$,*

$$\partial^{FM} f \subseteq \partial^q f \subseteq \partial f. \tag{3.1}$$

*Proof.* The second inclusion follows directly from Definition 5. To prove the first inclusion, consider any $x^* \in \partial^{FM} f(x)$. It is straightforward from (2.3) that $x^* \in N_f(x) \subseteq N_f^<(x)$. Note also that $N_f^<(x) \neq \{0\}$ (if $x^* = 0$, then (2.1) implies that $N_f^<(x) = X^*$). Hence (3.1) follows from Remark 2. □

*Remark* 14. In view of Proposition 8, the inclusion $\partial^q f \subseteq \partial f$ becomes an equality if the function $f$ is quasiconvex and continuous, while both inclusions in (3.1) become equalities if the function $f$ is convex.

We shall further compare $\partial^q$ with the subdifferential $\partial^Q$ introduced recently in [11, Definition 2.1]. Before recalling the definition of the latter, we provide a result concerning the representation of lsc quasiconvex functions by means of *quasiaffine* functions. We recall that a function $f$ is called quasiaffine if it is both quasiconvex and quasiconcave. In contrast to the rest of the paper, in the next proposition we allow the functions to take the value $-\infty$.

PROPOSITION 15. *A function* $f : X \to \mathbb{R}\cup\{+\infty, -\infty\}$ *is lsc quasiconvex if and only if it satisfies*

$$f(x) = \sup_{q\in Q} q(x),$$

*where* $Q$ *is the set of continuous quasiaffine minorants* $q : X \to \mathbb{R}\cup\{+\infty, -\infty\}$ *that are differentiable on* $q^{-1}(\mathbb{R})$.

*Proof.* The "if" part of the statement is obvious, since all continuous quasiaffine functions are lsc quasiconvex, and this class is closed under pointwise suprema. To prove the "only if" part, let $f : X \to \mathbb{R}\cup\{+\infty, -\infty\}$ be lsc quasiconvex and define $g : X \to \mathbb{R}\cup\{+\infty\}$ by $g(x) = e^{f(x)}$ (using the conventions $e^{+\infty} = +\infty$ and $e^{-\infty} = 0$). It follows that $g$ is quasiconvex and nonnegative. Combining [8, Theorem 5.15] with implication (ii)⇒(i) in [8, Theorem 5.1], we conclude that $g$ is the pointwise supremum of the collection of its real valued, differentiable, quasiaffine minorants with bounded derivatives. It follows that $g$ is also the supremum of a collection of continuous nonnegative quasiaffine functions, which are differentiable at all points where their value is positive. Let us observe that $f(x) = \ln g(x)$ (with the conventions $\ln 0 = -\infty$ and $\ln +\infty = +\infty$) and that the logarithmic function

$$\ln : [0, +\infty] \to \mathbb{R} \cup \{+\infty, -\infty\}$$

is continuous, differentiable on $]0, +\infty[$, and increasing. The proposition follows from the observation that the composition $q = \ln \circ\, r$ of ln with a continuous quasiaffine function $r$ which is differentiable at all points $x$ such that $r(x) \in ]0, +\infty[$ yields a continuous quasiaffine function $q$ differentiable on $q^{-1}(\mathbb{R})$. ∎

Given an lsc function $f : X \to \mathbb{R}\cup\{+\infty\}$, let us recall the definition of the subdifferential $\partial^Q f$ given in [11], as follows. The subdifferential $\partial^Q f(x)$ of $f$ at $x \in dom(f)$ is the set of all $x^* \in X^*$ such that for some nondecreasing differentiable function $\varphi : \mathbb{R} \to \mathbb{R}$ (depending on $x^*$), with $\varphi(0) = 0$ and $\varphi'(0) = 1$, the following relation holds:

$$f(y) \geq f(x) + \varphi(\langle x^*, y - x\rangle) \quad \forall y \in X. \tag{3.2}$$

Let us observe that the right-hand part of the above inequality defines a differentiable quasiaffine support function of $f$ at $x$ (i.e., a differentiable quasiaffine function $g$ satisfying $f \geq g$ and $f(x) = g(x)$). Therefore $\partial^Q f(x)$ is contained in the set of the derivatives at $x$ of the differentiable quasiaffine supports of $f$ at $x$.

PROPOSITION 16. *Let* $f : X \to \mathbb{R}\cup\{+\infty\}$ *be lsc, and suppose that* $\partial^F f \subseteq \partial f$.

(i) *If* $x^*$ *is the derivative of a continuous quasiaffine support of* $f$ *at* $x$ *differentiable at* $x$, *then* $x^* \in \partial^q f(x)$.

(ii) $\partial^Q f(x) \subseteq \partial^q f(x)$.

*Proof.* (i) From Theorem 2.31 of [8] it follows that a continuous function $h : X \to \mathbb{R}$ is quasiaffine if and only if there exist $y^* \in X^*$ and a nondecreasing continuous function $\psi : \mathbb{R} \to \mathbb{R}$ such that $h = \psi \circ y^*$. Thus if $h$ is a quasiaffine support of $f$ at $x$, and $x^*$ is the derivative of $h$ at $x$, then $x^* = \psi'(\langle y^*, x\rangle)y^*$. Since $h$ is a support of $f$ at $x$, we obviously have $x^* \in \partial^F f(x)$; thus $x^* \in \partial f(x)$.

Let us first assume that $x^* \neq 0$. Let $y \in X$ be such that $\langle x^*, y - x\rangle > 0$. Since $x^* \in \partial f(x)$ and $h$ is quasiconvex, using (2.8) we conclude that $f(y) \geq h(y) >$

$h(x) = f(x)$. Thus $y \notin S^{<}_{f(x)}(f)$, which proves that $x^* \in N_f(x) \subseteq N^{<}_f(x)$. Hence $x^* \in \partial f(x) \cap N_f(x) = \partial^q f(x)$.

Suppose now that $x^* = 0$. Then obviously $x^* \in \partial f(x) \cap N_f(x)$; hence it suffices to show that $N^{<}_f(x) \neq \{0\}$. This certainly holds if $x$ is a global minimum of $f$. If this is not the case, then $y^* \neq 0$. Let us prove that, in this case, $y^* \in N^{<}_f(x)$. Indeed, for $y \in S^{<}_{f(x)}(f)$ one has $\psi(\langle y^*, y\rangle) \leq f(y) < f(x) = \psi(\langle y^*, x\rangle)$, whence, as $\psi$ is nondecreasing, $\langle y^*, y\rangle < \langle y^*, x\rangle$.

(ii) This portion of the proof follows directly from (i) and (3.2). □

**3.2. Other properties of the subdifferential $\partial^q$.** In this section we establish calculus rules for the quasiconvex subdifferential $\partial^q$. Let us first remark that inside the class of lsc quasiconvex functions whose sublevel sets have nonempty interior, the quasiconvex subdifferential $\partial^q$ inherits calculus rules from the abstract subdifferential $\partial$; see Corollary 12. On the other hand, for any lsc function $f$, Definition 5 yields the following necessary condition for global optimality:

$$f \text{ has a global minimum at } x_0 \implies 0 \in \partial^q f(x_0). \tag{3.3}$$

*Remark.* Thanks to Proposition 8, relation (3.3) holds true also for local minima whenever $f$ is lsc quasiconvex, and for all $a > \inf f$ the sublevel sets $S_a(f)$ have nonempty interior.

Let us further show a calculus rule based on the "supremum," an operation important in quasiconvex analysis.

PROPOSITION 17. *Suppose that $\partial$ is either the upper Dini subdifferential $\partial^{D^+}$ or the Fréchet subdifferential $\partial^F$. Let $\{f_i\}_{i\in I}$ be a family of lsc functions on $X$, and set $f = \sup_{i\in I} f_i$. Then for every $x_0 \in X$*

$$\overline{\mathrm{co}}^{w^*}\left(\bigcup_{i\in I(x_0)} \partial^q f_i(x_0)\right) \subseteq \partial^q f(x_0), \tag{3.4}$$

*where $I(x_0) := \{i \in I : f_i(x_0) = f(x_0)\}$ and $\overline{\mathrm{co}}^{w^*}(K)$ denotes the $w^*$-closed convex hull of $K$.*

*Proof.* Let $x_0 \in X$. If $x_0 \notin dom(f)$, then for all $i \in I(x_0)$, $f_i(x_0) = f(x_0) = +\infty$ and $\partial^q f(x_0) = \partial^q f_i(x_0) = \emptyset$. Hence we may suppose that $x_0 \in dom(f)$. Let us observe that $\partial^q f(x_0)$ is a $w^*$-closed and convex subset of $X^*$. Thus it suffices to show that if $x^* \in \bigcup_{i\in I(x_0)} \partial^q f_i(x_0)$, then $x^* \in \partial^q f(x_0)$. To do so, let $i \in I(x_0)$ and $x^* \in \partial^q f_i(x_0)$. Since $\partial^q f_i(x_0) \neq \emptyset$, we deduce that $N^{<}_{f_i}(x_0) \neq \{0\}$. Using the fact that $f(x_0) = f_i(x_0)$ and $f(x) \geq f_i(x)$ for all $x \in X$, we obtain $N^{<}_f(x_0) \neq \{0\}$. Thus it remains to show (see Definition 5) that $x^* \in \partial^{D^+} f(x_0) \cap N_f(x_0)$ (resp., $x^* \in \partial^F f(x_0) \cap N_f(x_0)$). But this follows easily from the fact that $N_{f_i}(x_0) \subset N_f(x_0)$ and $\partial^{D^+} f_i(x_0) \subset \partial^{D^+} f(x_0)$ (resp., $\partial^F f_i(x_0) \subset \partial^F f(x_0)$). □

*Remark.* (i) Relation (3.4) holds true whenever $\partial$ is an abstract subdifferential satisfying $\partial f(x_0) \subset \partial g(x_0)$, whenever $f(x_0) = g(x_0)$ and $f \leq g$.

(ii) Equality in (3.4) is generally not true, even if $f$ is the supremum of two continuous quasiconvex functions. Indeed, let

$$f_1(x) = \begin{cases} \sqrt{-x} & \text{if } x \leq 0, \\ -\sqrt{x} & \text{if } x > 0, \end{cases}$$

and $f_2 = -f_1$. Then $f(x) = \max\{f_1(x), f_2(x)\} = \sqrt{|x|}$ and $\partial^q f(0) = \mathbb{R}$, while $\partial^q f_1(0) = \partial^q f_2(0) = \emptyset$.

Let us give a special case where (3.4) holds with equality. Suppose that $\{f_1, f_2, \dots, f_k\}$ is a finite family of locally Lipschitz quasiconvex functions on $X$ that are regular (resp., strongly regular) at $x_0$; that is, $\partial^{D^+} f_i(x_0) = \partial^o f_i(x_0)$ (resp., $\partial^F f_i(x_0) = \partial^o f_i(x_0)$), where $\partial^o f_i(x_0)$ stands for the Clarke subdifferential of $f_i$ at $x_0$ [3]. If $f = \max f_i$ and $x^* \in \partial^q f(x_0)$, then obviously $x^* \in \partial^o f(x_0)$; hence by [3, Proposition 2.3.12] $x^* \in \mathrm{co}(\bigcup_{i\in I(x_0)} \partial^o f_i(x_0))$. Thanks to Corollary 12(ii) and the regularity (resp., strong regularity) of each $f_i$, we infer that $\partial^o f_i(x_0) = \partial^q f_i(x_0)$, so equality in (3.4) follows.

A more general result is given in the following proposition.

Proposition 18. *Let $f = \max_{i\in I} f_i$, where $\{f_i\}_{i\in I}$ is a finite set of lsc quasiconvex functions such that for all $a > \inf f_i$ the sublevel sets $S_a(f_i)$ have nonempty interior, and let $x_0 \in X$. Further, let $\partial$ be the upper Dini subdifferential, and assume that for all $i \in I$ and $d \in X$*

$$f_i^{D^+}(x_0, d) = \sup\{\langle x^*, d\rangle : x^* \in \partial f_i(x_0)\}. \tag{3.5}$$

*(This condition is in particular satisfied whenever $f$ is regular, or (Pshenichnyi) quasidifferentiable at $x_0$ with nonempty subdifferential.) Then*

$$\overline{co}^{w^*}\left(\bigcup_{i\in I(x_0)} \partial^q f_i(x_0)\right) = \partial^q f(x_0), \tag{3.6}$$

*where $I(x_0) := \{i \in I : f_i(x_0) = f(x_0)\}$.*

*Proof.* Thanks to Proposition 17, we have only to show the right-hand side inclusion "$\supseteq$". Let us suppose, in seeking a contradiction, that there exists

$$x^* \in \partial^q f(x_0) \setminus \overline{co}^{w^*}\left(\bigcup_{i\in I(x_0)} \partial^q f_i(x_0)\right).$$

Then by the Hahn–Banach theorem there exist $d \in X$ and $\varepsilon > 0$ such that for all $z^* \in \overline{co}^{w^*}(\bigcup_{i\in I(x_0)} \partial^q f_i(x_0))$ we have $\langle x^*, d\rangle > \langle z^*, d\rangle + \varepsilon$. Since $I$ is finite, it can be easily shown that there exists $i \in I$ such that $f^{D^+}(x_0, d) \le f_i^{D^+}(x_0, d)$. Our assumptions imply (see Proposition 8(ii)) that $\partial f_i(x_0) = \partial^q f_i(x_0)$. Since $\partial^q f(x_0) \subseteq \partial f(x_0)$, we get $x^* \in \partial f(x_0)$; that is,

$$f_i^{D^+}(x_0, d) \ge f^{D^+}(x_0, d) \ge \langle x^*, d\rangle > \langle z^*, d\rangle + \varepsilon \quad \forall z^* \in \partial f_i(x_0).$$

This clearly contradicts (3.5). ∎

Note that whenever $X$ is finite-dimensional, the assumption on the sublevel sets is superfluous (see the remark after Proposition 8). The following example shows that the assumption that the family is finite cannot be overcome, even if all $f_i$ are convex and the supremum is actually a maximum at each point.

*Example.* Let $f : \mathbb{R} \to \mathbb{R}$ be the convex function

$$f(x) = \begin{cases} 0 & \text{if } x \le 0, \\ x + x^2 & \text{if } 0 < x. \end{cases}$$

For each $n \in \mathbb{N}$, let $g_n(x)$ be the equation of the straight line which is tangent to the graph of $f$ at $(1/n, f(1/n))$, and let $x_n \in ]0, 1/n[$ be the intersection of this tangent with the $x$-axis. Let us define

$$f_n(x) = \begin{cases} 0 & \text{if } x \le x_n, \\ g_n(x) & \text{if } x_n < x \le \frac{1}{n}, \\ f(x) & \text{if } \frac{1}{n} < x. \end{cases}$$

Then $f_n$ is convex, $f(x) = \max_{n \ge 1} f(x)$ for each $x \in \mathbb{R}$, and $\partial^q f_n(0) = \{0\}$ while $\partial^q f(0) = [0,1]$. Hence (3.6) does not hold.

In what follows, we shall show that $\partial^q$ obeys a chain rule. We start with the corresponding rule for classical subdifferentials.

PROPOSITION 19. *Suppose that $\partial$ is either $\partial^{D^+}$ or $\partial^F$, let $f : X \to \mathbb{R} \cup \{+\infty\}$, and suppose that $g : \mathbb{R} \cup \{+\infty\} \to \mathbb{R} \cup \{+\infty\}$ is nondecreasing.*

(i) *If $g$ is differentiable at $f(x_0)$ for some $x_0 \in dom(f)$, then*

$$g'(f(x_0))\, \partial f(x_0) \subseteq \partial (g \circ f)(x_0). \tag{3.7}$$

(ii) *If, moreover, $f$ is convex and $g'(f(x_0)) > 0$, then* (3.7) *holds with equality.*

*Proof.* (i) Assume first that $\partial = \partial^{D^+}$. Let $a < f^{D^+}(x_0, d)$. It follows from (2.5) that for any $\delta > 0$ there exists $0 < t < \delta$ satisfying

$$\frac{f(x_0 + td) - f(x_0)}{t} > a.$$

Hence $f(x_0 + td) > f(x_0) + at$ and $g(f(x_0 + td)) \ge g(f(x_0) + at)$. Since $g$ is differentiable at $f(x_0)$ it follows that

$$g(f(x_0) + at) = g(f(x_0)) + g'(f(x_0))\, at + o(at),$$

where $\lim_{t \to 0} \frac{o(t)}{t} = 0$. Hence

$$\frac{g(f(x_0 + td)) - g(f(x_0))}{t} \ge a g'(f(x_0)) + \frac{o(at)}{t},$$

which yields $(g \circ f)^{D^+}(x_0, d) \ge a g'(f(x_0))$. Consequently,

$$g'(f(x_0))\, f^{D^+}(x_0, d) \le (g \circ f)^{D^+}(x_0, d);$$

hence (3.7) holds.

Assume now that $\partial = \partial^F$ and take any $x^* \in \partial^F f(x_0)$. Then

$$\liminf_{\|u\| \searrow 0} \frac{f(x_0 + u) - f(x_0) - \langle x^*, u \rangle}{\|u\|} \ge 0.$$

Let $a < 0$. Then there exists $\delta > 0$ such that for all $u \in X$ with $\|u\| < \delta$

$$\frac{f(x_0 + u) - f(x_0) - \langle x^*, u \rangle}{\|u\|} > a.$$

Since $g$ is nondecreasing, the previous inequality implies

$$g(f(x_0 + u)) \ge g(f(x_0) + \langle x^*, u \rangle + a\|u\|),$$

and since $g$ is differentiable at $f(x_0)$,

$$g(f(x_0+u)) \geq g(f(x_0)) + g'(f(x_0))(\langle x^*, u\rangle + a\|u\|) + o(\langle x^*, u\rangle + a\|u\|),$$

where $\lim_{t\to 0} \frac{o(t)}{t} = 0$. Since $(\|x^*\| - a)\|u\| \geq |\langle x^*, u\rangle + a\|u\||$, it follows that

$$\liminf_{\|u\|\searrow 0} \frac{(g\circ f)(x_0+u) - (g\circ f)(x_0) - g'(f(x_0))\langle x^*, u\rangle}{\|u\|} \geq ag'(f(x_0)). \tag{3.8}$$

Since the above relation is true for all $a < 0$, the left-hand side is nonnegative. This implies that $g'(f(x_0))\ x^* \in \partial^F (g\circ f)(x_0)$; hence (3.7) holds.

(ii) Suppose now that $f$ is convex. Then the function $t \to f(x_0+td)$ is right differentiable; hence the same holds also for the function $t \to (g\circ f)(x_0+td)$. It follows from the usual chain rule for differentiable functions that

$$g'(f(x_0))\, f^{D^+}(x_0, d) = (g\circ f)^{D^+}(x_0, d). \tag{3.9}$$

Hence if $\partial = \partial^{D^+}$, then (3.7) holds with equality.

Suppose now that $\partial = \partial^F$. It is sufficient to show that if $x^* \notin \partial^F f(x_0)$, then $g'(f(x_0))x^* \notin \partial^F (g\circ f)(x_0)$. Since $f$ is convex we have $\partial^F f = \partial^{FM} f$; hence from (2.1) there exists $u \in X$ such that $f(x_0+u) - f(x_0) < \langle x^*, u\rangle$. Choose $a < 0$ such that

$$f(x_0+u) - f(x_0) < \langle x^*, u\rangle + a\|u\|. \tag{3.10}$$

Convexity of $f$ guarantees that the function $t \to \frac{f(x_0+tu)-f(x_0)}{t}$ is nondecreasing for all $t \geq 0$. Thus for any $0 < t < 1$ we infer from (3.10) that

$$f(x_0+tu) - f(x_0) < (\langle x^*, u\rangle + a\|u\|)\, t.$$

Since $g$ is nondecreasing we obtain

$$g(f(x_0+tu)) \leq g(f(x_0) + t\langle x^*, u\rangle + ta\|u\|),$$

and, since $g$ is differentiable at $f(x_0)$,

$$g(f(x_0+tu)) \leq g(f(x_0)) + tg'(f(x_0))(\langle x^*, u\rangle + a\|u\|) + o(t\langle x^*, u\rangle + ta\|u\|),$$

where $\lim_{t\to 0} \frac{o(t)}{t} = 0$. Dividing by $t\|u\|$ and letting $t \to 0$ we deduce

$$\liminf_{t\searrow 0} \frac{(g\circ f)(x_0+tu) - (g\circ f)(x_0) - g'(f(x_0))\langle x^*, tu\rangle}{\|tu\|} \leq ag'(f(x_0)).$$

Since $a < 0$ and $g'(f(x_0)) > 0$, it follows that the left-hand side of (3.8) is negative. Hence $g'(f(x_0))\, x^* \notin \partial^F (g\circ f)(x_0)$. □

Proposition 20. *Let $f : X \to \mathbb{R}\cup\{+\infty\}$ be lsc and $g : \mathbb{R}\cup\{+\infty\} \to \mathbb{R}\cup\{+\infty\}$ be nondecreasing. Assume that the subdifferential $\partial$ satisfies assertions* (i) *and* (ii) *of Proposition* 19 *(for instance, $\partial = \partial^F$ or $\partial^{D^+}$). If $g$ is differentiable at $f(x_0)$ with $g'(f(x_0)) > 0$ for some $x_0 \in dom(f)$, then*

$$g'(f(x_0))\, \partial^q f(x_0) \subseteq \partial^q (g\circ f)(x_0); \tag{3.11}$$

*the above inclusion becomes an equality whenever $f$ is convex.*

*Proof.* Since $g$ is nondecreasing and $g'(f(x_0)) > 0$, we can easily deduce that

$$N_f^{<}(x_0) = N_{g\circ f}^{<}(x_0) \tag{3.12}$$

and

$$N_f(x_0) = N_{g\circ f}(x_0). \tag{3.13}$$

Thus, if $x^* \in \partial^q f(x_0)$, then (3.12) yields $N_{g\circ f}^{<}(x_0) \neq \emptyset$. Since $\partial^q f \subseteq \partial f$, we infer from (3.7) that

$$g'(f(x_0))\, x^* \in \partial(g\circ f)(x_0).$$

Besides, since $x^* \in N_f(x_0)$ and $N_{g\circ f}(x_0)$ is a cone, (3.13) implies

$$g'(f(x_0))\, x^* \in N_{g\circ f}(x_0).$$

Hence (3.11) holds.

If now $f$ is convex, then, by Proposition 8, $\partial^q f = \partial^{FM} f = \partial f$. Hence, in order to show the equality in (3.11), we have to show that $\partial^q(g\circ f)(x_0) = \partial(g\circ f)(x_0)$. It suffices to show that if $x^* \in \partial(g\circ f)(x_0)$, then $x^* \in \partial^q(g\circ f)(x_0)$. Since (3.7) holds with equality, we have

$$\frac{x^*}{g'(f(x_0))} \in \partial f(x_0) = \partial^q f(x_0).$$

Hence $N_{g\circ f}^{<}(x_0) = N_f^{<}(x_0) \neq \{0\}$ and (since $N_f(x_0)$ is a cone) $x^* \in N_f(x_0) = N_{g\circ f}(x_0)$. It follows that $x^* \in \partial^q(g\circ f)(x_0)$. ∎

Let $C \subseteq X$ and let us define the (upper Dini tangent) cone $T_{D^+}(C, x_0)$ of $C$ at $x_0 \in C$ as follows:

$$T_{D^+}(C, x_0) = \{u \in X : \exists\delta > 0\text{: } \forall t \in ]0,\delta[,\ x_0 + tu \in C\}.$$

We have the following proposition.

PROPOSITION 21. *Let $f : X \to \mathbb{R}\cup\{+\infty\}$ and $x_0 \in f^{-1}(\mathbb{R})$. Then*

$$\begin{aligned}&\{x^* \in X^* : (x^*, -1) \in N_{\operatorname{epi} f}(x_0, f(x_0))\} \subseteq \partial^q f(x_0)\\ &\subseteq \{x^* \in X^* : (x^*, -1) \in (T_{D^+}(\operatorname{epi} f, (x_0, f(x_0))))^o\}.\end{aligned}$$

*Proof.* The first inclusion follows from (3.1) and the observation that

$$\partial^{FM} f(x_0) = \{x^* \in X^* : (x^*, -1) \in N_{\operatorname{epi} f}(x_0, f(x_0))\}.$$

To prove the second inclusion, since $\partial^q \subseteq \partial \subseteq \partial^{D^+}$ it suffices to show that

$$\partial^{D^+} f(x_0) = \{x^* \in X^* : (x^*, -1) \in (T_{D^+}(\operatorname{epi} f, (x_0, f(x_0))))^o\}.$$

To this end, let $x^* \in \partial^{D^+} f(x_0)$. For any $(u, v) \in T_{D^+}(\operatorname{epi} f, (x_0, f(x_0)))$ there exists $\delta > 0$ such that

$$f(x_0 + tu) \leq f(x_0) + tv$$

for all $t \in ]0, \delta[$. It follows that

$$\langle x^*, u\rangle \leq \limsup_{t\searrow 0} \frac{f(x_0 + tu) - f(x_0)}{t} \leq v,$$

i.e., $(x^*, -1) \in (T_{D^+}(\operatorname{epi} f, (x_0, f(x_0))))^o$.

Conversely, let $x^* \in X^*$ be such that $(x^*, -1) \in (T_{D^+}(\operatorname{epi} f, (x_0, f(x_0))))^o$. For each $u \in X$, set $v = f^{D^+}(x_0, u)$. Then for any $\lambda \in ]v, +\infty[$ we can find $\delta > 0$ such that for all $t \in ]0, \delta[$

$$\frac{f(x_0 + tu) - f(x_0)}{t} \leq \lambda.$$

It follows that $(u, \lambda) \in T_{D^+}(\operatorname{epi} f, (x_0, f(x_0)))$, and hence $\langle x^*, u\rangle \leq \lambda$. Since this is true for all $\lambda \in ]v, +\infty[$, we deduce that $\langle x^*, u\rangle \leq v$; hence $x^* \in \partial^{D^+} f(x_0)$. ∎

Let us finally state the following corollary.

COROLLARY 22. *Let $A \subseteq X$ and denote by $\delta_A : X \to \mathbb{R} \cup \{+\infty\}$ the indicator function of $A$ defined by*

$$\delta_A(x) = \begin{cases} 0 & \text{if } x \in A, \\ +\infty & \text{if } x \notin A. \end{cases}$$

*For all $x_0 \in A$ we have*

$$\partial^q \delta_A(x_0) = N_A(x_0).$$

*Proof.* We have the following equivalencies:

$$\begin{aligned} x^* \in \partial^{FM} \delta_A(x_0) &\Leftrightarrow \forall x \in X,\ \langle x^*, x - x_0\rangle \leq \delta_A(x) - \delta_A(x_0) \\ &\Leftrightarrow \forall x \in A,\ \langle x^*, x - x_0\rangle \leq 0 \Leftrightarrow x^* \in N_A(x_0). \end{aligned}$$

Hence (3.1) implies that $N_A(x_0) \subseteq \partial^q \delta_A(x_0)$. Conversely, if $x^* \in \partial^q \delta_A(x_0)$, then $x^* \in N_{\delta_A}(x_0)$. It is very easy to see that $N_{\delta_A}(x_0) = N_A(x_0)$, and the corollary follows. ∎

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# PARTIE II

# Analyse quasi-convexe

*Approche géométrique*

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**Article [9]**

"Normal characterization of the
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# Normal Characterization of the Main Classes of Quasiconvex Functions

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**Abstract.** In this article we explore the concept of the normal cone to the sublevel sets (or strict sublevel sets) of a function. By slightly modifying the original definition of Borde and Crouzeix, we obtain here a new (but strongly related to the already existent) notion of a normal operator. This technique turns out to be appropriate in Quasiconvex Analysis since it allows us to reveal characterizations of the various classes of quasiconvex functions in terms of the generalized quasimonotonicity of their 'normal' multifunctions.

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**Key words:** normal cone, quasiconvexity, normal characterization.

## 1. Introduction

In recent years, generalized convex functions appear naturally in optimization problems, forming a large domain for possible – but not always apparent – extensions of known results of Convex Analysis [6], e.g. Their particular interest in research stems from the fact that these functions usually enjoy remarkable stability properties, which are not shared by the class of convex functions (see, for instance, [12, 24]), while on the other hand, they retain important properties of convex functions.

The notion of quasiconvexity is one of the oldest and classical concepts in generalized convexity. Quasiconvex functions can be defined in mere geometric terms by postulating the convexity of their sublevel sets. Due to its simple definition this class is often the starting point of the investigations in generalized convexity (see for example [14, 15, 23, 27], etc.). At the same time, the class of quasiconvex functions (or eventually interesting subclasses of it as, for example, the class of the semistrictly quasiconvex functions) meets a large domain of applications in

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Mathematical Economy (see [1, 9] and references therein) and recently in Control Theory ([7, 8]).

Concurrently a great number of generalized monotonicity concepts have been introduced (originated by some works in complementarity problems, see [20]). In case of differentiable functions, these definitions are chosen to be appropriate to guarantee a correspondence between the generalized convexity of a function and the generalized monotonicity of its derivative [31].

After the recent developments in the subdifferential theory of convex analysis and its extensions to the class of lower semicontinuous (in short lsc) functions (see [4] and reference therein) there has been an effort to establish these dual characterizations of the various classes of quasiconvex functions in terms of the generalized monotonicity of their subdifferentials (see [19] for the Lipschitzian case, and [3, 21, 2, 26] and [17] for the lsc case). This interest was motivated by the fact that the subdifferentials of convex functions are monotone operators.

In the meanwhile a different line of research has been developed in the area of Generalized Convexity. In this line, which was mainly originated by the recent work of Borde and Crouzeix [10], one treats the generalized convex functions by considering and exploiting the notion of the normal cone to the sublevel sets (or strict sublevel sets) of the function, see also [25]. In [10] the authors presented some interesting continuity properties of this normal cone when it is applied on the class of quasiconvex functions, and they subsequently anticipated that this approach was more apt in the framework of generalized convex analysis, by arguing on some analogies with the generalized derivative of convex functions. In the same spirit Penot [25] showed how this notion is related to some known, and largely used in generalized convexity, subdifferentials of 'nonlocal' nature (as for example the subdifferentials of Plastria, of Greenberg-Pierskalla, etc.), revealing in this way a different aspect of the topic. Thereafter, it follows rather naturally that the normal cones of the sublevel sets of a function $f$ form a 'normal' multivalued operator $N_f$, which could eventually play the role of a subdifferential; moreover for this operator one can now anticipate that it could accomplish many needs in Quasiconvex Analysis. However this approach has an – a priori – disadvantage; namely under the originally given definition, this 'normal' operator is always (cyclically) quasimonotone for every function $f$.

This paper aims to overcome the aforementioned inconvenience. By slightly modifying the original idea of Borde and Crouzeix [10], we obtain here a new (but strongly related to the already existent) notion of a normal operator, the quasimonotonicity of which actually characterizes the classes of continuous quasiconvex functions. We are also able to provide similar characterizations for the classes of (semi)strictly quasiconvex functions by means of the (semi)strict quasimonotonicity of this 'normal' multifunction. The main difference in our approach is the fact that the normal cone is not applied directly to the sublevel sets (as was the case in [10, 25]), but it is considered to the Clarke tangential cone of them. This provides us the possibility to explore the properties of the latter cone, especially the fact that it is always closed and convex.

The results of this work can be partially considered as a confirmation of what was conjectured in [10], i.e. that this cone normality technique is rather naturally related with the quasiconvexity.

The paper is organized as follows: in Section 2 we fix our notations and we define the 'normal operator'; in Section 3 we establish characterizations of quasiconvexity, semistrict quasiconvexity and strict quasiconvexity in terms of the corresponding monotonicity properties of the normal operator. Finally, in Section 4, we show that, under additional assumptions, these characterizations can be obtained by replacing the normal operator by the 'strict normal operator', a notion based on the strict sublevel sets of the function.

## 2. Preliminaries

Let $X$ be a Banach space, $X^*$ its topological dual and $\langle \cdot, \cdot \rangle$ the duality pairing. For any function $f\colon X \to \mathbb{R} \cup \{+\infty\}$ its sublevel sets (resp. its strict sublevel sets) will be denoted by $S_\lambda(f) = \{x \in X : f(x) \leqslant \lambda\}$ (resp. $S^-_\lambda(f) = \{x \in X : f(x) < \lambda\}$). However the simplified notation $S_\lambda$ (resp. $S^-_\lambda$) will also be used whenever no confusion is possible. For any $A \subset X$, we denote by $\operatorname{int} A$ its interior and by $\operatorname{cl}(A)$ its closure. If $\delta > 0$ and $x \in X$, we will denote by $B_\delta(x)$ the open ball centered at $x$ with radius $\delta$. For $x, y \in X$ we set $[x, y] = \{tx + (1-t)y : 0 \leqslant t \leqslant 1\}$ and we define the segments $]x, y]$, $[x, y[$ and $]x, y[$ analogously.

Let us recall that for any nonempty subset $C$ of $X$ and any point $x$ of $X$, the Clarke tangent cone (cf. [11], e.g.) of $C$ at $x$ is defined by

$$d \in T_C(x) \Leftrightarrow \begin{cases} \forall \varepsilon > 0, \exists\, \delta > 0, \exists\, T > 0 \quad \text{such that} \\ \quad \forall x' \in B_\delta(x) \cap C, \forall t \in (0, T),\ (x' + tB_\varepsilon(d)) \cap C \neq \emptyset \end{cases}$$

and the corresponding normal cone $N_C(x)$ by

$$N_C(x) = \{x^* \in X^* : \langle x^*, d \rangle \leqslant 0,\ \forall d \in T_C(x)\}.$$

Essential and well known properties of the Clarke tangent cone are the following

- $T_C(x)$ is a closed convex cone,
- if $C$ is convex, the Clarke tangent cone coincides with the classical Bouligand tangent cone, that is,

$$T_C(x) = \operatorname{cl}\Bigl(\bigcup_{\lambda > 0} \lambda(C - \{x\})\Bigr).$$

We are now in position to define the main tool of our paper. Borrowing heavily from ideas developed in [10], we associate to any lower semicontinuous function $f\colon X \to \mathbb{R} \cup \{+\infty\}$ a multivalued operator $N_f\colon X \to 2^{X^*}$ defined as follows:

$$N_f(x) = \begin{cases} N_{S_{f(x)}}(x) & \text{if } x \in \operatorname{dom} f, \\ \emptyset & \text{otherwise}. \end{cases}$$

In the sequel we will often use the term *normal multifunction* or *normal operator* in order to refer to this operator.

This normal multifunction is a natural extension of the corresponding concepts developed in Borde and Crouzeix [10]. Indeed, in that paper, a normal operator $N(x)$ was defined to be the set of all $x^* \in X^*$ such that $\langle x^*, y - x\rangle \leqslant 0$ whenever $f(y) \leqslant f(x)$. Hence if the function is quasiconvex both concepts coincide.

However, as recently observed in [25, Prop. 18], the normal operator defined in [10] is always (cyclically) quasimonotone. Under our definition, the latter happens only if the function $f$ is quasiconvex, and it is exactly in this case that the normal operator $N_f$ enjoys almost all the nice properties of the operator introduced in [10]. On the other hand this passage through the tangential cone provides an efficient tool to treat the general case, since one can now exploit the convexity of this latter cone.

## 3. Normal characterizations

In this section we establish 'normal' characterizations for the class of quasiconvex functions (Subsection 3.1) and for the classes of semistrictly and strictly quasiconvex functions (Subsection 3.2), in terms of the normal multifunction $N_f$. In the last part (Subsection 3.3) we show by means of a simple example that this 'normal' duality technique is no longer useful in the convex case. By a second example of the same spirit, we provide a negative answer to the eventual question of integrating the normal operator.

From now on we shall always assume that the function $f$ is at least lower semicontinuous (in short lsc). A function $f$ is called radially continuous if, for any $x$, the restriction of $f$ to any segment through $x$ is continuous at $x$.

We recall the definition of the Clarke–Rockafellar (in short CR) subdifferential $\partial^{\mathrm{CR}}$ of a lsc function $f$, see [29]:

$$\partial^{\mathrm{CR}} f(x) = \left\{x^* \in X^* : \langle x^*, d\rangle \leqslant f^{\uparrow}(x, d) \text{ for all } d \in X\right\},$$

where

$$f^{\uparrow}(x, d) = \sup_{\varepsilon>0} \limsup_{\substack{t\searrow 0\\ y\to_f x}} \inf_{d'\in B_\varepsilon(d)} \frac{1}{t}\big(f\left(y + td'\right) - f(y)\big).$$

It is recalled that $t \searrow 0$ indicates the fact that $t > 0$ and $t \to 0$, while $x \to_f x_0$ means that both $x \to x_0$ and $f(x) \to f(x_0)$.

Another useful subdifferential is the lower Hadamard subdifferential (also called contingent or Dini–Hadamard) $\partial^{H^-}$, which is defined as follows:

$$\partial^{H^-} f(x) = \{x^* \in X^* : \langle x^*, d\rangle \leqslant f^{H^-}(x, d) \text{ for all } d \in X\},$$

where

$$f^{H^-}(x, d) = \liminf_{\substack{t\searrow 0\\ d'\to d}} \frac{1}{t}\big(f\left(x + td'\right) - f(x)\big).$$

Finally we say that a Banach space admits a Gâteaux-smooth renorm if it admits a renorm which is Gâteaux differentiable on $X \setminus \{0\}$. Classical examples of such spaces are separable Banach spaces and reflexive Banach spaces.

Before proceeding to the normal characterizations of the quasiconvex functions, we prove the following useful lemma:

LEMMA 3.1. *Let $C$ be a nonempty closed subset of $X$ and let $\psi_C$ be its indicator function (i.e. $\psi_C(x) = 0$, if $x \in C$ and $\psi_C(x) = +\infty$, if $x \notin C$). Then*

(a) *$N_{\psi_C}(x) = \partial^{\mathrm{CR}}\psi_C(x)$, for any $x \in X$.*
(b) *Consider any $x \in C$ and $d \in X$. Then the following statements are equivalent:*

  (i) *$\psi_C^{\uparrow}(x, d) = +\infty$,*
  (ii) *$d \notin T_C(x)$,*
  (iii) *$\exists\, x^* \in N_{\psi_C}(x) : \langle x^*, d\rangle > 0$.*

*Proof.* (a) Let us first suppose that $x \in C$. Let $x^* \in \partial^{\mathrm{CR}}\psi_C(x)$ and $h \in X$. If $\langle x^*, h\rangle > 0$ then $\psi_C^{\uparrow}(x, h) > 0$. But, since

$$\psi_C^{\uparrow}(x, d) = \begin{cases} +\infty, & \text{if } d \notin T_C(x), \\ 0, & \text{if } d \in T_C(x) \end{cases} \tag{1}$$

(see [29], e.g.) we obtain that $h \notin T_C(x)$, i.e. $x^* \in N_C(x) = N_{\psi_C}(x)$.

On the other hand, if $x \in C$ and $x^* \in N_{\psi_C}(x) = N_C(x)$, then $\langle x^*, h\rangle \leqslant +\infty = \psi_C^{\uparrow}(x, h)$ if $h \notin T_C(x)$ and $\langle x^*, h\rangle \leqslant 0 = \psi_C^{\uparrow}(x, h)$ if $h \in T_C(x)$. Therefore $x^* \in \partial^{\mathrm{CR}}\psi_C(x)$. If now $x \notin C$ then both $N_{\psi_C}(x)$ and $\partial^{\mathrm{CR}}\psi_C(x)$ are empty.

(b) The implication (i) $\Rightarrow$ (ii) is a direct consequence of (1). (ii) $\Rightarrow$ (iii) can be easily proved by applying a strong separation argument to the closed convex subsets $T_C(x)$ and $\{d\}$. Finally, (iii) $\Rightarrow$ (i) follows from (a). □

### 3.1. QUASICONVEX FUNCTIONS

Let us now recall that a function $f\colon X \to \mathbb{R} \cup \{+\infty\}$ is called *quasiconvex* if its sublevel sets $S_\lambda$ are convex subsets of $X$. We also recall that a (multivalued) operator $F$ is called *quasimonotone* (on a nonempty subset $K$) if for all $x, y \in K$, $x^* \in F(x)$ and $y^* \in F(y)$ we have

$$\langle x^*, y - x\rangle > 0 \Rightarrow \langle y^*, y - x\rangle \geqslant 0.$$

As mentioned in the introduction, quasiconvexity has the following dual characterization in terms of its subdifferential:

THEOREM 3.2. *Let $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be a lsc function. We consider $\partial$ to be either the CR or the lower Hadamard subdifferential (in the latter case we should also assume that $X$ is a Banach space with a Gâteaux-smooth renorm). The following are equivalent:*

(i) *The function $f$ is quasiconvex.*

(ii) *The operator $\partial f$ is quasimonotone.*

For a proof, see [2]. Note that in the second case, since Gâteaux and lower Hadamard subdifferentials coincide on the class of locally Lipschitz functions, we conclude that $X$ also admits a lower Hadamard smooth renorm (in the sense of [5]).

*Remark*. (1) As a direct consequence of Lemma 3.1 and Theorem 3.2 we obtain

$$C \text{ is convex } \iff u \mapsto N_C(u) \text{ is quasimonotone}$$

providing that $C$ is a closed subset of $X$. It follows that a function $f$ is quasiconvex if, and only if, for any $x \in X$, the mapping $u \mapsto N_{S_{f(x)}}(u)$ is quasimonotone. But this easy characterization will be of no use in this paper since our aim is to characterize quasiconvexity by the quasimonotonicity of one mapping (and not of a familly of mappings).

(2) Clearly, the forthcoming characterizations of generalized convexity do not follow from the corresponding subdifferential characterizations. Indeed, the relations between $N_f$ and classical subdifferentials (like $\partial^{\text{CR}} f$, $\partial^{H^-} f, \ldots$) are not simple even if the function is supposed to be Lipschitz and quasiconvex. For example, considering the function $f(x) = x^3$ we have

$$\partial^{\text{CR}} f(x) \subsetneq N_f(x), \quad \forall x \in \mathbb{R}$$

and $\text{cone}(\partial^{\text{CR}} f(0)) = \{0\} \subsetneq N_f(0) = \mathbb{R}_+$ whereas for $g$ defined by $g(x) = x$ if $x < 0$ and $g(x) = 0$ if $x \geqslant 0$ we obtain

$$N_f(0) = \{0\} \subsetneq \partial^{\text{CR}} f(0) = [0, 1].$$

(3) In [17] it was shown that, under the assumptions of the previous theorem, $\partial f$ is not only quasimonotone, but also cyclically quasimonotone, in the sense that for every $x_1, x_2, \ldots, x_n \in X$, we have

$$\min_{i \in \{1,2,\ldots,n\}} \sup_{x_i^* \in \partial f(x_i)} \langle x_i^*, x_{i+1} - x_i \rangle \leqslant 0$$

(where $x_{n+1} := x_1$). As shown in [17], the latter notion refines essentially the notion of quasimonotonicity and in fact is an intrinsic property of the subdifferential of a quasiconvex function. As we shall see in the sequel (see Theorem 3.3), this property still holds for the normal multifunction $N_f$ of a quasiconvex function $f$.

It is easily seen that a function $f$ is quasiconvex if and only if for all $x \in \text{dom}\, f$ the function $\psi_x$ is (quasi)convex (where $\psi_x$ denotes the indicator function of the subset $S_{f(x)}$).

THEOREM 3.3. *Let $f \colon X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. Consider the following statements:*

(i) *$f$ is a quasiconvex function.*

(ii) $x, y \in \operatorname{dom} f$, $x^* \in N_f(x)$ *and* $\langle x^*, y - x\rangle > 0 \Rightarrow f(x) < f(y)$.
(iii) $N_f$ *is a cyclically quasimonotone operator.*
(iv) $N_f$ *is a quasimonotone operator.*

*Then we always have* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). *Moreover if, either $f$ is continuous or, $X$ admits a Gâteaux-smooth renorm, then we also have* (iv) $\Rightarrow$ (i), *hence all these four conditions are equivalent.*

*Proof.* (i) $\Rightarrow$ (ii) Indeed, if for some $x^* \in N_f(x) = N_{S_{f(x)}}(x)$ we have $\langle x^*, y - x\rangle > 0$, then $y - x$ is not an element of $T_{S_{f(x)}}(x)$. Since $S_{f(x)}$ is convex, the Clarke tangent cone $T_{S_{f(x)}}(x)$ coincides with the Bouligand cone $\mathrm{cl}(\bigcup_{\lambda>0} \lambda(S_{f(x)} - \{x\}))$ and consequently $y$ cannot be an element of $S_{f(x)}$. Hence $f(x) < f(y)$.

(ii) $\Rightarrow$ (iii) Take any finite family $\{x_1, \dots, x_n\}$ of points of $X$ and suppose that for $i \in \{1, \dots, n\}$, there exists $x_i^* \in N_f(x_i)$ such that $\langle x_i^*, x_{i+1} - x_i\rangle > 0$ where $x_{n+1} = x_1$. The contradiction immediately occurs since from property (ii) we obtain $f(x_1) < f(x_2) < \cdots < f(x_{n+1}) = f(x_1)$.

(iii) $\Rightarrow$ (iv) is obvious.

(iv) $\Rightarrow$ (i) Let us suppose (for a contradiction) that $f$ is not quasiconvex, i.e. for some $x_0 \in \operatorname{dom} f$ the function $\psi_{x_0}$ is not quasiconvex. In view of Theorem 3.2 this implies that the operator $\partial^{\mathrm{CR}}\psi_{x_0}$ is not quasimonotone. Hence there exist $x, y \in \operatorname{dom}\psi_{x_0} = S_{f(x_0)}$, $x^* \in \partial^{\mathrm{CR}}\psi_{x_0}(x)$ and $y^* \in \partial^{\mathrm{CR}}\psi_{x_0}(y)$ satisfying $\langle x^*, y - x\rangle > 0$ and $\langle y^*, x - y\rangle > 0$.

Let us first suppose that $f$ is continuous.

CLAIM. *We have* $f(x) = f(y) = f(x_0)$.

[We obviously have $f(x) \leqslant f(x_0)$. Let us now suppose that $f(x) < f(x_0)$. Then we may find $\delta > 0$ such that $f(u) < f(x_0)$ for all $u \in B_\delta(x)$. It follows that the function $\psi_{x_0}$ is locally constant on $x$, which contradicts the fact that $(\psi_{x_0})^{\uparrow}(x, y - x) > 0$. We thus conclude that $f(x) = f(x_0)$. The equality $f(x_0) = f(y)$ can be proved in the same way.]

Since now $\psi_{x_0} = \psi_x = \psi_y$, $x^*$ is an element of $\partial^{\mathrm{CR}}\psi_{x_0}(x) = \partial^{\mathrm{CR}}\psi_x(x)$, thus by Lemma 3.1(a) $x^* \in N_f(x)$. We similarly conclude that $y^* \in N_f(y)$, furnishing thus a contradiction to the quasimonotonicity of $N_f$.

Suppose now that $X$ has a Gâteaux smooth renorming. Then by Theorem 3.2 we conclude that $\partial^{H^-}\psi_{x_0}$ is not quasimonotone, i.e. there exist $x, y \in \operatorname{dom}\psi_{x_0} = S_{f(x_0)}$, $x^* \in \partial^{H^-}\psi_{x_0}(x)$ and $y^* \in \partial^{H^-}\psi_{x_0}(y)$ satisfying $\langle x^*, y - x\rangle > 0$ and $\langle y^*, x - y\rangle > 0$. Since $S_{f(x)} \subseteq S_{f(x_0)}$, it follows that $\psi_{x_0}(\cdot) \leqslant \psi_x(\cdot)$. We easily conclude that $\psi_{x_0}^{H^-}(x, d) \leqslant \psi_x^{H^-}(x, d)$ for all $d$ in $X$, hence $\partial^{H^-}\psi_{x_0}(x) \subseteq \partial^{H^-}\psi_x(x)$. Hence $x^* \in \partial^{H^-}\psi_x(x) \subseteq \partial^{\mathrm{CR}}\psi_x(x)$ and by Lemma 3.1 $x^* \in N_f(x)$. Similarly $y^* \in N_f(y)$, hence we obtain again a contradiction. □

*Remark.* (1) Implication '(ii) $\Rightarrow$ (iv)' has also been proved in [25].

(2) The last part of the proof of Theorem 3.3 essentially provides all necessary arguments for establishing an analogous statement in terms of the Bouligand normal cone to sublevel sets, associated with the lower Hadamard subdifferential. In

such a case a regularity assumption on the space (i.e. Gâteaux smooth norm) has to be imposed, but in return, the assumption on $f$ can be weakened (from continuity to lsc). However in several results in the sequel where this remark applies (for example in Theorem 3.4), a continuity assumption on $f$ is needed (independently of the considered concept of normal cone), so that a consideration of the Bouligand normal cone (instead of the Clarke normal cone) would yield more restrictive results.

### 3.2. SEMISTRICTLY AND STRICTLY QUASICONVEX FUNCTIONS

We now recall the definitions of semistrict and strict quasiconvexity. A function $f\colon X \to \mathbb{R} \cup \{+\infty\}$ is said to be

– *semistrictly quasiconvex* if $f$ is quasiconvex and for any $x, y \in \operatorname{dom} f$ we have

$$f(x) < f(y) \Longrightarrow f(z) < f(y), \quad \forall z \in [x, y[.$$

– *strictly quasiconvex* if $f$ is quasiconvex and for any $x, y \in \operatorname{dom} f$ and any $z \in ]x, y[$ we have

$$f(z) < \max\{f(x), f(y)\}.$$

From a geometrical point of view if a function $f$ is strictly quasiconvex, then its epigraph does not contain any horizontal part (including segments) whereas if it is semistrictly quasiconvex, then all ‘full dimensional horizontal parts’ of its epigraph correspond to points where the infimal value of the function is attained. These two classes meet many applications in Multicriteria Optimization (see for instance [1, 6, 9, 30]).

Let now $K$ be a nonempty subset of $X$. We recall from [22, Def. 5.1] (see also [16]) the following definitions:

A multivalued map $F\colon K \to 2^{X^*}$ is said to be

– *semistrictly quasimonotone* (on the set $K$), if $F$ is quasimonotone on $K$ and for all $x, y \in K$ and $x^* \in F(x)$ we have

$$\langle x^*, y - x\rangle > 0 \Longrightarrow \exists\, z \in ](x+y)/2, y[,\ \exists\, z^* \in F(z)\ :\ \langle z^*, y - z\rangle > 0.$$

– *strictly quasimonotone* (on the set $K$), if $F$ is quasimonotone on $K$ and for all $x, y \in K$ we have

$$\exists\, z \in ]x, y[,\ \exists\, z^* \in F(z)\ :\ \langle z^*, y - x\rangle \neq 0.$$

It is easy to check (cf. [16, Prop. 4.1]) that every strictly quasimonotone operator is semistrictly quasimonotone.

In [16] it was shown that the (semi)strict quasiconvexity of a locally Lipschitzian function is characterized by the (semi)strict quasimonotonicity of its Clarke–Rockafellar subdifferential. In the following theorem we show that this is also the case for any continuous function, if we consider the normal multifunction $N_f$

instead of the subdifferential $\partial f$ of the function, thus pointing out again that the normal operator is an efficient tool in quasiconvex analysis.

THEOREM 3.4. *Let* $f\colon X \to \mathbb{R} \cup \{+\infty\}$ *be lsc and continuous on its domain* dom $f$. *Then the following statements are equivalent:*

(i) $f$ *is a semistrictly quasiconvex function.*
(ii) $x, y \in \operatorname{dom} f$, $x^* \in N_f(x)$ *and* $\langle x^*, y - x\rangle > 0 \Rightarrow f(z) < f(y)$, $\forall z \in [x, y)$.
(iii) $N_f$ *is a semistrictly quasimonotone operator on* dom $f$.

*Proof.* (i) $\Rightarrow$ (iii) Let us suppose that $f$ is semistrictly quasiconvex. From Theorem 3.3 we conclude that $N_f$ is a quasimonotone operator. Assume now that $x, y \in \operatorname{dom} f$ and $x^* \in N_f(x)$ satisfy $\langle x^*, y - x\rangle > 0$ and take any point $z \in ](x+y)/2, y[$. According to Theorem 3.3(ii) we have $f(x) < f(y)$. Since the function $f$ is semistrictly quasiconvex, it follows that $y \notin S_{f(z)}$. Since now $S_{f(z)}$ is closed and convex, there exists $\delta > 0$ such that $B_\delta(y) \cap S_{f(z)} = \varnothing$. This, together with semistrict quasiconvexity of $f$ yields:

$$\operatorname{conv}(\{z\} \cup B_\delta(y)) \cap S_{f(z)} = \{z\}. \tag{2}$$

Now the first part of the proof is complete since $y - z$ is not an element of $T_{S_{f(z)}}(z)$ and therefore according to Lemma 3.1 there exists $z^* \in N_f(z)$ satisfying $\langle z^*, y - x\rangle > 0$.

(iii) $\Rightarrow$ (ii) Assume that $N_f$ is a semistrictly quasimonotone operator. Then $N_f$ is in particular quasimonotone, hence by Theorem 3.3 we conclude that $f$ is quasiconvex. Let now $x, y \in X$ and $x^* \in N_f(x)$ be such that $\langle x^*, y - x\rangle > 0$. Applying again Theorem 3.3(ii), we get $f(x) < f(y)$. From the definition of semistrict quasimonotonicity, we may conclude (see also [16, Prop. 3.1]) that there exists a dense subset $D$ of the line segment $]x, y[$, such that for all $z' \in D$, we have $\langle z'^*, y - z'\rangle > 0$ for some $z'^* \in N_f(z')$.

Hence for every $z \in [x, y)$ there exists $z' \in D$ and $z'^* \in N_f(z')$ such that $z \in [x, z']$ and $\langle z'^*, y - z'\rangle > 0$. From Theorem 3.3(ii), we immediately obtain $f(z') < f(y)$ and thus, by quasiconvexity of $f$, $f(z) \leqslant \operatorname{Max}\{f(x), f(z')\} < f(y)$.

(ii) $\Rightarrow$ (i). Assumption (ii) clearly implies condition (ii) in Theorem 3.3 and thus the quasiconvexity of $f$. Let now $x, y \in \operatorname{dom} f$ such that $f(x) < f(y)$. We will show that for any $z \in ]x, y[$, we have $f(z) < f(y)$.

We first note that due to the continuity of $f$, there is no loss of generality to assume that $f(x) < f(z)$ for any $z \in ]x, y]$.

Now let us fix an element $\bar{z} \in ]x, y[$, such that $f(\bar{z}) < f(y)$. We may again suppose (with no loss of generality) that $f(\bar{z}) < f(z')$ for any $z' \in ]\bar{z}, y[$. Since $S_{f(\bar{z})}$ is a closed convex subset of $X$ with nonempty interior we can separate (in a large sense) the sets $S_{f(\bar{z})}$ and $]\bar{z}, y]$. Hence there exist $\bar{z}^* \in X^*\backslash\{0\}$ and $\alpha \in \mathbb{R}$ such that $\langle \bar{z}^*, z'\rangle \geqslant \alpha \geqslant \langle \bar{z}^*, x'\rangle$ for all $z' \in ]\bar{z}, y]$ and all $x' \in S_{f(\bar{z})}$. It follows easily that $\langle \bar{z}^*, \bar{z}\rangle = \alpha$, hence we may conclude that $\bar{z}^* \in N_f(\bar{z})$ and $\langle \bar{z}^*, y - \bar{z}\rangle \geqslant 0$.

We claim that the equality $\langle \bar{z}^*, y - \bar{z} \rangle = 0$ is impossible. Indeed, suppose that $\langle \bar{z}^*, y \rangle = \langle \bar{z}^*, \bar{z} \rangle$. Then we also have $\langle \bar{z}^*, x \rangle = \langle \bar{z}^*, \bar{z} \rangle$. On the other hand, for any $u \in S_{f(\bar{z})}$, we have $\langle \bar{z}^*, u \rangle \leqslant \langle \bar{z}^*, \bar{z} \rangle$. Thus since $x$ is an interior point of $S_{f(\bar{z})}$ which maximizes $\bar{z}^*$, we deduce that $\bar{z}^*$ is constant on $S_{f(\bar{z})}$, which means that $\bar{z}^* = 0$, a contradiction.

Thus $\langle \bar{z}^*, y - \bar{z} \rangle > 0$ and by hypothesis (ii) $f(z) < f(y)$ for every $z \in [\bar{z}, y[$.

By combining $f(x) < f(\bar{z})$ and the quasiconvexity of $f$, we get $f(z) \leqslant f(\bar{z}) < f(y)$ for every $z \in [x, \bar{z}]$ and therefore the proof is complete. □

The radial continuity assumption is an – a priori – weak assumption comparing with (full) continuity. However if $X$ is a Banach space it has been proved in [18] (extending a previous result of Crouzeix [13] in finite dimension) that every lsc radially continuous quasiconvex function $f\colon X \to \mathbb{R} \cup \{+\infty\}$ is actually continuous on its domain.

PROPOSITION 3.5 [18]. *Let $X$ be a Banach space and $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be a quasiconvex, lsc and radially continuous function. Then $f$ is continuous on its domain.*

If we assume that $X$ admits a Gâteaux-smooth renorming then, using the above result, we can replace the continuity assumption by 'radial continuity' in the statement of Theorem 3.4, concluding to the following corollary.

COROLLARY 3.6. *Let $X$ be a Banach space with a Gâteaux-smooth renorm and $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be a lsc and radially continuous function. The following statements are equivalent:*

(i) *$f$ is semistrictly quasiconvex and continuous on* dom $f$.
(ii) *$N_f$ is a semistrictly quasimonotone operator on* dom $f$.

Indeed, if $N_f$ is semistrictly quasimonotone then, according to Theorem 3.3, $f$ is quasiconvex and by Proposition 3.5 it is also continuous. The semistrict quasiconvexity is now a direct consequence of Theorem 3.4.

*Remark*. The following example shows that there is no hope to characterize semistrictly quasiconvex functions by the semistrict quasimonotonicity of the normal operator $N_f$ if the function is only assumed to be lower semicontinuous.

Indeed, consider the function $f\colon \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f(x, y) = \begin{cases} |x| + |y| & \text{if } |x| + |y| \leqslant 1, \\ |y| - 2\sqrt{1 - x^2} + 2 & \text{if } |x| \leqslant 1 \text{ and } x^2 + y^2/4 > 1, \\ +\infty & \text{if } |x| > 1, \\ \dfrac{|y| + |x| - 1}{2\sqrt{1 - x^2} - 1 + |x|} + 1 & \text{otherwise.} \end{cases}$$

Since the sublevel sets of $f$ are closed and convex, it follows that $f$ is quasiconvex and lower semicontinuous. Moreover one can also verify that this function is in particular semistrictly quasiconvex. However the associate operator $N_f$ is not semistrictly quasimonotone: indeed, consider for example the points $X = (1, 0)$ and $Y = (1, 1)$. We have

$$N_f(X) = \{\lambda_1 U_1 + \lambda_2 U_2 \text{ with } \lambda_1, \lambda_2 \geqslant 0, U_1 = (1, -1) \text{ and } U_2 = (1, 1)\}$$

while

$$N_f(Y) = \mathbb{R}\{X\} \text{ and } \forall\, Z \in ]X, Y],\ N_f(Z) = \mathbb{R}\{X\}. \qquad \square$$

Let us now state the following result concerning the class of strictly quasiconvex functions.

THEOREM 3.7. *Let $f: X \to \mathbb{R} \cup \{+\infty\}$ be lsc and continuous on* dom $f$.
*Then $f$ is strictly quasiconvex if and only if $N_f$ is strictly quasimonotone on* dom $f$.

*Proof.* Assume first that $f$ is strictly quasiconvex. By Theorem 3.3, we have that $N_f$ is quasimonotone. Let $x, y$ be any two points of dom $f$ and $\bar{x}$ be any element of $]x, y[$. Without loss of generality we can assume that $f(\bar{x}) < f(y)$. Let $z \in ]\bar{x}, y[$ be such that $f(\bar{x}) < f(z) < f(y)$. Arguing as in the first part of the proof of Theorem 3.4 we conclude that there exists $z^* \in N_f(z)$ with $\langle z^*, y - z\rangle > 0$, hence in particular $\langle z^*, y - x\rangle \neq 0$.

For the converse implication let us assume that $N_f$ is strictly quasimonotone. From Theorem 3.3 we have that $f$ is quasiconvex. Let $x, y \in$ dom $f$. Then for some $z \in ]x, y[$ and some $z^* \in N_f(z)$ we have $\langle z^*, y - x\rangle \neq 0$. With no loss of generality we suppose that $\langle z^*, y - x\rangle > 0$. Then by Theorem 3.3(ii), we get that $f(z) < f(y)$. We have shown that $f$ cannot be constant on any segment $]x, y[$, hence it is strictly quasiconvex. $\square$

*Remark*. It is worth noting that, as in Theorem 3.4, the continuity assumption in Theorem 3.7 cannot be replaced by a lower semicontinuity one. Indeed, it is possible to modify the function $f$ defined in the remark following Theorem 3.4 in order to get a strictly quasiconvex function such that its associated normal operator is not semistrictly quasimonotone (and thus not strictly quasimonotone).

On the other hand, we may observe as before that the (full) continuity hypothesis is not needed to state the sufficient part of Theorem 3.7 if we assume that $X$ admits a Gâteaux-smooth renorm. In particular we have the following corollary:

COROLLARY 3.8. *Let $X$ be a Banach space with a Gâteaux-smooth renorm and $f: X \to \mathbb{R} \cup \{+\infty\}$ be a lsc and radially continuous function. The following statements are equivalent:*

(i) *$f$ is strictly quasiconvex and continuous on* dom $f$.
(ii) *$N_f$ is strictly quasimonotone on* dom $f$.

### 3.3. ON THE PARTICULAR FEATURE OF THE NORMAL OPERATOR

In this subsection we show that the previous technique of considering the normal operator $N_f$, is no longer appropriate if one moves from the quasiconvex to the convex functions.

Our first example shows that $N_f$ is not necessarily a monotone operator, even if the function $f$ is convex.

EXAMPLE 1. Consider the convex function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$. One may easily check that for $x \in \mathbb{R}$, the corresponding level set is $S_{f(x)} = [-|x|, |x|]$, hence

$$N_f(x) = \begin{cases} [0, +\infty) & \text{if } x > 0, \\ (-\infty, 0] & \text{if } x < 0, \\ \mathbb{R} & \text{if } x = 0. \end{cases}$$

Obviously $N_f$ is a (maximal cyclically) quasimonotone operator, without being monotone.

In the following example we see that a convex function may share the same normal multifunction $N_f$ with a quasiconvex (and not convex) one. In particular one cannot expect any chance to 'integrate' a (cyclically) quasimonotone operator $N_f$, even if he knows – a priori – that $N_f$ is a normal operator of some function $f$.

EXAMPLE 2. Consider the quasiconvex functions $f, g\colon \mathbb{R} \to \mathbb{R}$, with

$$f(x) = x \quad \text{and} \quad g(x) = \begin{cases} x & \text{if } x \geqslant 0, \\ 2x & \text{if } x < 0. \end{cases}$$

Note that $f$ is in particular convex. It follows easily that for every $x \in \mathbb{R}$, we have $N_f(x) = N_g(x) = [0, +\infty)$. Actually this equality holds for every strictly increasing function $g$ from $\mathbb{R}$ to $\mathbb{R}$.

## 4. Strict normal operator

The previous characterizations are based on generalized monotonicity properties of the normal cone to sublevel sets $S_{f(x)}$. A natural question is whether it is possible to obtain analogous characterizations by considering the normal cone to strict sublevel sets $S^-_{f(x)} = \{y \in X \ : \ f(y) < f(x)\}$.

The corresponding operator considered in [10] was

$$\tilde{N}(x) = \{x^* \in X^* \ : \ \langle x^*, y - x\rangle \leqslant 0, \ \forall y \in S^-_{f(x)}\}.$$

In case where $f$ is quasiconvex, the operator $\tilde{N}$ enjoys certain continuity properties ([10]) and can be associated with the Dini directional derivatives ([14]).

However, $\tilde{N}$ is not an appropriate notion for characterizing quasiconvexity, as shows the example of the following real valued continuous quasiconvex function $f$ defined on $\mathbb{R}^2$ by

$$f(x,y)=\begin{cases}\max\{x,y\} & \text{if } x<0 \text{ and } y<0,\\ 0 & \text{otherwise.}\end{cases}$$

The strict sublevel set $S^-_{f(0,0)} = S^-_{f(0,1)} = S^-_{f(1,0)}$ is reduced to $\mathbb{R}^-_* \times \mathbb{R}^-_*$. Consequently the points $(0,1)$ and $(1,0)$ are respectively elements of $\tilde{N}(1,0)$ and $\tilde{N}(0,1)$ and thus $\tilde{N}$ is not a quasimonotone operator.

In order to overcome this difficulty we propose the following improvement for the operator $\tilde{N}$:

To any lower semicontinuous function $f\colon X \to \mathbb{R}\cup\{+\infty\}$ we associate the multivalued operator $\tilde{N}_f\colon X \to 2^{X^*}$ defined by

$$\tilde{N}_f(x) = \begin{cases}\emptyset & \text{if } x \notin \operatorname{dom} f,\\ X^* & \text{if } x \in \operatorname{Argmin} f,\\ N_{S^-_{f(x)}}(x) & \text{if } x \in \operatorname{cl}(S^-_{f(x)}),\\ \{0\} & \text{otherwise.}\end{cases}$$

This definition has a double advantage. Firstly, using this operator we shall be able to characterize quasiconvexity for continuous functions. Secondly, since the $\tilde{N}_f$ coincide with $X^*$ at any point of the subset Argmin $f$, $\tilde{N}_f$ inherits the continuity properties established in [10] for $\tilde{N}$, providing that $f$ is real valued quasiconvex and that every local minimum of $f$ is a global minimum (or equivalently $\forall\lambda > \inf_X f$, $\operatorname{cl}(S^-_\lambda) = S_\lambda$) since, in this case, $\tilde{N}_f$ and $\tilde{N}$ coincide. This situation occurs, for example, whenever $f$ is continuous and semistrictly quasiconvex.

On the other hand, if $f$ is such that the subset Argmin $f$ is a singleton or empty and every local minimum is a global one (for example whenever $f$ is continuous and strictly quasiconvex) then $\tilde{N}_f$ and $N_f$ coincide.

Before establishing the ‘strict normal characterization’ of quasiconvexity, let us quote the following lemma.

LEMMA 4.1. *Let $f\colon X \to \mathbb{R}$ be a lsc function. Then $f$ is quasiconvex if, and only if,* $\operatorname{cl}(S^-_\lambda)$ *convex for all* $\lambda \in \mathbb{R}$.

*Proof.* The ‘only if’ part of the proof is a immediate consequence of the definition of quasiconvexity.

Suppose now that $\operatorname{cl}(S^-_\lambda)$ convex, for all $\lambda \in \mathbb{R}$. Then for all $\mu \in \mathbb{R}$ we have

$$S^-_\mu = \bigcup_{\lambda<\mu} S^-_\lambda = \bigcup_{\lambda<\mu} S_\lambda. \tag{3}$$

Since $S^-_\lambda \subset S_\lambda \subset S^-_\mu$ and $f$ is lsc we infer that $\operatorname{cl}(S^-_\lambda) \subset S_\lambda \subset S^-_\mu$, hence combining with (3) we conclude

$$S^-_\mu = \bigcup_{\lambda<\mu} S^-_\lambda \subseteq \bigcup_{\lambda<\mu} \operatorname{cl}(S^-_\lambda) \subseteq \bigcup_{\lambda<\mu} S_\lambda = S^-_\mu. \tag{4}$$

It follows that all inclusions in (4) are equalities and that $S_\mu^-$ is convex (as increasing union of the convex sets $\mathrm{cl}(S_\lambda^-)$, for $\lambda < \mu$). Since $\mu$ is arbitrarily chosen, we conclude that $f$ is quasiconvex. □

In the above lemma the lsc assumption cannot be dropped. Indeed, for any dense subset $D$ of $\mathbb{R}$ let us consider the function $f\colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in D, \\ +\infty & \text{if } x \notin D. \end{cases} \tag{5}$$

It follows that $f$ is not quasiconvex (unless $D = \mathbb{R}$) and that for all $\lambda \in \mathbb{R}$, $\mathrm{cl}(S_\lambda^-)$ is either empty or the whole space (hence convex). Let us also note that the restriction $f|_{\mathrm{dom}\, f}$ of $f$ on its domain $\mathrm{dom}\, f = D$ is constant, hence continuous.

THEOREM 4.2. *Let $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be lsc and continuous on its domain* $\mathrm{dom}\, f$*. Then $f$ is quasiconvex if and only if $\tilde{N}_f$ is quasimonotone on* $\mathrm{dom}\, f \setminus \mathrm{Argmin}\, f$.

*Proof.* ($\Rightarrow$) Let us suppose to a contradiction that there exist $x, y \in \mathrm{dom}\, f \setminus \mathrm{Argmin}\, f$, $x^* \in \tilde{N}_f(x)$ and $y^* \in \tilde{N}_f(y)$ satisfying

$$\langle x^*, y - x\rangle > 0 \quad \text{and} \quad \langle y^*, x - y\rangle > 0.$$

From the definition of the operator $\tilde{N}_f$, we immediately conclude that $x \in \mathrm{cl}(S_{f(x)}^-)$ and $y \in \mathrm{cl}(S_{f(y)}^-)$. Consequently, considering $\varepsilon > 0$ small enough, one can find $x_\varepsilon \in S_{f(x)}^- \cap B_\varepsilon(x)$ and $y_\varepsilon \in S_{f(y)}^- \cap B_\varepsilon(y)$ such that

$$\langle x^*, y_\varepsilon - x\rangle > 0 \quad \text{and} \quad \langle y^*, x_\varepsilon - y\rangle > 0.$$

On the other hand, $\tilde{N}_f(x) = N_{S_{f(x)}^-}(x)$ and, since $f$ is quasiconvex, $y_\varepsilon - x$ is not an element of $T_{S_{f(x)}^-}(x) = \mathrm{cl}(\bigcup_{\lambda>0} \lambda(S_{f(x)}^- - \{x\}))$. Thus $y_\varepsilon \notin S_{f(x)}^-$ and $f(x) \leqslant f(y_\varepsilon) < f(y)$. Using the same arguments with $x$, $x_\varepsilon$ and $y$ the contradiction is obtained.

($\Leftarrow$) Now assume that $f$ is not quasiconvex. Then, according to Lemma 4.1, there exists $\lambda > \inf_X f$ such that $\mathrm{cl}(S_\lambda^-)$ is not convex. Thus the operator $\partial^{\mathrm{CR}}\psi_{\mathrm{cl}(S_\lambda^-)}$ is not quasimonotone and there exist $x, y \in \mathrm{dom}\, f$, $x^* \in \partial^{\mathrm{CR}}\psi_{\mathrm{cl}(S_\lambda^-)}(x)$ and $y^* \in \partial^{\mathrm{CR}}\psi_{\mathrm{cl}(S_\lambda^-)}(y)$ satisfying

$$\langle x^*, y - x\rangle > 0 \quad \text{and} \quad \langle y^*, x - y\rangle > 0. \tag{6}$$

Since $x^*$ and $y^*$ are nonzero elements of $X^*$, $x$ and $y$ are not elements of $\mathrm{int}(\mathrm{cl}(S_\lambda^-))$ (see for example formula (1)) which contains $S_\lambda^-$ since $f$ is continuous. This immediately yields $f(x) = f(y) = \lambda$ (and $x, y \notin \mathrm{Argmin}\, f$). Then, according to Lemma 3.1, $x^* \in \partial^{\mathrm{CR}}\psi_{\mathrm{cl}(S_\lambda^-)}(x) = N_{\psi_{\mathrm{cl}(S_\lambda^-)}}(x) = N_{\mathrm{cl}(S_{f(x)}^-)}(x) = \tilde{N}_f(x)$. In the same way $y \in \tilde{N}_f(y)$. Relation (6) implies that $\tilde{N}_f$ is not quasimonotone on $\mathrm{dom}\, f \setminus \mathrm{Argmin}\, f$. □

*Remark*. It is important to mention that, unlike the case of the operator $N_f$, there is no hope to obtain the previous characterization if $f$ is only assumed to be lower semicontinuous. Indeed, the following real valued function $g\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$g(x, y) = \begin{cases} \max\{x, y\} & \text{if } x < 0 \text{ and } y < 0, \\ 1 & \text{if } x > 0 \text{ and } y > 0, \\ 0 & \text{otherwise} \end{cases}$$

is not quasiconvex although $\tilde{N}_f$ is quasimonotone.

This shows that $N_f$ is more apt for the characterization of the different kinds of quasiconvexity.

THEOREM 4.3. *Let* $f\colon X \to \mathbb{R} \cup \{+\infty\}$ *be lsc and continuous on its domain* dom $f$. *Then the following statements are equivalent:*

(i) $f$ *is a semistrictly quasiconvex function.*

(ii) $x, y \in \operatorname{dom} f \setminus \operatorname{Argmin} f$, $x^* \in \tilde{N}_f(x)$ *and* $\langle x^*, y - x\rangle > 0$

$$\Longrightarrow f(z) < f(y), \quad \forall z \in [x, y[$$

(iii) $\tilde{N}_f$ *is a semistrictly quasimonotone operator on* dom $f \setminus$ Argmin $f$.

Let us first remark that for any semistrictly quasiconvex function $f$ and any point $x$ of dom $f \setminus$ Argmin $f$, one has $\tilde{N}_f(x) = N_f(x)$.

*Proof.* (i) $\Rightarrow$ (iii) is a direct consequence of implication (i) $\Rightarrow$ (iii) of Theorem 3.4. Indeed, if $f$ is continuous and semistrictly quasiconvex, then $N_f$ is semistrictly quasimonotone on dom $f$ and thus on dom $f \setminus$ Argmin $f$, subset on which $N_f$ and $\tilde{N}_f$ coincide.

(iii) $\Rightarrow$ (ii) Hypothesis (iii) implies that $\tilde{N}_f$ is quasimonotone on dom $f \setminus$ Argmin $f$ and thus, according to Theorem 4.2, $f$ is quasiconvex.

Now let $x, y \in \operatorname{dom} f \setminus \operatorname{Argmin} f$ and $x^* \in \tilde{N}_f(x)$ be such that $\langle x^*, y - x\rangle > 0$. Actually $x^* \in N_{S^-_{f(x)}}(x)$ and $y - x \notin T_{S^-_{f(x)}}(x) = \operatorname{cl}(\bigcup_{\lambda>0} \lambda(S^-_{f(x)} - \{x\}))$ which implies that $f(z) \leqslant f(y)$, $\forall z \in [x, y]$.

Let us suppose that there exists $\bar{z} \in [x, y[$ such that $f(u) = f(y)$, for any $u \in [\bar{z}, y]$. According to [16, Prop. 3.1], one can find $z_1 \in ]\bar{z}, y[$, $z_2 \in ]z_1, y[$, $z_1^* \in \tilde{N}_f(z_1)$ and $z_2^* \in \tilde{N}_f(z_2)$ verifying

$$\langle z_1^*, z_2 - z_1\rangle > 0 \quad \text{and} \quad \langle z_2^*, y - z_2\rangle > 0. \tag{7}$$

Since $f(z_2) = f(y) > \inf_X f$, this immediately implies that $z_2 \in \operatorname{cl}(S^-_{f(z_2)})$. Thus there exists a sequence $(z_2^k)_{k\in\mathbb{N}} \subset S^-_{f(z_2)}$ converging to $z_2$. From (7) we deduce that for $k$ large enough we have $\langle z_1^*, z_2^k - z_1\rangle > 0$ and then, using again the quasiconvexity of $f$, $f(z_2) = f(z_1) \leqslant f(z_2^k)$ which is impossible.

(ii) $\Rightarrow$ (i) Assumption (ii) implies that $\tilde{N}_f$ is quasimonotone on dom $f \setminus$Argmin $f$ and thus, by Theorem 4.2, that $f$ is quasiconvex.

To prove the semistrict quasiconvexity of $f$, it suffices to modify the part (ii) $\Rightarrow$ (i) of the proof of Theorem 3.4. Let us suppose (without loss of generality) that $x, y \in \operatorname{dom} f$ are such that $f(x) < f(z)$, for any $z \in ]x, y]$. Due to the continuity of $f$, one can find $\bar{z} \in ]x, y[$ such that $f(z') > f(\bar{z})$, for any $z' \in ]\bar{z}, y]$ and $\bar{z}$ is not a local minimum of the restriction of $f$ on the segment $[x, y]$. Now using a separation argument with the subsets $]\bar{z}, y]$ and $S^-_{f(\bar{z})}$, we deduce (as in the proof of Theorem 3.4) the existence of $\bar{z}^* \in N_{S^-_{f(\bar{z})}}(\bar{z})$ satisfying $\langle \bar{z}^*, y - \bar{z} \rangle > 0$. Since $\bar{z}$ is an element of $\operatorname{cl}(S^-_{f(\bar{z})})$, $\bar{z}^* \in \tilde{N}_f(\bar{z})$ and, according to hypothesis (ii), $f(z) < f(y)$, for all $z \in [\bar{z}, y[$. Now the proof is complete since, using the quasiconvexity of $f$, the previous inequality holds for all $z \in [x, y[$. □

THEOREM 4.4. *Let* $f\colon X \to \mathbb{R} \cup \{+\infty\}$ *be lsc and continuous on* dom $f$. *Then* $f$ *is strictly quasiconvex if and only if* $\tilde{N}_f$ *is strictly quasimonotone on* dom $f$.

*Proof.* Assume that $f$ is strictly quasiconvex. Then Argmin $f$ is a singleton or empty and, as it has been already observed, $N_f$ and $\tilde{N}_f$ coincide. According to Theorem 3.7, $N_f$ (and thus $\tilde{N}_f$) is strictly quasimonotone on dom $f$.

The proof of the converse implication follows the same lines as the last part of the proof of Theorem 3.7. Indeed, according to Theorem 4.3, $f$ is semistrictly quasiconvex. We now observe that Argmin $f$ is at most a singleton. Indeed, if $x$, $y$ are distinct elements of Argmin $f$, then the fact that $\tilde{N}_f(x) = \tilde{N}_f(y) = X^*$ contradicts the quasimonotonicity of $\tilde{N}_f$. So let $x, y \in \operatorname{dom} f$. Then we can find $\bar{x}, \bar{y} \in ]x, y[$ such that $[\bar{x}, \bar{y}] \cap \operatorname{Argmin} f = \emptyset$. Now invoking Theorem 4.3(ii) the proof can be completed as in Theorem 3.7. □

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# Normal cones to sublevel sets: an axiomatic approach. Applications in quasiconvexity and pseudoconvexity

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**Abstract.** An axiomatic approach of normal operators to sublevel sets is given. Considering the Clarke-Rockafellar subdifferential (resp. quasiconvex functions), the definition given in [4] (resp. [5]) is recovered. Moreover, the results obtained in [4] are extended in this more general setting. Under mild assumptions, quasiconvex continuous functions are classified, establishing an equivalence relation between functions with the same normal operator. Applications in pseudoconvexity are also discussed.

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## 1 Introduction

The notion of a "normal cone to sublevel sets", i.e. a multivalued operator associating with every function $f$ and every point $x$ of its domain the normal cone to the sublevel set $S_{f(x)}$ has first been introduced and studied in [5], where the authors discussed continuity properties of this operator (or variants of it) when applied to quasiconvex functions. Subsequently, several authors used this notion (see [13], [10], [11] e.g.) for dealing with quasiconvex optimization problems.

In [4], a modification on the original definition ([5]) of the normal operator has been proposed, consisting in considering for every $x$ the polar cone of the Clarke tangent cone of $S_{f(x)}$ at $x$. This new definition coincides with the previous one whenever the function $f$ is quasiconvex, whereas it has the advantage to allow simple characterizations of various types of quasiconvexity in terms of corresponding types of quasimonotonicity of the normal operator.

In this work, following the lines of [4], we give an axiomatic formulation for the concept of normal operator, based on an abstract notion of subdifferential, see Section 2. Subsequently, we present some applications in quasiconvexity (Sections 3 and 5) and in pseudoconvexity (Section 4).

Throughout this paper, $X$ will be a Banach space with dual $X^*$, and $f$ a lower semicontinuous (lsc) function on $X$ with values in $\mathbb{R}\cup\{+\infty\}$. For any $x\in X$ and any $x^*\in X^*$ we denote by $\langle x^*,x\rangle$ the value of the functional $x^*$ at the point $x$. We also use the standard notation: $B_\delta(x)$ for the closed ball centered at $x$ with radius $\delta>0$, $\operatorname{dom} f := \{x\in X : f(x)\neq +\infty\}$ for the domain of the function $f$ and $S_{f(x)} := \{x'\in X : f(x')\le f(x)\}$ (resp. $S^-_{f(x)} = \{x'\in X : f(x') < f(x)\}$) for the sublevel and the strict sublevel sets of $f$. For $x,y\in X$ we set $[x,y]=\{tx+(1-t)y : 0\le t\le 1\}$ and we define the segments $]x,y]$, $[x,y[$ and $]x,y[$ analogously.

## 2 Abstract subdifferential and normal operator

Let us first recall from [2] the definition of an abstract subdifferential.

**Definition 1.** We call *subdifferential operator*, any operator $\partial$ associating to any Banach space $X$, any lower semicontinuous function $f : X\to\mathbb{R}\cup\{+\infty\}$ and any $x\in X$, a subset $\partial f(x)$ of $X^*$, and satisfying the following properties:

(P1) $\partial f(x)=\{x^*\in X^* \ : \ \langle x^*,y-x\rangle + f(x)\le f(y),\quad \forall y\in X\}$, whenever $f$ is convex;

(P2) $0\in\partial f(x)$, whenever $f$ attains a local minimum at $x\in dom f$;

(P3) $\partial(f+g)(x)\subset\partial f(x)+\partial g(x)$, whenever $g$ is real-valued convex continuous, and $\partial$*-differentiable at* $x$,

where $g$ $\partial$*-differentiable at* $x$ means that $\partial g(x)$ and $\partial(-g)(x)$ are nonempty.

In the sequel, we shall assume in addition that

$$\partial\subset\partial^{\uparrow}\quad\text{or}\quad\partial\subset\partial^{D+}$$

where $\partial^{\uparrow}$ is the Clarke-Rockafellar and $\partial^{D+}$ the upper Dini subdifferential.
Let us recall that the definitions:

$$\partial^{\uparrow} f(x)=\{x^*\in X^* : \langle x^*,d\rangle\le f^{\uparrow}(x,d)\,,\ \text{for all } d\in X\}$$

where

$$f^{\uparrow}(x,d)=\sup_{\varepsilon>0}\limsup_{\substack{t\searrow 0\\ y\to_f x}}\inf_{d'\in B_\varepsilon(d)}\frac{1}{t}\left(f(y+td')-f(y)\right).$$

and

$$\partial^{D+} f(x) = \{x^* \in X^* : \langle x^*, d\rangle \le f^{D+}(x,d), \text{ for all } d \in X\}$$

where

$$f^{D+}(x,d) = \limsup_{t \searrow 0^+} \frac{1}{t}\left(f(x+td) - f(x)\right).$$

It is recalled that $t \searrow 0^+$ indicates the fact that $t > 0$ and $t \to 0$, while $x \to_f x_o$ means that both $x \to x_o$ and $f(x) \to f(x_o)$.

We further recall from [2] the following definition.

**Definition 2.** A norm $\|.\|$ on $X$ is said to be $\partial$*-smooth* if the functions of the following form are $\partial$-differentiable:

$$x \mapsto \Delta_2(x) := \sum_n \mu_n \|x - v_n\|^2,$$

where $\mu_n \ge 0$, the series $\sum_n \mu_n$ is convergent, and the sequence $(v_n)$ converges in $X$.

Let us also introduce the notion of an "abstract" normal cone, based on the subdifferential $\partial$.

**Definition 3.** Let $\partial$ be a subdifferential operator. For any closed subset $C$ of $X$ and any point $x \in X$ we associate the normal cone to $C$ at the point $x$ defined by

$$N_C(x) = \begin{cases} \partial\psi_C(x) & \text{if } x \in C \\ \emptyset & \text{otherwise} \end{cases}$$

where $\psi_C$ denotes the indicator function of $C$ (i.e. $\psi_C(x) = 0$ if $x \in C$ and $+\infty$ if $x \notin C$).

For all classical subdifferentials (Clarke, lower and upper Hadamard, lower and upper Dini, Frèchet, proximal...) the subset $N_C(x)$ is effectively a cone. Although this property will not be used in the sequel, to be in accordance with the term "normal cone" of the above definition, we can assume that the abstract subdifferential fulfills the following property:

For any function $f$, any $\lambda > 0$ and any $x \in X$, $\quad \partial(\lambda f)(x) = \lambda \partial f(x)$.

Whenever the subdifferential operator is the lower Hadamard subdifferential $\partial^{H-}$, the corresponding normal cone is the classical Bouligand normal cone defined as follows

$$NK_C(x) = \{x^* \in X^* \ : \ \langle x^*, d\rangle \le 0, \ \forall d \in K_C(x)\} \tag{1}$$

with

$$K_C(x) = \{y = \lim_{k\to\infty} y_k \ : \ \exists\, t_k \searrow 0 \text{ with } x + t_k y_k \in C, \ \ \forall\, k \in \mathbb{N}\}$$

On the other hand, if $\partial = \partial^{\uparrow}$, then we recover the Clarke normal cone

$$N_C^{\uparrow}(x) = \{x^* \in X^* \ : \ \langle x^*, d\rangle \le 0, \ \forall d \in T_C(x)\} \tag{2}$$

with

$$d \in T_C(x) \Leftrightarrow \begin{cases} \forall \varepsilon > 0, \exists\, \delta > 0 \ \text{ such that} \\ \forall x' \in B_\delta(x) \cap C, \forall t \in ]0, \delta[, \ (x' + tB_\varepsilon(d)) \cap C \neq \emptyset. \end{cases}$$

We are now in a position to define the normal operator associated with a function.

**Definition 4.** Let $\partial$ be a subdifferential operator. For any lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$ we associate a multivalued operator $N_f : X \to 2^{X^*}$ - called normal operator - defined by

$$x \mapsto \begin{cases} N_{S_{f(x)}}(x) & \text{if } x \in \operatorname{dom} f \\ \emptyset & \text{otherwise} \end{cases}$$

**Remark:** 1) In the particular case $\partial = \partial^{\uparrow}$, we recover the definition used in [4] (see relation (2)).

2) Based on the strict sublevel sets (i.e. $S_\lambda^- = \{x \in X \ : \ f(x) < \lambda\}$) an analogous concept of normal operator (called strict normal operator) has been considered in [4] (extending the original definition of [5]) :

$$\tilde{N}_f(x) = \begin{cases} \emptyset & \text{if } x \notin \operatorname{dom} f \\ X^* & \text{if } x \in \operatorname{Argmin} f \\ N_{cl(S^-_{f(x)})}(x) & \text{if } x \in cl(S^-_{f(x)}) \\ \{0\} & \text{otherwise} \end{cases}$$

Since, as showed in [4], the operator $N_f$ is more appropriate than $\tilde{N}_f$ for the normal characterization of the different types of quasiconvexity, the use of (large) sublevel sets has been preferred for the purpose of this paper.

A natural question immediately arises concerning the relation between the multivalued operators $N_f$ and $\partial f$ and in particular, the possible equality between $N_f(x)$ and *cone* $(\partial f(x)) := \{tx^* \ : \ t \ge 0 \text{ and } x^* \in \partial f(x)\}$. This equality is not true in general. In fact several counterexamples have been given in [4] for the case $\partial = \partial^{\uparrow}$. In the following proposition we shed more light on this topic.

Let us recall that a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is called *quasiconvex* if its sublevel sets $S_\lambda$ are convex subsets of $X$. Following [6], a locally Lipschitz function is said to be *regular* at a point $x$, if for any $d \in X$ the classical directional derivative $f'(x, d)$ exists and is equal to the Clarke directional derivative $f^\circ(x, d)$ defined as follows:

$$f^\circ(x, d) = \limsup_{\substack{t \searrow 0^+ \\ y \to x}} \frac{1}{t}\left(f(y + td) - f(y)\right)$$

**Proposition 1.** *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be lsc such that $0 \notin \partial f(X)$.*

*i) If $f$ is quasiconvex then, for any $x \in X$,*

$$cone(\partial f(x)) \subset N_f(x).$$

*ii) Let us suppose, in addition, that $f$ is Lipschitz continuous and $\partial \subset \partial^{\uparrow}$. If $f$ is quasiconvex or $f$ is regular, then for any $x \in X$,*

$$N_f(x) = cone(\partial f(x)).$$

*Proof.* For $i$) let us suppose, for a contradiction, that $x \in \mathrm{dom}\, f$ is such that $\partial f(x) \not\subset N_f(x) = NK_{S_f(x)}$. Hence there exists $y \in S_f(x)$ and $x^* \in \partial f(x)$ verifying $\langle x^*, y - x\rangle > 0$. Let $\delta > 0$ be such that $\langle x^*, u - x\rangle > 0$ for all $u \in B_\delta(y)$. Since $f$ is quasiconvex, it follows (see [3] e.g.) that $f(u) \geq f(x)$ for all $u \in B_\delta(y)$. But, since $y$ is an element of $S_f(x)$, $y$ is a local minimum of $f$ and therefore $0 \in \partial f(y)$ which contradicts the hypothesis.

$ii$) is a direct consequence of [6, Th. 2.4.7]. □

*Remark 1.* $a$) As proved in [14, Lemma 5.3], if $\partial$ is the Fréchet subdifferential, then assertion $i$) can be obtained without the assumption "$0 \notin \partial f(X)$".

$b$) In assertion $ii$) of the previous proposition, the Lipschitz assumption can not be dropped. Indeed, if we define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sqrt{x}$ if $x \geq 0$ and $f(x) = -\sqrt{-x}$ otherwise, then for any $x \neq 0$, $cone(\partial f(x)) = N_f(x)$, while for $x = 0$ we have $\partial f(0) = \emptyset$ and $N_f(0) = [0, +\infty[$.

## 3 Normal characterizations of quasiconvexity

In this section we establish 'normal' characterizations for quasiconvex and strictly (semistrictly) quasiconvex functions in terms of the abstract normal operator $N_f$. These characterizations have been derived in [4] in the particular case $\partial = \partial^{\uparrow}$.

Let us first recall the relevant definitions. A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *semistrictly quasiconvex* if $f$ is quasiconvex and for any $x$, $y \in \mathrm{dom}\, f$ we have

$$f(x) < f(y) \Longrightarrow f(z) < f(y),\ \forall\, z \in [x, y[.$$

Similarly, $f$ is called *strictly quasiconvex*, if it is quasiconvex and for any $x$, $y \in \mathrm{dom}\, f$ and $z \in ]x, y[$ we have

$$f(z) < \max\{f(x), f(y)\}.$$

For any subset $K$ of $X$, let us also recall that a multivalued operator $T : X \to 2^{X^*}$ is called *quasimonotone* on $K$ if for all $x, y \in K$ we have

$$\exists\, x^* \in T(x),\ \langle x^*, y - x\rangle > 0 \Rightarrow \forall\, y^* \in T(y)\ :\ \langle y^*, y - x\rangle \geq 0.$$

Following [8] $T$ is called *cyclically quasimonotone* (on $K$), if for every $x_1, x_2, ..., x_n \in X$ (resp. $x_1, x_2, ..., x_n \in K$), there exists $i \in \{1, 2, ..., n\}$ such that

$$\langle x_i^*, x_{i+1} - x_i \rangle \leq 0, \forall x_i^* \in T(x_i)$$

(where $x_{n+1} := x_1$).

Furthermore ([7]), the operator $T$ is called *semistrictly quasimonotone* on $K$, if $T$ is quasimonotone on $K$ and for any $x$, $y \in K$ we have

$$\exists\, x^* \in T(x),\ \langle x^*, y - x \rangle > 0 \Longrightarrow \exists\, z \in \left]\frac{x+y}{2}, y\right[,\ \exists\, z^* \in T(z)\ :\ \langle z^*, y - z \rangle > 0.$$

Finally $T$ is called *strictly quasimonotone* if $T$ is quasimonotone and for any $x$, $y \in K$ we have

$$\exists\, z \in ]x, y[,\ \exists\, z^* \in T(z)\ :\ \langle z^*, y - x \rangle \neq 0.$$

Let us now recall from [3] the following characterization.

**Proposition 2.** *Let $X$ be a Banach space admitting a $\partial$-smooth renorm and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lsc function. Then $f$ is quasiconvex iff $\partial f$ is quasimonotone.*

For the forthcoming characterization we need the following lemmas:

**Lemma 1.** *Let $C$ be a nonempty subset of $X$. The following statements are equivalent:*

*i) $C$ is closed and convex.*
*ii) The indicator function $\psi_C$ is convex and lsc.*
*iii) The indicator function $\psi_C$ is quasiconvex and lsc.*

*Proof.* The proof is straightforward and will be omitted. □

**Lemma 2.** *For any lsc quasiconvex function $f$, and any $x \in dom(f)$ we have:*

$$N_f(x) = NK_{S_{f(x)}}(x)$$

*Proof.* For every $x \in domf$, the set $C = S_{f(x)}$ is convex and closed, hence from Lemma 1 it follows that the function $\psi_C$ is convex and lsc. Property (P1) of Definition 1 implies that $\partial \psi_C$ does not depend on the subdifferential operator. In particular $\partial \psi_C(x)$ coincides with the cones defined in (1) and (2) respectively. □

**Theorem 1.** *Let $X$ be a Banach space admitting a $\partial$-smooth renorm and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lsc function. Consider the following statements:*

*i) $f$ is a quasiconvex function.*

*ii) $\exists\, x^* \in N_f(x),\ \langle x^*, y - x \rangle > 0 \Longrightarrow f(y) > f(x)$*

*iii) $N_f$ is a (cyclically) quasimonotone operator.*

*Then we always have i) $\Rightarrow$ ii) $\Rightarrow$ iii). Moreover if, either $\partial^{H-} \subset \partial$ and $X$ admits a Gâteaux-smooth renorm or, $\partial \subset \partial^{D+}$ and $f$ is radially continuous or, $\partial \subset \partial^{\uparrow}$ and $f$ is continuous, then iii) $\Rightarrow$ i), hence all these three conditions are equivalent.*

*Proof.* *i)* $\Rightarrow$ *ii)*. Let us suppose that for some $x^* \in N_f(x)$ we have $\langle x^*, y-x\rangle > 0$. It follows from Lemma 2 that $N_f(x) = NK_{S_{f(x)}}(x)$. Consequently $y - x$ is not an element of $K_{S_{f(x)}}(x) = cl(\cup_{\lambda>0}\lambda(S_{f(x)} - \{x\}))$. Hence, in particular, $y$ is not an element of $S_{f(x)}$, i.e. $f(x) < f(y)$.

*ii)* $\Rightarrow$ *iii)*. Take any finite family $\{x_1, \ldots, x_n\}$ of points of $X$ and suppose that for $i \in \{1, \ldots, n\}$, there exists $x_i^* \in N_f(x_i)$ such that $\langle x_i^*, x_{i+1} - x_i\rangle > 0$ where $x_{n+1} = x_1$. A contradiction immediately occurs since *ii)* yields $f(x_1) < f(x_2) < \cdots < f(x_{n+1}) = f(x_1)$.

*iii)* $\Rightarrow$ *i)*. Let us suppose, to a contradiction, that $f$ is not quasiconvex. Then from Lemma 1 it follows that for some $x_0 \in \operatorname{dom} f$, the function $\psi_{x_0} := \psi_{S_{f(x_0)}}$ is not quasiconvex.

If $\partial^{H-} \subset \partial$ (and $X$ admits a Gâteaux-smooth renorm) then, in view of Proposition 2, its lower Hadamard subdifferential $\partial^{H-}\psi_{x_0}$ is not quasimonotone. Hence there exist $x$, $y \in \operatorname{dom} \psi_{x_0} = S_{f(x_0)}$, $x^* \in \partial^{H-}\psi_{x_0}(x)$ and $y^* \in \partial^{H-}\psi_{x_0}(y)$ satisfying $\langle x^*, y - x\rangle > 0$ and $\langle y^*, x - y\rangle > 0$. Note now that $S_{f(x)} \subseteq S_{f(x_0)}$, from which it follows that $\psi_{x_0}(\cdot) \le \psi_x(\cdot)$. We can easily conclude that $\psi_{x_0}^{H^-}(x, d) \le \psi_x^{H^-}(x, d)$ for all $d$ in $X$, hence $\partial^{H^-}\psi_{x_0}(x) \subseteq \partial^{H^-}\psi_x(x)$. Hence $x^* \in N_f(x)$ and (similarly) $y^* \in N_f(y)$ and we obtain the desired contradiction.

In both other cases, using again Proposition 2, we conclude to the existence of $x$, $y \in \operatorname{dom} \psi_{x_0} = S_{f(x_0)}$, $x^* \in \partial\psi_{x_0}(x)$ and $y^* \in \partial\psi_{x_0}(y)$ satisfying $\langle x^*, y - x\rangle > 0$ and $\langle y^*, x - y\rangle > 0$.

Now we claim that $f(x) = f(y) = f(x_0)$.

[We obviously have $f(x) \le f(x_0)$. Let us now suppose that $f(x) < f(x_0)$.

If $\partial \subset \partial^{D+}$, then from the radial continuity of $f$ we may find some $\delta > 0$ such that $f(u) < f(x_0)$ for any element $u$ in the segment $(x-\delta(y-x), x+\delta(y-x))$. Then it follows that the function $\psi_{x_0}$ is constant on this segment, which is not compatible with the inequality $\langle x^*, y - x\rangle > 0$. Hence $f(x) = f(x_0)$ and for the same reasons $f(y) = f(x_0)$.

If now $\partial \subset \partial^{\uparrow}$ (and the function $f$ is continuous), then we may take a $\delta > 0$ such that $f(u) < f(x_0)$ for all $u \in B_\delta(x)$, hence the function $\psi_{x_0}$ is locally constant on $x$, which contradicts the fact that $\langle x^*, y - x\rangle > 0$. Again we conclude that $f(x) = f(x_0) = f(y)$. The claim is proved.]

Now the proof is complete. Indeed $\psi_{x_0} = \psi_x = \psi_y$. Hence, in both cases $x^*$ is an element of $\partial\psi_{x_0}(x) = \partial\psi_x(x) = N_{S_{f(x)}}(x) = N_f(x)$ and $y^*$ is an element of $N_f(y)$ thus furnishing a contradiction with the quasimonotonicity of $N_f$. □

Using essentially the same proof as in [4] it is possible to obtain the following characterizations of semistrict and strict quasiconvexity in this more general framework. Let us thus state - without proof - these results.

**Theorem 2.** *Let $X$ be a Banach space admitting a $\partial$-smooth renorm and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be lsc and continuous on its domain $dom f$. Then the following statements are equivalent:*

*i)* $f$ *is a semistrictly quasiconvex function.*

*ii)* $\exists\, x^* \in N_f(x) \ : \ \langle x^*, y - x\rangle > 0 \Longrightarrow f(y) > f(z), \ \forall\, z \in [x, y)$

*iii)* $N_f$ *is a semistrictly quasimonotone operator on* $dom f$.

**Theorem 3.** *Let $X$ be a Banach space admitting a $\partial$-smooth renorm and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lsc and continuous on $dom f$.*

*Then $f$ is strictly quasiconvex if and only if $N_f$ is strictly quasimonotone on $dom f$.*

## 4 Normal cones and pseudoconvexity.

In this section we shall discuss relations between normal operators and pseudoconvexity. In [1], a *differentiable* function $f$ was called *pseudoconvex*, if for every $x, y \in dom(f)$ the inequality $\langle df(x), y - x\rangle \geq 0$ ensures $f(y) \geq f(x)$. The notion of pseudoconvexity was subsequently extended into non-smooth functions, based on the concept of subdifferential (see [12], [3]). Let us further give the definition of pseudoconvexity in an even more abstract setting.

**Definition 5.** Given an operator $T : X \to 2^{X^*}$, a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is called $T$-pseudoconvex, if for any $x, y \in dom(f)$ and $x^* \in T(x)$, the inequality $\langle x^*, y - x\rangle \geq 0$ implies $f(y) \geq f(x)$.

In case $T := \partial f$, we recover the definition given in [12] (see also [9] for a summary).

Since Definition 5 of $N_f \setminus \{0\}$-pseudoconvexity and Theorem 1 *ii)* are very similar, one may wonder whether quasiconvexity and $N_f \setminus \{0\}$-pseudoconvexity differ. It is shown below (Proposition 3) that for some particular case these concepts coincide. However this is not the case in general, as shows the example of the function $f : \mathbb{R} \to \mathbb{R}$, with

$$f(x) = \begin{cases} 1, \text{ if } \ | x | < 1 \\ 0, \text{ if } \ | x | \geq 1 \end{cases} \tag{3}$$

The above function is lower semicontinuous and $T$-pseudoconvex (for $T = N_f \setminus \{0\}$), without being quasiconvex.

A more general example of a lsc function satisfying for all $x, y \in \operatorname{dom} f$ the property:

$$\forall x^* \in T(x),\ \langle x^*, y - x\rangle \geq 0 \Longrightarrow f(y) \geq f(z), \text{ for all } z \in [x, y] \tag{4}$$

without being quasiconvex is given below. (Relation (4) was taken as definition for $T$-pseudoconvexity in [9]).

**Example:** Let us consider the lsc function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} 1\ , \text{ if } & x<0 \text{ and } y>0 \\ 0\ , \text{ if } & xy \geq 0 \\ -x\ , \text{ if } & x>0,\ y<0 \text{ and } -y \geq x \\ y\ , \text{ if } & x>0,\ y<0 \text{ and } -y \leq x. \end{cases} \tag{5}$$

It is easily seen that $f$ is $N_f \setminus \{0\}$-pseudoconvex, provided that $\partial \subset \partial^{\uparrow}$. On the other hand, since

$$S_{f(0,0)} = \mathbb{R}^2 \setminus \{(x,y)\ :\ x<0,\ y>0\}$$

the function $f$ is not quasiconvex.

**Proposition 3.** *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lsc radially continuous function with convex domain. Then*

*i) $f$ quasiconvex $\Rightarrow$ $f$ $N_f \setminus \{0\}$-pseudoconvex.*

*ii) if, moreover, $X = \mathbb{R}^n$ and $\partial = \partial^{\uparrow}$ then $f$ is quasiconvex iff $f$ is $N_f(x) \setminus \{0\}$-pseudoconvex.*

*Proof.* $i)$ Let us assume that $x,\ y \in \operatorname{dom} f$ and $x^* \in N_f(x) \setminus \{0\}$ are such that $\langle x^*, y-x\rangle \geq 0$. Since $x^* \neq 0$, there exists $d \in X$ such that $\langle x^*, d\rangle > 0$. Then for $y_n = y + \frac{1}{n}d$ (with $n \in \mathbb{N}$) we have $\langle x^*, y_n - x\rangle > 0$ which implies, by $ii)$ of Theorem 1 that $f(y_n) > f(x)$. Since $f$ is radially continuous this yields $f(y) \geq f(x)$ and $f$ is $N_f \setminus \{0\}$-pseudoconvex.

$ii)$ To prove the converse implication, let us suppose that $f$ is $N_f(x)\setminus\{0\}$-pseudoconvex and (towards a contradiction) $z$ is an element of $]x,y[$ verifying

$$f(z) > \max[f(x), f(y)].$$

Since $f$ is radially continuous, we may assume that $f(x) > f(y)$ and that there exists $\tilde{z} \in ]z,y[$ such that $f(x) < f(\tilde{z}) < f(z)$. It is also no loss of generality in assuming that $f(u) > f(\tilde{z})$ for all $u \in ]z,\tilde{z}]$. Thus $\tilde{z}$ is on the boundary of the closed subset $S_{f(\tilde{z})}$ and consequently $N_f(\tilde{z})$ contains a nonzero element $\tilde{z}^*$ (see [6] e.g.). On the other hand, since $f(\tilde{z}) > f(x)$, we have $\langle \alpha^*, x-\tilde{z}\rangle < 0$ for any $\alpha^* \in N_f(\tilde{z}) \setminus \{0\}$. In particular, $\langle \tilde{z}^*, y-\tilde{z}\rangle > 0$ and, according to the $N_f \setminus \{0\}$-pseudoconvexity, $f(y) > f(\tilde{z})$ which is a contradiction. □

We also recall ([8]) that an operator $T$ is called *cyclically pseudomonotone*, if for every $x_1, x_2, ..., x_n \in X$, the following implication holds:

$$\exists i \in \{1,2,...,n\}, \exists x_i^* \in T(x_i) : (x_i^*, x_{i+1} - x_i) > 0 \Longrightarrow$$
$$\exists j \in \{1,2,...,n\}, \forall x_j^* \in T(x_j) : (x_j^*, x_{j+1} - x_j) < 0$$

(where $x_{n+1} := x_1$).

Let us now state the following result, to be compared with Theorem 1.

**Proposition 4.** *Let $X$ be a Banach space admitting a $\partial$-smooth renorm and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a continuous function. The following statements are equivalent:*

*i)* *$f$ is quasiconvex.*

*ii)* *$N_f \setminus \{0\}$ is (cyclically) pseudomonotone.*

*Proof.* *i)* $\Rightarrow$ *ii)*. Set $T(x) = N_f(x) \setminus \{0\}$ for all $x \in X$. Let any finite subset $\{x_1, x_2, ..., x_n\}$ of $X$ and suppose (for a contradiction) that $\langle x_1^*, x_2 - x_1 \rangle > 0$ whereas for all $j \geq 2$, and all $x_j^* \in T(x_j)$, $(x_j^*, x_{j+1} - x_j) \geq 0$ (where $x_{n+1} := x_1$). Since $x_j^* \neq 0$, using the same arguments as in part *i)* of the previous proof, we obtain $f(x_{j+1}) \geq f(x_j)$, for $j \geq 2$. On the other hand, since $\langle x_1^*, x_2 - x_1 \rangle > 0$ we infer by Theorem 1 *ii)* that $f(x_2) > f(x_1)$. The contradiction follows easily, since $x_{n+1} := x_1$. Hence $T$ is cyclically pseudomonotone.

*ii)* $\Rightarrow$ *i)*. This implication follows from Theorem 1 (*iii)* $\Rightarrow$ *i)*), since the pseudomonotonicity of $N_f \setminus \{0\}$ obviously implies the quasimonotonicity of $N_f$. □

It is well known (see [8] e.g.) that every $\partial f$-pseudoconvex lsc function is quasiconvex. Combining with Proposition 3 *i)* and proposition 1 *i)* we thus recover easily the following known result:

**Corollary 1.** *Suppose that $f$ is continuous and $0 \notin \partial f(X)$. Then*
*$f$ is quasiconvex $\Longleftrightarrow$ $f$ is $\partial$-pseudoconvex*

## 5 Normally equivalent functions

As observed in [4], two functions with the same normal operator may differ by more than an additive constant. Nevertheless, using the previous definition of $T$-pseudoconvexity (with $T = N_f \setminus \{0\}$), it is possible to characterize, under certain regularity assumptions, the set of quasiconvex functions having the same normal operator as a given quasiconvex function. This is the aim of Theorem 4.

Let us first define an equivalent relation on the set of all real-valued functions on $X$ as follows:

$$f \sim g \Leftrightarrow N_f(x) = N_g(x), \quad \forall\, x \in X.$$

**Remark:** It follows directly from the definition that $f \sim \varphi \circ f$ for every $f : X \to \mathbb{R}$ and every strictly increasing function $\varphi : \mathbb{R} \to \mathbb{R}$, since the functions $f$ and $\varphi \circ f$ have the same sublevel sets.

We now denote by $\mathcal{C}$ the class of continuous quasiconvex functions $f : X \to \mathbb{R}$ satisfying the following two regularity conditions:

$(a)$ every local minimum is a global minimum (*b*) the subset

$$Argmin\ f := \{x \in X : f(x) = \inf_X f\}$$

is included in a closed hyperplane of $X$.

Let us remark that assumption (*a*) can be rewritten as follows:

($a'$) For every $\lambda \in f(X)$, $\lambda > \inf_X f$ : $cl(S_\lambda^-) = S_\lambda$

and that, in finite dimensional spaces, (*b*) is equivalent to

($b'$) the subset $Argmin\ f$ has an empty interior.

Hypothesis (*a*) has been used in [5] in order to obtain continuity results for the normal operator.

In the following theorem, we characterize the equivalent class, denoted by $\bar{f}$, of a given function $f$ in $\mathcal{C}$.

**Theorem 4.** *The equivalent class* $\bar{f}$ *of a given function* $f$ *in* $\mathcal{C}$ *is the set of all* $N_f \setminus \{0\}$*-pseudoconvex functions, that is*

$$\bar{f} = \{g \in \mathcal{C}\ :\ \exists\, x^* \in N_f(x) \setminus \{0\}\ :\ \langle x^*, y - x\rangle \geq 0 \Longrightarrow g(y) \geq g(x)\}.$$

Another way to express this result is to say that a function $g$ of $\mathcal{C}$ has the same normal operator as a given function $f$ of $\mathcal{C}$ if, and only if, $g$ is $N_f \setminus \{0\}$-pseudoconvex.

*Proof.* Let us denote by $\mathcal{C}_f$ the subset of $\mathcal{C}$ defined by

$$\mathcal{C}_f = \{g \in \mathcal{C}\ :\ \exists\, x^* \in N_f(x) \setminus \{0\}\ :\ \langle x^*, y - x\rangle \geq 0 \Longrightarrow g(y) \geq g(x)\}.$$

(*i*) Let us first show $\bar{f} \subseteq \mathcal{C}_f$ :

Suppose that $g \in \bar{f}$ and let $x$, $y \in X$ and $x^* \in N_f(x) \setminus \{0\} = N_g(x) \setminus \{0\}$ be such that

$$\langle x^*, y - x\rangle \geq 0. \tag{6}$$

If the inequality (6) is strict, then from Theorem 1 we conclude $g(y) > g(x)$.

In case where equality holds in (6), there exists a sequence $(y_n)_n \subset X$ converging to $y$ such that $\langle x^*, y_n - x\rangle > 0$, for any $n \in \mathbb{N}$. It follows $g(y_n) \geq g(x)$, which together with the upper semicontinuity of $g$ yields $g(y) \geq g(x)$.

(*ii*) We shall now show $\mathcal{C}_f \subseteq \bar{f}$ :

Let any $g \in \mathcal{C}_f$.

<u>*Step 1:*</u> $N_f(x) \subseteq N_g(x)$, for all $x \in X$.

Assume, for a contradiction, that there exists $x \in X$ and $x^* \in N_f(x)$ such that $x^* \notin N_g(x)$.

Claim: $x \in Argmin\ g$

[ Indeed, if $x$ is not an element of $Argmin\ g$ then, using assumption $(a')$ and the fact that $x^*$ is not an element of $N_g(x)$, we immediately obtain the existence of a point $y$ of $S^-_{g(x)}$ satisfying $\langle x^*, y - x\rangle \geq 0$. A contradiction occurs since the definition of $\mathcal{C}_f$ now yields $g(y) \geq g(x)$. The claim is proved.]

Since $x^* \notin N_g(x)$, there exists $\bar{y} \in S_{g(x)} = Argmin\ g$ such that

$$\langle x^*, \bar{y} - x\rangle > 0. \tag{7}$$

Obviously

$$g(x) = g(\bar{y}) = \min g \tag{8}$$

On the other hand, $x^*$ is an element of $N_f(x)$ and therefore, (7) implies, $f(\bar{y}) > f(x)$.

Pick now any $\lambda$ in $]f(x), f(\bar{y})[$. Since $f$ is continuous, there exists $\varepsilon > 0$ such that

$$B_\varepsilon(x) \subset S^-_\lambda(f). \tag{9}$$

Due to the closedness of $S_\lambda(f)$, one can find $t \in ]0,1[$ such that

$$C_t \cap S_\lambda(f) = \emptyset \tag{10}$$

where $C_t = \{t\bar{y} + (1-t)u\ ;\ u \in B_\varepsilon(x)\}$. Since $int(C_t) \neq \emptyset$, assumption $(b)$ implies the existence of a point $\tilde{x} \in B_\varepsilon(x)$ such that for $\tilde{y} = t\bar{y} + (1-t)\tilde{x}$ we have:

$$g(\tilde{y}) > g(\bar{y}) = g(x) \tag{11}$$

Thanks to (10), it is no loss of generality to assume that $f(z) > f(\tilde{x})$ for all $z$ in $]\tilde{x}, \bar{y}]$. Applying thus a separation argument to the disjoint convex sets $]\tilde{x}, \bar{y}]$ and $S_{f(\tilde{x})}(f)$, we conclude that there exists $\tilde{x}^* \in N_f(\tilde{x}) \setminus \{0\}$ such that $\langle \tilde{x}^*, \bar{y} - \tilde{x}\rangle \geq 0$.

The definition of $\mathcal{C}_f$ now yields $g(\bar{y}) \geq g(\tilde{x})$. The contradiction is obtained, since, using (8) with the quasiconvexity of $g$ we get $g(\tilde{y}) = g(\bar{y})$, which is not compatible with (11). Hence $N_f(x) \subseteq N_g(x)$, for all $x \in X$.

*Step 2:* $N_g(x) \subset N_f(x)$, for all $x \in X$.

We shall also proceed by contradiction. So let us suppose that there exist $x \in X$ and $x^* \in N_g(x)$ such that $x^*$ is not an element of $N_f(x)$. This implies the existence of a point $y$ of $S_{f(x)}(f)$ which is not in $S_{g(x)}(g)$, i.e. $g(y) > g(x)$.

Case 1: The interior of $S_{f(x)}(f)$ is nonempty.

In this case we claim that there exists $\bar{z}$ such that $f(\bar{z}) < f(x)$ and $g(\bar{z}) > g(x)$.

Indeed if $f(y) < f(x)$, then take $\bar{z} = y$. Otherwise we have $f(x) = f(y)$, and thanks to hypothesis $(a')$ there exists a sequence $\{y_n\}_{n\geq 1}$ in $S^-_{f(x)}(f)$ converging to $y$. Since $g$ is continuous and $g(y) > g(x)$, the claim follows for $\bar{z} = y_n$ and $n$ sufficiently large.

Now one can separate (in a large sense) the subsets $S_{f(\bar{z})}(f)$ and $\{x\}$. Hence there exists $\bar{z}^* \in N_f(\bar{z}) \setminus \{0\}$ such that

$$\langle \bar{z}^*, x - \bar{z}\rangle \geq 0.$$

This immediately implies, from the definition of $\mathcal{C}_f$, that $g(\bar{z}) \leq g(x)$ which is impossible.

Case 2. The set $S_{f(x)}(f)$ has an empty interior.

In this case we have $f(x) = f(y) = \min f$. We shall conclude again to a contradiction. Indeed, by hypothesis $(b)$ there exists $\alpha^* \in X^* \setminus \{0\}$ such that

$$Argmin\ f \subseteq H_{\alpha^*} = \{u \in X\ :\ \langle \alpha^*, u - y \rangle = 0\}$$

Thus $\alpha^* \in N_f(y) \setminus \{0\}$, hence according to the definition of $\mathcal{C}_f$, $g(x) \geq g(y)$ which is impossible.

Consequently $N_f$ coincides with $N_g$ and the proof is complete. □

**Example:** If $X = \mathbb{R}$, the class $\mathcal{C}$ consists of the equivalent classes determined by the functions $\bar{f}_1(x) = x$, $\bar{f}_2(x) = -x$ and $\bar{f}_{3,\alpha}(x) = |x - \alpha|$ (for $\alpha \in \mathbb{R}$). For example, the function defined in Remark 1 is an element of $\bar{f}_1$.

**Remarks:** 1. Two equivalent functions $f, g \in \mathcal{C}$ do not necessarily have the same family of sublevel sets. Consider for instance the functions $f(x) = |x|$ and $g(x) = \max\{x, -2x\}$. Note that both functions belong to the class defined by $\bar{f}_{3,0}$ (see the previous example).

2. It is possible to consider quasiconvex functions taking the value $+\infty$. In this case one can obtain a result similar to Theorem 4 under the assumption that all functions have the same domain. Without this assumption, the fore mentioned result is not true, as can be shown by easy counterexamples.

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# PARTIE III

# Applications aux inéquations variationnelles

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**Article [11]**

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# EXISTENCE THEOREMS FOR VECTOR VARIATIONAL INEQUALITIES

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Given two real Banach spaces $X$ and $Y$, a closed convex subset $K$ in $X$, a cone with nonempty interior $C$ in $Y$ and a multivalued operator from $K$ to $2^{L(X,Y)}$, we prove theorems concerning the existence of solutions for the corresponding vector variational inequality problem, that is the existence of some $x_0 \in K$ such that for every $x \in K$ we have $A(x - x_0) \notin -\operatorname{int} C$ for some $A \in Tx_0$. These results correct previously published ones.

## 1. INTRODUCTION

Let $X$, $Y$ be real Banach spaces, $K$ be a closed, convex subset of $X$ and $L(X, Y)$ be the set of all continuous linear operators from $X$ to $Y$. Let further $T\colon K \to 2^{L(X,Y)} \setminus \{\emptyset\}$ be a multivalued operator and $C\colon K \to 2^Y$ be a multivalued mapping such that for each $x \in K$, $C(x)$ is a cone with nonempty interior $\operatorname{int} C(x)$. The purpose of this paper is to study the existence of solutions for the vector variational inequality problem (VVIP):

$$(1) \qquad \exists x_0 \in K\colon \forall x \in K, \exists A \in Tx \text{ such that } A(x - x_0) \notin -\operatorname{int} C(x_0).$$

In case $Y = \mathbb{R}$, $C(x) = \mathbb{R}^+$, the VVIP reduces to the well-known variational inequality problem [13]. The VVIP was introduced by Gianessi [8] for the case $Y = \mathbb{R}^n$ and was subsequently studied by many other authors [2, 3, 4, 14, 17] in connection with vector optimisation. Theorems asserting the existence of solutions of the VVIP are contained in [3, Theorem 2.1] for single-valued, monotone operators $T$, where $Y$ has a constant cone $C$ (that is, not depending on $x$), in [2, Theorem 2.1] for $T$ a single-valued, monotone operator, where $Y$ is equipped with a non-constant $C(x)$ and in [14, Theorem 2.1] for multivalued, pseudomonotone operators $T$, with $C(x)$ constant. However, the proofs of all these theorems contain a mistake: a certain set defined in these papers in asserted to be weakly compact, while this is not the case (see Remark 2 at the end of the present paper for details).

In the following paragraph we prove the existence of a solution of the VVIP for a multi-valued, monotone operator [9] with constant cone $C$ (Theorem 3). We also prove

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the existence of solutions for multivalued, pseudomonotone or quasimonotone operators with values consisting of completely continuous operators.

We now recall some definitions and fix our notation. A cone $C$ in $Y$ is a nonempty, convex, proper subset of $Y$, such that for all $\lambda \geqslant 0$, $y \in C$, we have $\lambda y \in C$. The dual cone $C^*$ of $C$ is the set of all $f$ in the dual space $Y^*$ such that $f(y) \geqslant 0$, for all $y \in C$.

If $C$ is closed, then

$$y \in C \Leftrightarrow f(y) \geqslant 0, \text{ for all } f \in C^*. \tag{2}$$

On the other hand, if $\operatorname{int} C \neq \emptyset$, then

$$y \in \operatorname{int} C \Leftrightarrow f(y) > 0, \text{ for all } f \in C^* \setminus \{0\}. \tag{3}$$

Note that in both cases we have $C^* \neq \{0\}$. We refer the reader to [11] for these and other properties of cones.

Now let $C\colon K \to 2^Y$ be a multivalued mapping such that for each $x \in K$, $C(x)$ is a cone with nonempty interior. A multivalued operator $T\colon K \to 2^{L(X,Y)} \setminus \{\emptyset\}$ is called:

(i) *monotone* [9], if for all $x, y \in K$ and all $A \in Tx$, $B \in Ty$ we have $(B - A)(y - x) \in C(x)$.

(ii) *(weakly) pseudomonotone* [14], if for all $x, y \in K$ and $A \in Tx$, $A(y - x) \notin -\operatorname{int} C(x)$ implies $B(y - x) \notin -\operatorname{int} C(x)$, for all (for some) $B \in Ty$.

(iii) *(weakly) quasimonotone*, if for all $x, y \in K$ and $A \in Tx$, $A(y - x) \notin -C(x)$ implies $B(y - x) \notin -\operatorname{int} C(x)$, for all (for some) $B \in Ty$.

It is obvious that (weak) quasimonotonicity is implied by (weak) pseudomonotonicity, which in turn, is implied by monotonicity. These notions generalise the well-known corresponding ones for the case $Y = \mathbb{R}$ [12, 15].

The strong operator topology (SOT) on $L(X, Y)$ is the weakest topology for which the functions $L(X, Y) \ni A \to Ax \in Y$ are continuous, for every $x \in X$. The multivalued operator $T$ is called upper hemicontinuous, if its restriction on line segments is SOT-upper semicontinuous. An operator $A \in L(X, Y)$ is called *completely continuous*, if it maps weakly convergent sequences to strongly convergent ones [5]. Any compact operator is completely continuous. The converse is not true, since the identity mapping in $\ell_1$ is completely continuous without being compact [6]. If $Y$ is finite-dimensional, all elements of $L(X, Y)$ are obviously completely continuous operators.

A point $x_0 \in K$ is called an *inner point* [10] or *relative quasi-interior point* [1] of $K$, if for all $f \in X^*$, we have

$$\forall x \in K,\ f(x - x_0) \geqslant 0 \Rightarrow \forall x \in K,\ f(x - x_0) = 0.$$

In other words, $x_0$ is an inner point of $K$ if every closed hyperplane which supports $K$ at $x_0$, necessarily contains $K$.

The set of inner points of $K$ is denoted by $\operatorname{inn} K$. Note that interior points of $K$ are also inner points, since in this case the above implication holds vacuously. In fact, whenever $\operatorname{int} K \neq \emptyset$, it can be shown that $\operatorname{int} K = \operatorname{inn} K$. However, for any separable $K$ we have $\operatorname{inn} K \neq \emptyset$, even if $\operatorname{int} K = \emptyset$ [1, 10]. In [1, 10] it was also shown that $\operatorname{inn} K$ is *lineally full* in $K$, that is for every $x \in \operatorname{inn} K$ and every $y \in K$, we have $\{tx + (1-t)y : t \in (0, 1]\} \subseteq \operatorname{inn} K$.

For any $S \subseteq L(X, Y)$ and $x \in X$, $S(x)$ will denote the set $\{Ax : A \in S\}$.

## 2. The main results

In what follows, $X$ and $Y$ will be Banach spaces. Unless explicitly mentioned, we shall always consider the *weak* topology on $X$, the norm topology on $Y$ and the strong operator topology on $L(X, Y)$. $K$ will be a nonempty closed, convex subset of $X$ and $C: K \to 2^Y$ a multifunction, such that $C(x)$ is a cone with nonempty interior for each $x \in K$. We set $D(x) = Y \setminus (-\operatorname{int} C(x))$ and for any operator $T: K \to 2^{L(X,Y)} \setminus \{0\}$ we define the multifunctions:

$$G(y) = \{x \in K : \exists A \in Tx \text{ such that } A(y - x) \in D(x)\} \tag{4}$$

$$F(y) = \{x \in K : \exists B \in Ty \text{ such that } B(y - x) \in D(x)\}. \tag{5}$$

Let $S$ be the set of all $x \in K$ such that relation (1) holds, that is, $S$ is the solution set of the VVIP. We note that $S = \bigcap_{y \in K} G(y)$.

We begin with some lemmas:

**Lemma 1.** *Let $K$ be (weakly) compact. Then* $\bigcap_{y \in K} \overline{G(y)} \neq \emptyset$.

Proof: According to K. Fan's lemma [7], it is sufficient to show that for any $x = \sum_{i=1}^{n} \lambda_i x_i$, with $x_i \in G(x_i)$, $\lambda_i \in [0, 1]$, $\sum_{i=1}^{n} \lambda_i = 1$, we have $x \in \bigcup_{i=1}^{n} G(x_i)$. Indeed, were this not the case, we would have $x \notin G(x_i)$ for all $i$'s, so for all $A \in Tx$ we would have $A(x_i - x) \in -\operatorname{int} C(x)$. Since $-\operatorname{int} C(x)$ is convex, this would imply $0 = \sum_{i=1}^{n} \lambda_i A(x_i - x) \in -\operatorname{int} C(x)$, a clear contradiction. □

**Lemma 2.** *Let $T$ be upper hemicontinuous. Then $\bigcap_{y \in K} F(y) \subseteq \bigcap_{y \in K} G(y)$. If, in addition, $\operatorname{inn} K \neq \emptyset$ and $T$ has compact values, then $\bigcap_{y \in K} F(y) = \bigcap_{y \in \operatorname{inn} K} F(y)$.*

Proof: Assume first that there exists $x \in \bigcap_{y \in K} F(y)$ such that $x \notin \bigcap_{y \in K} G(y)$. Then there would exist $y \in K$ such that $(Tx)(y - x) \subseteq -\operatorname{int} C(x)$. Set $x_t = ty +$

$(1-t)x$, $t \in (0,1)$. Since $-\operatorname{int} C(x)$ is open and $T$ is upper hemicontinuous, there exists $\delta > 0$ such that $(Tx_t)(y-x) \subseteq -\operatorname{int} C(x)$, for all $t \in (0,\delta)$. Since $t(y-x) = x_t - x$ and $-\operatorname{int} C(x)$ is a cone, we deduce that $(Tx_t)(x_t - x) \subseteq -\operatorname{int} C(x)$, that is $x \notin F(x_t)$, a contradiction. This proves the inclusion.

Now suppose that $\operatorname{inn} K \neq \emptyset$. Suppose that there exists $x \in \bigcap_{y \in \operatorname{inn} K} F(y)$ such that $x \notin \bigcap_{y \in K} F(y)$. Then for some $y \in K$, we would have

$$(Ty)(y-x) \subseteq -\operatorname{int} C(x). \tag{6}$$

Since $(Ty)(y-x)$ is compact by assumption, relation (6) implies that there exists $\varepsilon > 0$ such that

$$(Ty)(y-x) + B_\varepsilon + B_\varepsilon \subseteq -\operatorname{int} C(x) \tag{7}$$

where $B_\varepsilon = \{x \in X : \|x\| \leqslant \varepsilon\}$.

We choose $z \in \operatorname{inn} K$ and set $y_t = tz + (1-t)y$, $t \in (0,1]$. Since $\operatorname{inn} K$ is lineally full, we have $y_t \in \operatorname{inn} K$, so $x \in F(y_t)$. We also have

$$(Ty_t)(y_t - x) \subseteq (Ty_t)(y-x) + (Ty_t)(y_t - y). \tag{8}$$

Upper hemicontinuity shows that for $t$ sufficiently small, $(Ty_t)(y-x) \subseteq (Ty)(y-x) + B_\varepsilon$. On the other hand, since $T$ has compact values and is upper hemicontinuous, the image of any line segment by $T$ is compact; hence, for small $t$ we have: $(Ty_t)(y_t - y) = t(Ty_t)(z-y) \subseteq B_\varepsilon$. Hence, relations (7) and (8) imply $(Ty_t)(y_t - x) \subseteq -\operatorname{int} C(x)$, that is, $x \notin F(y_t)$, a contradiction. This shows that $\bigcap_{y \in K} F(y) = \bigcap_{y \in \operatorname{inn} K} F(y)$. □

LEMMA 3. *Suppose that $K$ is compact and for some $y \in K$, $T(y)$ is norm compact and its elements are completely continuous operators. Suppose further that the graph of $D$ is sequentially closed in $X \times Y$. Then $F(y)$ is closed.*

PROOF: Let $x \in \overline{F(y)}$. By Eberlein's theorem, there exists a sequence $(x_n)_{n \in \mathbb{N}} \subset F(y)$ converging to $x$. Then for any $n \in \mathbb{N}$, there exists $B_n \in Ty$ such that $B_n(y - x_n) \in D(x_n)$. Since $Ty$ is norm compact, we may assume with no loss of generality that $(B_n)_{n \in \mathbb{N}}$ norm-converges to some $B \in Ty$. Since $B$ is completely continuous, we have $Bx_n \to Bx$, so using a standard argument, we conclude that $B_n(y - x_n) \to B(y-x)$. The sequential closedness of the graph of $D$ implies that $B(y-x) \in D(x)$, that is $x \in F(y)$, so $F(y)$ is closed. □

LEMMA 4. *Suppose that $T$ is weakly quasimonotone and upper hemicontinuous, with compact values. Then for all $y \in \operatorname{inn} K$ we have $G(y) \subseteq F(y) \cup S$.*

PROOF: Let $x \in G(y)$ be such that $x \notin F(y)$. We shall show that $x \in S$. The assumption on $x$ implies that there exists $A \in Tx$ such that $A(y-x) \notin -\operatorname{int} C(x)$.

In addition, $A(y-x) \in -C(x)$, since otherwise the weak quasimonotonicity would imply that $x \in F(y)$. Hence $A(y-x)$ belongs to the boundary of $-C(x)$, so by the Hahn-Banach theorem there exists an $f \in Y^*$ such that $f(A(y-x)) \geqslant f(z)$, for all $z \in -C(x)$. Since $-C(x)$ is a cone containing $A(y-x)$, we easily deduce that

$$(9) \qquad (f \circ A)(y-x) = 0 \geqslant f(z), \text{ for all } z \in -C(x)$$

so, in particular

$$(f \circ A)(y) = (f \circ A)(x).$$

We now show that

$$(10) \qquad (f \circ A)(x) = (f \circ A)(y) \geqslant (f \circ A)(z), \quad \forall z \in K.$$

Indeed, suppose to the contrary, that $(f \circ A)(z) > (f \circ A)(x)$ for some $z \in K$. Set $y_t = tz + (1-t)y$, $t \in (0,1)$. Obviously $(f \circ A)(y_t - x) > 0$, for all $t \in (0,1)$, so (9) implies $A(y_t - x) \notin -C(x)$. Using the weak quasimonotonicity, we get

$$(11) \qquad (Ty_t)(y_t - x) \cap D(x) \neq \emptyset.$$

On the other hand, $x \notin F(y)$, which means that $(Ty)(y-x) \subset -\operatorname{int} C(x)$. Using the same argument as in the second part of the proof of Lemma 2, we conclude that for $t$ sufficiently small we have $x \notin F(y_t)$, a contradiction.

Hence (10) holds. Since $y \in \operatorname{inn} K$, we deduce that $(f \circ A)(x) = (f \circ A)(y) = (f \circ A)(z)$, $\forall z \in K$; that is, $(f \circ A)(z - x) = 0$, $\forall z \in K$. According to (9), $f$ belongs to the polar cone of $C(x)$, hence relation (3) implies $A(z-x) \notin -\operatorname{int} C(x)$, for all $z \in K$, that is, $x \in S$. □

THEOREM 1. *Suppose that $T$ is upper hemicontinuous and for all $y \in K$, $T(y)$ is norm compact and its elements are completely continuous operators. Let the graph of $D$ be sequentially closed in $X \times Y$ and $K$ be compact. Then in each of the following cases, the VVIP has a solution:*

- ($\alpha$) *$T$ is weakly pseudomonotone,*
- ($\beta$) *$T$ is weakly quasimonotone and $\operatorname{inn} K \neq \emptyset$.*

PROOF: ($\alpha$). If $T$ is weakly pseudomonotone, then for all $y \in K$ we have: $G(y) \subseteq F(y)$, so invoking Lemma 3 we get $\overline{G(y)} \subseteq F(y)$. Combining now Lemmas 1 and 2 we get

$$\emptyset \neq \bigcap_{y \in K} \overline{G(y)} \subseteq \bigcap_{y \in K} F(y) \subseteq \bigcap_{y \in K} G(y) = S,$$

hence $S$ is nonempty.

($\beta$). Let $T$ be weakly quasimonotone. Suppose $S = \emptyset$. Then Lemmas 3 and 4 show that $\overline{G(y)} \subseteq F(y)$, for all $y \in \operatorname{inn} K$. Hence an application of Lemmas 1 and 2 gives

$$\emptyset \neq \bigcap_{y\in K} \overline{G(y)} \subseteq \bigcap_{y\in \operatorname{inn} K} \overline{G(y)} \subseteq \bigcap_{y\in \operatorname{inn} K} F(y) = \bigcap_{y\in K} F(y) \subseteq \bigcap_{y\in K} G(y) = S,$$

which is a contradiction. Thus $S \neq \emptyset$. □

Theorem 2 replaces the hypothesis of (weak!) compactness of $K$ by a coercivity condition. We assume for simplicity that $X$ is reflexive.

THEOREM 2. *Let $X$ be a reflexive Banach space. The conclusion of the Theorem 1 still holds if the assumption "$K$ is compact" is replaced by the following coercivity condition:*

*"There exists an $R > 0$ such that for all $x \in K$, $\|x\| \geqslant R$, there exists a $z \in K$, $\|z\| < R$, such that $(Tx)(z - x) \subseteq -C(x)$."*

PROOF: Define $K_1 = \{x \in K : \|x\| \leqslant R\}$. Then $K_1$ is a nonempty, convex, compact subset of $X$.

We consider two cases:

($\alpha$) If $T$ is pseudomonotone, then by Theorem 1 the VVIP on $K_1$ has a solution $x_0$. By the coercivity condition, there exists a $z \in K$, $\|z\| < R$, such that

(12) $$(Tx_0)(x_0 - z) \subseteq C(x_0)$$

(if $\|x_0\| < R$, we may take $z = x_0$). Now given $x \in K$, there exists $t \in (0,1)$ such that $x_t = tz + (1-t)x \in K_1$. By the definition of $x_0$, there exists $A \in Tx_0$, such that $A(x_t - x_0) \notin -\operatorname{int} C(x_0)$. Combining the latter with (12), we easily deduce that $tA(x_0 - z) + A(x_t - x_0) \notin -\operatorname{int} C(x_0)$, that is, $A(x - x_0) \notin -\operatorname{int} C(x_0)$. Hence $x_0$ is also a solution of the VVIP on $K$.

($\beta$) Let $T$ be quasimonotone and $\operatorname{inn} K \neq \emptyset$. Since $\operatorname{inn} K$ is lineally full, there exists $z \in \operatorname{inn} K$ such that $\|z\| < R$. Then it is easy to prove that $z \in \operatorname{inn} K_1$ (see also the proof of Theorem 3.1 in [10]), so $\operatorname{inn} K_1 \neq \emptyset$. Hence, by Theorem 1, the VVIP on $K_1$ has a solution $x_0$, which is in fact, as in the previous case, a solution on $K$. □

Note that for a pseudomonotone operator $T$, the assumption of the norm compactness of $Ty$ may be replaced by that of compactness. Indeed, if the latter is the case, we set

$$F_1(y) = \{x \in K : (Ty)(y - x) \subseteq D(x)\},\ y \in K.$$

Then obviously

$$F_1(y) \subseteq F(y),\ \forall y \in K.$$

Hence Lemma 2 gives

$$\bigcap_{y \in K} F_1(y) \subseteq \bigcap_{y \in K} G(y).$$

An analogous proof to that of Lemma 3 shows that $F_1(y)$ is closed for all $y \in K$. Finally, the proof of Theorem 1 goes through if we consider $F_1(y)$ instead of $F(y)$.

If the cone $C$ does not depend on $x$ and $T$ is monotone, then the existence of solutions for the VVIP is a trivial consequence of the analogous theorem for the (scalar) variational inequality problem, as the following shows:

THEOREM 3. *Let $T\colon K \to 2^{L(X,Y)} \setminus \{\emptyset\}$ be a monotone, upper hemicontinuous operator with compact values and let $C$ be a cone with nonempty interior in $Y$. Suppose that $K$ is compact or that $X$ is reflexive and $T$ satisfies the coercivity condition of Theorem 2. Then the VVIP*

$$\forall y \in K,\ \exists A \in Tx \text{ such that } A(y-x) \notin -\operatorname{int} C$$

*has a solution $x$ on $K$.*

PROOF: Choose $f \in C^* \setminus \{0\}$. Then the operator $f \circ T\colon K \to 2^{X^*} \setminus \{\emptyset\}$ is obviously monotone, upper hemicontinuous with $w^*$-compact values, so there exists a solution $x \in K$ of the variational inequality

$$\forall y \in K,\ \exists u \in (f \circ T)(x)\colon (u,\, y-x) \geqslant 0$$

(see, for instance, [16]). Obviously, $u = f \circ A$ for some $A \in Tx$ and this according to relation (3) shows that $A(y-x) \notin -\operatorname{int} C$, that is, $x$ is also a solution for the VVIP. □

REMARK 1. In the case $Y = \mathrm{R}$, the set of solutions for the (scalar) V.I.P. of the pseudomonotone operator is known to be convex. This does not hold for the VVIP even if the operator $T$ is constant, as the following example shows: Let $X = Y = \mathrm{R}^2$, $C(x) = C = \mathrm{R}^2_+$, $K = \{x \in \mathrm{R}^2\colon \|x\|_2 \leqslant 1\}$ and $Tx$ be the identity operator for all $x \in K$. Then $x_1 = (0,-1)$ and $x_2 = (-1,0)$ are solutions for the VVIP while all convex combinations of them are not.

REMARK 2. The set $F(y)$ defined by relation (5) is not compact under the assumptions of Theorem 3, as it is asserted to be in the proof of Theorem 2.1 in [3, 2, 14] (where it is denoted by $F_2(y)$). Here is a counterexample: Let $X = Y = \ell_2$ and let $B$ be the closed unit ball. Let $(e_n)_{n\in\mathrm{N}}$ be the canonical basis of $\ell_2$ and $K = e_1 + B$. For each

$z \in K$, let $C(z) = C$, where $C$ is the cone $\bigcup_{\lambda \geqslant 0} \lambda(e_1 + (1/4)B)$. Note that $\operatorname{int} C \neq \emptyset$. For any $y$, $z$ in $B$ the scalar product $\langle e_1 + y/4, e_1 + z/4 \rangle$ is positive; it follows that the scalar product of any two elements of $C$ is nonnegative. Hence, $C \subseteq C^*$, so in particular $\operatorname{int} C^* \neq \emptyset$. (This was an additional assumption in [3, Theorem 2.1]). Finally, let $T\colon K \to 2^{L(\ell_2, \ell_2)}$ be such that $Tz$ is the identity operator on $\ell_2$ for each $z \in K$. Then $T$ is of course single-valued and monotone. One may immediately check that $F(0) = K \setminus \operatorname{int} C$. It follows that for all $n > 1$ we have $e_1 + e_n \in F(0)$ (indeed, otherwise we would have $e_1 + e_n = \lambda(e_1 + z/4)$ for some $z \in B$; this is impossible, since the norm of $(1 - \lambda)e_1 + e_n$ is easily seen to be greater than $\lambda/4$). However, $e_1$ is the weak limit of $e_1 + e_n$; on the other hand, since $e_1 \in \operatorname{int} C$, we have $e_1 \notin F(0)$, that is, $F(0)$ is not weakly closed.

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# Coercivity conditions and variational inequalities

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**Abstract**. Various coercivity conditions appear in the literature in order to guarantee solutions for the Variational Inequality Problem. We show that these conditions are equivalent to each other and that they are not only sufficient, but also necessary for the set of solutions to be non-empty and bounded.

**Key words.** variational inequalities – coercivity – pseudomonotonicity

## 1. Introduction

The study of the existence of solutions of Variational Inequalities on unbounded domains usually involves the same sufficient assumptions as for bounded domains, together with a coercivity condition. It is of course desirable to have hypothesis as weak as possible; for this reason various different coercivity conditions have been proposed. Non-coercive problems have also been studied.

In a recent article, Crouzeix [5] studied the variational inequality problem in finite dimensions for multivalued operators which are pseudomonotone in the sense of Karamardian (see [7], [12]). He introduced a new kind of coercivity condition and showed that the latter is not only sufficient, but also necessary for the set of solutions to be non-empty and compact (so in this sense coercivity cannot be relaxed). In this article we extend Crouzeix's results in infinite dimensions: We show that in reflexive Banach spaces if the assumptions used for bounded domains hold, then various coercivity conditions introduced in the literature are equivalent to each other, and also to the fact that the set of solutions is non-empty and bounded. In the finite dimensional case we show in particular that these conditions are also equivalent to the one introduced in [5].

## 2. Solution sets of the variational inequality problem and coercivity conditions

In what follows $K$ will be a non-empty, closed and convex subset of a real Banach space $X$. Let $T : K \to 2^{X^*} \backslash \{\emptyset\}$ be a multivalued operator with non-empty values. We recall that $T$ is called upper hemicontinuous [1], if its restriction to line segments of $K$ is upper semicontinuous, where $X^*$ is equiped with the $w^*$-topology. The operator $T$ is called pseudomonotone (according to Karamardian [7], to be distinguished from the

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*Mathematics Subject Classification (1991):* 49J40 notion defined by Brezis [3]), if for every $x, y \in K$ and $x^* \in T(x)$, $y^* \in T(y)$ the following implication holds:

$$(x^*, y - x) \geq 0 \Rightarrow (y^*, y - x) \geq 0 \tag{1}$$

where $(u^*, u)$ denotes the value of $u^* \in X^*$ at the point $u \in X$.

The Variational Inequality Problem (VIP) for the operator $T$ consists in finding $x \in K$ such that:

$$\forall y \in K, \exists x^* \in T(x) : (x^*, y - x) \geq 0 \tag{2}$$

The set of solutions of the VIP will be denoted by $S$. A solution $x \in S$ will be called a strong solution, if $x^*$ in (2) does not depend on $y$. The set of strong solutions will be denoted by $S_{str}$. It is well known (see for example [10]) that VIP is closely related to the following Dual Variational Inequality Problem (DVIP), which consists in finding $x \in K$ such that:

$$\forall y \in K, \forall y^* \in T(y) : (y^*, y - x) \geq 0 \tag{3}$$

We denote by $S_D$ the set of solutions of the DVIP.

The relations between $S$, $S_{str}$ and $S_D$ are given in the following well known proposition. We include a proof for the sake of completeness.

**Proposition 1.** *(i) If $T$ is pseudomonotone, then $S \subseteq S_D$.*
*(ii) If $T$ is upper hemicontinuous, then $S_D \subseteq S$.*
*(iii) If $T$ has $w^*$-compact and convex values, then $S = S_{str}$.*

*Proof.* (i) is obvious. For (ii), let $x \in S_D$ and suppose to the contrary that for some $y \in K$ and all $x^* \in T(x)$, we have $(x^*, y - x) < 0$. Since in that case the set $\{x^* \in X^* : (x^*, y - x) < 0\}$ is a $w^*$-open neighbourhood of $T(x)$ and $T$ is upper hemicontinuous, then setting $x_t = ty + (1 - t)x$ and taking $t$ close to zero, we obtain the relation $(x_t^*, y - x) < 0$, for all $x_t^* \in T(x_t)$. This in particular implies that $(x_t^*, x_t - x) < 0$, which contradicts the fact that $x \in S_D$. Finally, (iii) is a direct application of the minimax Theorem of Sion [11].

□

In order to show the existence of a solution for unbounded sets $K$, various coercivity conditions have been used. We single out three of these. Denoting by $\Re(K)$ the set of all weakly compact and convex subsets of $K$, we have:

$$\exists A \in \Re(K), \forall x \in K \backslash A, \forall x^* \in T(x), \exists y \in A : (x^*, x - y) > 0 \tag{C1}$$

$$\exists A \in \Re(K), \forall x \in K \backslash A, \exists y \in A, \forall x^* \in T(x) : (x^*, x - y) > 0 \tag{C2}$$

$$\exists A \in \Re(K), \forall x \in K \backslash A, \exists y \in A, \exists y^* \in T(y) : (y^*, x - y) > 0 \tag{C3}$$

Condition (C1) is standard (see for instance [5]). Condition (C2) is a weaker version of various coercivity conditions (see [4], [6], [12]). Condition (C3) was recently used by Konnov [8] to treat the quasimonotone DVIP.

*Remark 1.* It is obvious that conditions (C1), (C2) and (C3) imply respectively that the (possibly empty) solution sets $S_{str}$, $S$ and $S_D$ are included in the weakly compact set $A$.

*Remark 2.* Condition (C2) implies (C1). If $T$ has convex values, then Sion's Minimax Theorem [11] shows that conditions (C1) and (C2) are equivalent. Finally if $T$ is pseudomonotone, then (C3) clearly implies (C2).

The idea of the proof of the following Theorem is well known. However, we include a proof, since this theorem is usually stated (see [12, Theorem 2.3]) under stronger coercivity assumptions and the additional hypothesis that the values of $T$ are $w^*$-compact.

**Theorem 1.** *Let $T$ be an upper hemicontinuous, pseudomonotone operator. Suppose also that (C2) holds. Then $S \neq \emptyset$.*

*Proof.* Let $A \in \Re(K)$ be the one given by (C2). For every finite subset $F$ of $K$, the set $K_F = co(A \cup F)$ is a nonempty, convex and $w$-compact subset of $K$ (where $co(A \cup F)$ denotes as usual the convex hull of the set $A \cup F$). For each $x \in K_F$ we define $G(x) = \{y \in K_F : (x^*, y - x) \leq 0, \forall x^* \in T(x)\}$. The sets $G(x)$ are convex and $w$-compact. If $y = \sum_{i=1}^n \lambda_i x_i$, with $\sum_{i=1}^n \lambda_i = 1$ and $\lambda_i \geq 0$, then for some $i$ we have $y \in G(x_i)$. Indeed, otherwise for all $i$'s there would exist $x_i^* \in T(x_i)$ such that $(x_i^*, y - x_i) > 0$. Since $T$ is pseudomonotone, we would infer that $(y^*, y - x_i) > 0$, for all $y^* \in T(y)$, hence $0 = \sum_{i=1}^n \lambda_i (y^*, y - x_i) > 0$, a contradiction. By Ky Fan's Lemma [9, Lemma 1], $\bigcap_{x \in K_F} G(x) \neq \emptyset$. It is obvious that this intersection coincides with the set $S(F)$ of solutions of DVIP for the operator $T$ in $K_F$. By Proposition 1, $S(F)$ also coincides with the set of the solutions for VIP in $K_F$. Since (C2) holds, we have in particular (see Remark 1) that $S(F) \subseteq A$.

Let now $F_1$ and $F_2$ be two finite subsets of $K$. Since the set $S(F_1 \cup F_2)$ is always contained in the intersection $S(F_1) \cap S(F_2)$, we conclude by induction that the family of all weakly compact sets $S(F)$ (where $F$ is a finite subset of $K$) has the finite intersection property. Hence $\bigcap_{F \text{ finite}} S(F) \neq \emptyset$. It is straighforward to see that the above intersection coincides with the set $S$ of solutions of VIP for the operator $T$ in $K$.

□

From now on we assume that the Banach space $X$ is reflexive. In this framework we consider the weakly compact and convex set $K_R = \{x \in K : \|x\| \leq R\}$. It is now easily seen that (C1), (C2) and (C3) can be restated respectively as:

$$\exists R > 0, \forall x \in K \backslash K_R, \forall x^* \in T(x), \exists y \in K_R : (x^*, x - y) > 0$$

$$\exists R > 0, \forall x \in K \backslash K_R, \exists y \in K_R, \forall x^* \in T(x) : (x^*, x - y) > 0$$

$$\exists R > 0, \forall x \in K \backslash K_R, \exists y \in K_R, \exists y^* \in T(y) : (y^*, x - y) > 0$$

We proceed to show that under the usual assumptions on $T$ these coercivity conditions are not only sufficient, but also necessary for the solution set to be weakly compact.

**Theorem 2.** *Let $X$ be a reflexive Banach space. Suppose that $T$ is pseudomonotone and $S_D$ is non-empty and bounded. Then (C3) holds.*

*Proof.* Let $x_0 \in S_D$. Set $K_n = \{x \in K : \|x\| \leq n\}$. If (C3) does not hold, then for any $n \in N$ (in particular $n > \|x_0\|$), there exists $x \in K \backslash K_n$, such that for all $y \in K_n$ and all $y^* \in T(y)$, we have

$$(y^*, x - y) \leq 0 \tag{4}$$

Choose $z = \lambda x_0 + (1 - \lambda)x$ with $\lambda \in (0, 1)$ and $n - 1 \leq \|z\| < n$. Since $x_0 \in S_D$, then for any $y \in K_n$ and $y^* \in T(y)$ we have

$$(y^*, y - x_0) \geq 0 \tag{5}$$

which together with (4) implies

$$(y^*, y - z) \geq 0, \forall y^* \in T(y) \tag{6}$$

Hence, $z$ is a solution of DVIP in $K_n$. Now for any $y_1 \in K \backslash K_n$ we can find a $y \in K_n$, $(y \neq z)$ on the line segment joining $z$ and $y_1$. Then (6) implies $(y^*, y - z) \geq 0$, $\forall\, y^* \in T(y)$, hence $(y^*, y_1 - y) \geq 0, \forall\, y^* \in T(y)$. Since $T$ is pseudomonotone, we get $(y_1^*, y_1 - y) \geq 0, \forall\, y_1^* \in T(y_1)$, which in particular implies $(y_1^*, y_1 - z) \geq 0, \forall y_1^* \in T(y_1)$, i.e. $z \in S_D$. Given that $\|z\| \geq n - 1$ and $n$ is arbitrarily chosen, this contradicts the assumption that $S_D$ is bounded.

□

**Corollary 1.** *Let $X$ be a reflexive Banach space and $T$ be an upper hemicontinuous, pseudomonotone operator with $w^*$-compact convex values. Then each of the conditions (C1), (C2) and (C3) is equivalent to the fact that the set $S_{str}$ is non-empty and bounded.*

*Proof.* By Proposition 1, we have $S = S_{str} = S_D$. If $S_{str}$ is non-empty and bounded, then by Theorem 2 condition (C3) holds and so by Remark 2, (C1) and (C2) also hold. Conversely, if any of the coercivity conditions holds, then by Remark 2, conditions (C1) and (C2) hold, so $S = S_{str}$ is bounded (see Remark 1). By Theorem 1, $S$ is also non-empty.

□

## 3. The finite-dimensional case

Let $K$ and $T$ be as before. In this section we limit ourselves to the case $X = R^n$. In [5], Crouzeix considered the following coercivity assumption:

$$K_\infty \bigcap T(K)^o = \{0\} \tag{CR}$$

where $T(K)$ is the image of $K$ under $T$ and $T(K)^o = \{d : (x^*, d) \leq 0, \forall x^* \in T(K)\}$ is the polar cone of $T(K)$. Further, $K_\infty$ is the recession cone of $K$, which in the case of a closed, convex set is defined as follows (see for example [2]):

$$\begin{aligned} K_\infty &= \{d \in R^n : d = \lim_n \frac{x_n}{t_n}, x_n \in K, t_n \to +\infty\} \\ &= \{d \in R^n : \alpha + td \in K, \forall t \geq 0\} \end{aligned} \tag{7}$$

where $\alpha$ is arbitrarily chosen in $K$.

In [5] the following theorem is proved:

**Theorem 3.** *Let $T$ be an upper semicontinuous, pseudomonotone operator with compact, convex values. Then (CR) holds if and only if $S_{str}$ is nonempty and compact.*

We intend to show that even if we replace the upper semicontinuity by upper hemicontinuity, (CR) is equivalent to (C1), (C2), (C3), so Theorem 3 remains true.

**Theorem 4.** *(i) (CR) implies (C3).*

*(ii) If $S_D \neq \emptyset$, then (C3) implies (CR).*

*(iii) If $T$ is upper hemicontinuous and pseudomonotone, then (C2), (C3) and (CR) are equivalent.*

*(iv) If $T$ is upper hemicontinuous and pseudomonotone with convex values, then (C1), (C2), (C3) and (CR) are equivalent.*

*Proof.* (i) Suppose that (C3) does not hold. Then we can find a sequence $(x_n)_{n\in N} \subset K$, with $\|x_n\| > n$ , such that for every $n \in N$ and all $y \in K$, $\|y\| \leq n$, we have

$$(y^*, x_n - y) \leq 0, \forall y^* \in T(y) \tag{8}$$

We may assume with no loss of generality that $\frac{x_n}{\|x_n\|} \to d \neq 0$. If to the contrary (CR) holds, then for some $y \in K$, $y^* \in T(y)$ we should have $(y^*, d) > 0$. The latter implies that for sufficiently large $n$, we get $(y^*, \frac{x_n}{\|x_n\|}) > \frac{1}{2}(y^*, d)$, so in particular $(y^*, x_n) \to +\infty$, which clearly violates (8).

(ii) Let $x \in S_D$, and suppose that (CR) does not hold, i.e. there exists a $d \neq 0$, $d \in K_\infty$ such that for all $y \in K$, $y^* \in T(y)$, we have $(y^*, d) \leq 0$. Let $y \in K$ be arbitrarily chosen. Then for any $t > 0$, we obviously have

$$(y^*, y - (x + td)) = (y^*, y - x) - t(y^*, d) \geq 0, \forall y^* \in T(y) \tag{9}$$

which implies that $x + td \in S_D$. In particular $S_D$ cannot be bounded, hence (C3) does not hold (see Remark 1).

(iii) According to (i), (CR) implies (C3); by Remark 2, (C3) implies (C2). Thus we have only to show that (C2) implies (CR). Suppose that (C2) holds. Then Theorem 1 implies that $S \neq \emptyset$. By Remark 1, the set $S$ is bounded. By Proposition 1 we have $S = S_D$. Hence, Theorem 2 implies that (C3) holds. Using (ii), we conclude that (CR) holds.

(iv) This is an immediate consequence of (iii) and Remark 2.

□

Combining Corollary 1, Theorem 4(iv) and the fact that $S_{str} = S_D$ is closed, we get the following stronger version of Theorem 3.

**Corollary 2.** *Let $T$ be an upper hemicontinuous, pseudomonotone operator with compact, convex values. Then $S_{str}$ is nonempty and compact if and only if any of the conditions (C1), (C2), (C3) or (CR) holds.*

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# ON GENERALIZED CYCLICALLY MONOTONE OPERATORS AND PROPER QUASIMONOTONICITY

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In this paper we consider an abstract subdifferential that fulfills *a priori* a weak type of a mean value property. We survey and extend some recent results connecting the generalized convexity of nonsmooth functions with the generalized cyclic monotonicity of their subdifferentials. It is shown that, for a large class of subdifferentials, a lsc function is quasiconvex if and only if its subdifferential is a cyclically quasimonotone operator. An analogous property holds for pseudoconvexity. It is also shown that the subdifferential of a quasiconvex function is properly quasimonotone. This property is slightly stronger than quasimonotonicity, and is more useful in applications connected with variational inequalities.

*Keywords:* Generalized convexity; generalized monotonicity; cyclically monotone operators; abstract subdifferential

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## 1. INTRODUCTION

In recent years a great number of generalized monotonicity concepts have been introduced. The definitions of these concepts were usually chosen so that to guarantee a correspondence between the generalized convexity of a function and the generalized monotonicity of its derivative (if the function is smooth) or its subdifferential (if it is nonsmooth). This was motivated by the fact that the subdifferential of a convex function is a monotone operator.

However, the subdifferential of a convex function is also cyclically monotone. Actually this property is even more fundamental than monotonicity, since it is well known that an operator is cyclically monotone if and only if it is contained in the subdifferential of a convex function [1]. In this paper we survey some recent results which extend – at least partially – this connection to the classes of quasiconvex and pseudoconvex functions, by showing that their subdifferentials are, respectively, cyclically quasimonotone and cyclically pseudomonotone. At the same time we extend these results to include not only the Clarke-Rockafellar subdifferential, but also a large class of "local" subdifferentials.

We also survey and extend the properties of the "properly quasimonotone" operators. Proper quasimonotonicity is stronger than quasimonotonicity but weaker than, say, semistrict quasimonotonicity and cyclic quasimonotonicity. It is a property that applies to both single-valued and multivalued operators. Its interesting feature is that it characterizes the derivatives (or subdifferentials) of the quasiconvex functions, and seems to be more apt than quasimonotonicity to be applied to variational inequalities.

In the sequel $X$ will be a Banach space and $X^*$ its topological dual. We denote by $\langle \cdot,\cdot \rangle$ the duality pairing. We shall always deal with extended real valued functions $f: X \to R \cup \{+\infty\}$ with (effective) domain $dom(f) := \{x \in X : f(x) < +\infty\}$. The function will be called radially continuous if its restriction to line segments of $dom(f)$ is continuous. For any $x$, $y \in X$ we shall denote by $[x, y]$ the closed line segment $\{tx + (1-t)y : t \in [0, 1]\}$, while the segments $(x, y]$, $[x, y)$ and $(x, y)$ are defined analogously. Finally for every $x \in X$ and any $\varepsilon > 0$ we shall denote by $B_\varepsilon(x)$ the closed ball centered at $x$ with radius $\varepsilon$.

## 2. SUBDIFFERENTIALS AND GENERALIZED CONVEXITY

We recall that the Fenchel-Moreau subdifferential $\partial^{FM}(x_0)$ of a function $f$ at a point $x_0 \in dom(f)$ is given by the formula

$$\partial^{FM} f(x_0) := \{x^* \in X^* : f(x) \geq f(x_0) + \langle x^*, x - x_0 \rangle, \forall x \in X\}.$$

This subdifferential was mainly used for convex (usually lsc) functions. During the last two decades there has been an effort to generalize the Fenchel-Moreau subdifferential in a way that would be appropriate not only for convex functions but also for the class of all lsc functions. To this end, many concepts of subdifferentials have been proposed (see [2] for a survey). We focus our attention only on "local" subdifferentials, *i.e.*, those defined by using only local properties of the function (in contrast to some specific subdifferentials surveyed also in [2]). In order to increase generality, we shall use throughout this paper the following definition for the subdifferential concept:

DEFINITION 1 A subdifferential of a lsc function $f$ is any operator $\partial$ which associates to each $x \in X$ a subset $\partial f(x)$ of $X^*$, and has the properties:

(a) If $f$ is convex, then $\partial = \partial^{FM}$;
(b) If $x$ is a local minimum of $f$, then $0 \in \partial f(x)$;
(c) For any $v^* \in X^*$ one has $\partial(f + v^*)(x) = \partial f(x) + v^*$;
(d) If $x, y \in X$ and $f(y) > f(x)$, then there exists $z \in [x, y)$ and sequences $(x_n) \subseteq dom\, f$, $(x_n^*) \subseteq X^*$, such that $x_n \to z$, $x_n^* \in \partial f(x_n)$ and

$$\langle x_n^*, z + t(y - x) - x_n \rangle > 0, \quad \text{for all } t > 0.$$

Properties (a) and (b) are shared by all local subdifferentials. Property (d) is a weak form of a mean value property, which – under some standard regularity assumptions on the Banach space $X$ – is shared by the majority of local subdifferentials, and in particular by those included in the definition-scheme of Aussel – Corvellec – Lassonde [3]. The above definition is inspired by [2].

A subdifferential $\partial$ is said to be larger than a subdifferential $\partial_1$, if $\partial f(x) \subseteq \partial_1 f(x)$ for all $x \in X$. In this case we shall write $\partial \subseteq \partial_1$. One of the largest subdifferentials is the *dag subdifferential* [4], defined as follows:

$$\partial^{\dagger} f(x) = \{x^* \in X^* : \langle x^*, v \rangle \leq f^{\dagger}(x, v) \text{ for all } v \in X\}$$

where

$$f^{\dagger}(x, v) = \limsup_{\substack{t \searrow 0^-, y \to x \\ y \in dom f}} \frac{1}{t}(f(y + t(v + x - y)) - f(y)).$$

For instance, $\partial^{\dagger}$ is larger than the widely used and quite large Clarke-Rockafellar and upper Dini subdifferentials [2]. In the sequel we shall often assume that our subdifferential $\partial$ (in the sense of the Definition (1) is smaller than $\partial^{\dagger}$.

During the recent years, there has been an effort to establish an equivalence (just as in the smooth case [5]) between the generalized convexity of nonsmooth functions and the corresponding generalized monotonicity of their subdifferentials. This was initially done for the Clarke-Rockafellar sub-differential (see [6, 7]), and subsequently for more general cases [2, 8, 9]. We first recall the definitions of monotonicity, quasimonotonicity and pseudomonotonicity. An operator $T: X \to 2^{X^*}$ is called:

*Monotone*, if for all $x$, $y \in X$ and all $x^* \in T(x)$, $y^* \in T(y)$ one has

$$\langle y^* - x^*, y - x \rangle \geq 0. \tag{1}$$

*Pseudomonotone*, if for all $x$, $y \in X$ and all $x^* \in T(x)$, $y^* \in T(y)$ one has

$$\langle x^*, y - x \rangle \geq 0 \Rightarrow \langle y^*, y - x \rangle \geq 0 \tag{2}$$

or equivalently

$$\langle x^*, y - x \rangle > 0 \Rightarrow \langle y^*, y - x \rangle > 0 \tag{3}$$

*Quasimonotone*, if for all $x$, $y \in X$ and all $x^* \in T(x)$, $y^* \in T(y)$ one has

$$\langle x^*, y - x \rangle > 0 \Rightarrow \langle y^*, y - x \rangle \geq 0. \tag{4}$$

Among the many kinds of generalized convexity we shall consider here only quasiconvexity and pseudoconvexity. The definition of quasiconvexity is fairly standard: a function is quasiconvex if its sublevel sets are convex. Quasiconvexity has the following characterizations *via* the subdifferential:

PROPOSITION 2 *Let $f$ be a lsc function and $\partial$ a subdifferential. Consider the following statements:*

(i) *$f$ is quasiconvex;*
(ii) *If $\langle x^*, y - x \rangle > 0$ for some $x^* \in \partial f(x)$, then $f(z) \leq f(y)$ for all $z \in [x, y]$;*
(iii) *$\partial f$ is quasimonotone.*

*Then* (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). *If* $\partial \subseteq \partial^{\dagger}$, *then all three conditions are equivalent.*

For the proof of the implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) see [9]. The equivalence of all conditions under the assumptions $\partial \subseteq \partial^{\dagger}$ is shown in [4].

An analogous proposition holds for convexity:

PROPOSITION 3 *Let $f$ be a lsc function. The following are equivalent:*

(i) *$f$ is convex.*
(ii) *For all $x$, $y \in dom(f)$ and all $x^* \in \partial f(x)$ one has:*

$$\langle x^*, y - x\rangle \leq f(y) - f(x). \tag{5}$$

(iii) *The subdifferential $\partial f$ is a monotone operator.*

*Proof* Implication (i) $\Rightarrow$ (ii) follows from the fact that for convex functions we have $\partial = \partial^{FM}$. Implication (ii) $\Rightarrow$ (iii) is contained in almost all books dealing with monotone operators, see for instance [10]. Implication (iii) $\Rightarrow$ (i) is shown in [4]. ■

In contrast to quasiconvexity, the definition of pseudoconvexity is not standard, and depends essentially on the given subdifferential. Here we shall use the following definition:

DEFINITION 4 A lsc function $f$ is called pseudoconvex (with respect to a given subdifferential $\partial$), if for every $x$, $y \in X$, the following implication holds:

$$\exists x^* \in \partial f(x) : \langle x^*, y - x\rangle \geq 0 \Rightarrow \forall z \in [x, y] : f(z) \leq f(y) \tag{6}$$

An alternative definition of pseudoconvexity was given in [11]. According to that definition a lsc function $f$ is called pseudoconvex, if for every $x$, $y \in X$, the following implication holds:

$$\exists x^* \in \partial f(x) : \langle x^*, y - x\rangle \geq 0 \Longrightarrow f(x) \leq f(y) \tag{7}$$

We now compare the two definitions:.

PROPOSITION 5 *A lsc function $f$ is pseudoconvex if and only if its domain $dom\, f$ is convex and implication* (7) *holds.*

*Proof* If $f$ is pseudoconvex, then it obviously satisfies implication (7). If $dom\, f$ is not convex, then there exist $x$, $y \in dom\, f$ and $z \in [x, y]$ such that $f(z) = +\infty$. Since $f$ is lsc at $z$, there exists $\varepsilon > 0$ such that $f(z') > f(y)$ for all $z' \in B_\varepsilon(z)$. According to property (d) of Definition 1, there exist $c \in [x, z)$ and sequences $x_n \to c$ and $x_n^* \in \partial f(x_n)$ such that $\langle x_n^*, y - x_n \rangle > 0$. For $n$ sufficiently large, $B_\varepsilon(z)$ intersects $(x_n, y]$. Pick $z' \in B_\varepsilon(z) \cap [x_n, y]$. Using the pseudoconvexity of $f$, we deduce that $f(y) \geq f(z')$, a contradiction. Hence, *dom f* is convex.

Conversely, if a lsc function $f$ has a convex domain and satisfies (7), then (according to Proposition 2.2 of [12]) $f$ is also quasiconvex. In this case (7) clearly implies (6), thus $f$ is pseudoconvex. ■

Note that from the last part of the previous proof it follows that every pseudoconvex function is also quasiconvex. This is an interesting conclusion since it joins two ostensibly different classes: the class of pseudoconvex functions which is defined *via* a "mixed" property involving elements of the dual space, and on the other hand, the class of quasiconvex functions which is defined in pure geometric properties of the function.

The following lemma will be useful in the sequel:

LEMMA 6 *Let $f$ be a lsc function such that its subdifferential $\partial f$ is a pseudomonotone operator. If $x^* \in \partial f(x)$ and $\langle x^*, y - x \rangle > 0$, then $f(y) > f(x)$.*

*Proof* There exists $\varepsilon > 0$ such that $\langle x^*, y' - x \rangle > 0$ for all $y' \in B_\varepsilon(y)$. Obviously $\partial f$ is also quasimonotone, hence by Proposition 2 (implication (iii) $\Rightarrow$ (ii)) we deduce that $f(y') \geq f(x)$ for all $y' \in B_\varepsilon(y)$. In particular $f(y) \geq f(x)$. Suppose that $f(y) = f(x)$. Then $y$ would be a local minimum of $f$, hence $0 \in \partial f(y)$ by the property (b) of Definition 1. This is not possible since it contradicts implication (3). ■

Pseudoconvexity of $f$ is also related to pseudomonotonicity of $\partial f$:

PROPOSITION 7 *Suppose that $f$ is a lsc function. If $f$ is pseudoconvex and $\partial \subseteq \partial^\dagger$, then $\partial f$ is pseudomonotone.. Conversely, if $f$ is radially continuous and $\partial f$ is pseudomonotone, then $f$ is pseudoconvex.*

*Proof* The first assertion was proved in [2]. The second one was also essentially proved in [2] with the only difference that, under these assumptions, it was merely shown that $f$ satisfies condition (7). Involving now the fact that $\partial f$ is also quasimonotone, we get that $f$ is a quasiconvex function. It follows that $f$ is pseudoconvex. ■

## 3. GENERALIZED CYCLIC MONOTONICITY

One can strengthen the concept of monotonicity by postulating a strict inequality (for $x \neq y$) in (1) instead of a mere inequality; this leads to the notion of strict monotonicity. A more important and essentially different strengthening of monotonicity is cyclic monotonicity. We recall that an operator $T$ is called *cyclically monotone* if it has the following property: For every $x_1, x_2, \ldots, x_n \in X$ and every $x_1^* \in T(x_1)$, $x_2^* \in T(x_2), \ldots, x_n^* \in T(x_n)$ one has

$$\sum_{i=1}^{n} \langle x_i^*, x_{i+1} - x_i \rangle \leq 0$$

(where $x_{n+1} := x_1$).

Monotonicity describes the behavior of an operator along line segments. By contrast, cyclic monotonicity describes the behavior of an operator around a cycle determined by a finite sequence of points. It is well known that the subdifferential $\partial f$ of a lsc, convex function $f$ (which in this case, according to property (a) of Definition 1 coincides with $\partial^{FM}$), is not only monotone, but also a cyclically monotone operator.

It is easy to show that in one-dimensional spaces, monotonicity is equivalent to cyclic monotonicity [13, Chapter V, Example 4.3]. This is no longer true in more than one dimensions, as shown by the following easy example of Phelps [10]: the single-valued operator $T: R^2 \to R^2$ with $T(x_1, x_2) = (x_2, -x_1)$ is linear and positive (hence in particular monotone); however it is not cyclically monotone.

We shall now define corresponding notions of cyclic quasimonotonicity and cyclic pseudomonotonicity and show that any subdifferential (in the sense of Definition 1) of a quasimonotone or a pseudomonotone function has these properties respectively. We begin with cyclic quasimonotonicity, see also [12].

DEFINITION 8 An operator $T: X \to 2^{X^*}$ is called cyclically quasimonotone, if for every $x_1, x_2, \ldots, x_n \in X$, there exists an $i \in \{1, 2, \ldots, n\}$ such that

$$\langle x_i^*, x_{i+1} - x_i \rangle \leq 0, \forall x_i^* \in T(x_i)$$

(where $x_{n+1} := x_1$).

Clearly, every cyclically monotone operator is cyclically quasimonotone, while every cyclically quasimonotone operator is quasimonotone. Cyclic quasimonotonicity is considerably more restrictive than quasimonotonicity (see Example 11 below). However, this property characterizes all subdifferentials of quasiconvex functions, provided they are smaller than $\partial^\dagger$, as shown by the next theorem.

THEOREM 9 *Let $f$ be lsc. If $f$ is quasiconvex and $\partial \subseteq \partial^\dagger$, then $\partial f$ is cyclically quasimonotone*

*Proof* It is sufficient to show that $\partial^\dagger f$ is cyclically quasimonotone. Suppose to the contrary that there exist $x_1, x_2, \ldots, x_k \in dom\ f$ and $x_i^* \in \partial^\dagger f(x_i), i = 1, 2, \ldots, k$ such that $\langle x_i^*, x_{i+1} - x_i \rangle > 0$ for all $i = 1, 2, \ldots, k$, where $x_{k+1} \equiv x_1$. By Proposition 2, we have $f(x_{i+1}) \geq f(x_i)$ for all $i$'s. It follows that $f(x_1) = f(x_2) = \cdots = f(x_k)$.

From $\langle x_1^*, x_2 - x_1 \rangle > 0$ we deduce that $f^\dagger(x_1, x_2 - x_1) > 0$. Hence there exist sequences $y_n \to x_1$ and $t_n \to 0^+$ such that $f(y_n + t_n(x_2 - y_n)) - f(y_n) > 0$. Since $y_n + t_n(x_2 - y_n) \in [y_n, x_2]$ and $f$ is quasiconvex, we deduce that $f(y_n) < f(x_2) = f(x_1)$. On the other hand, since $\langle x_k^*, x_1 - x_k \rangle > 0$, for $n$ sufficiently large we have $\langle x_k^*, y_n - x_k \rangle > 0$. Hence, by Proposition 2, one has $f(y_n) \geq f(x_k) = f(x_1)$, a contradiction. ■

*Remark 10* The converse of the above theorem (*i.e.*, $\partial f$ is cyclically quasimonotone implies that $f$ is quasiconvex) is also true in view of Proposition 2 and the fact that every cyclically quasimonotone operator is quasimonotone.

In [1] it was proved that the subdifferentials of convex functions are maximal (cyclically) monotone operators. However, similar properties do not hold for the subdifferentials of quasiconvex function (see [12]).

As was the case for monotonicity, quasimonotonicity is equivalent to cyclic quasimonotonicity in one-dimensional spaces [12, Proposition 3.3]. We reproduce here from [12] the following example which shows that if the dimension of the space is greater than 1, then cyclic quasimonotonicity not only is not equivalent to quasimonotonicity, but also is not implied even by strong monotonicity:

*Example 11* Let $T: R^2 \to R^2$ be defined by $T(a, b) = ((a/2) - b, a + (b/2))$. Then the operator $T$ is strongly monotone (*i.e.*, satisfies

$\langle T(x) - T(y), x - y \rangle \geq k \| x - y \|^2$ for all $x, y \in R^2$ where $k > 0$ is a constant). In particular, $T$ is pseudomonotone and quasimonotone. However, it is not cyclically quasimonotone, as one sees by considering the points $x_1 = (1,0)$, $x_2 = (0,1)$, $x_3 = (-1,0)$ and $x_4 = (0,-1)$.

We now recall from [12] the notion of cyclic pseudomonotonicity:

DEFINITION 12 An operator $T : X \to 2^{X^*}$ is called cyclically pseudomonotone, if for every $x_1, x_2, \ldots, x_n \in X$, the following implication holds:

$$\exists i \in \{1,2,\ldots,n\}, \exists x_i^* \in T(x_i) : \langle x_i^*, x_{i+1} - x_i \rangle > 0 \Rightarrow$$
$$\exists j \in \{1,2,\ldots,n\}, \forall x_j^* \in T(x_j) : \langle x_j^*, x_{j+1} - x_j \rangle < 0$$

(where $x_{n+1} := x_1$).

Equivalently, $T$ is cyclically pseudomonotone if for every $x_1$, $x_2, \ldots, x_n \in X$ and $x_i^* \in T(x_i), i = 1,2,\ldots,n$, one has the implication:

$$\langle x_i^*, x_{i+1} - x_i \rangle \geq 0 \text{ for } i = 1,2,\ldots,n-1 \Rightarrow \langle x_n^*, x_1 - x_n \rangle \leq 0$$

(we thank Professor R. John for this remark). It can be easily checked that every cyclically monotone operator is cyclically pseudomonotone, while every cyclically pseudomonotone operator is pseudomonotone and cyclically quasimonotone.

The following theorem was proved in [12, Theorem 3.6] for the Clarke-Rockafellar subdifferential. The same proof applies to any subdifferential satisfying Definition 1:

THEOREM 13 *Let $f$ be a lsc function. If $f$ is pseudoconvex and $\partial \subseteq \partial^\dagger$, then $\partial f$ is cyclically pseudomonotone. Conversely, if $f$ is radially continuous and $\partial f$ is cyclically pseudomonotone, then $f$ is pseudoconvex.*

## 4. PROPER QUASIMONOTONICITY

As we already noted above, cyclic generalized monotonicity describes the behavior of an operator around a "cycle" consisting of a finite number of points. If we consider the convex hull of such a cycle, we are lead to interesting restatements of the standard definitions of a monotone and a pseudomonotone operator (see [12]):

PROPOSITION 14

(i) *An operator $T$ is monotone, if and only if for every finite sequence $(x_i)_{i=1}^n \subset X$, every $x_i^* \in T(x_i)$ and every $y = \Sigma_{i=1}^n \lambda_i x_i$, with $\Sigma_{i=1}^n \lambda_i = 1$ and $\lambda_i > 0$, one has*

$$\sum_{i=1}^{n} \lambda_i \langle x_i^*, y - x_i \rangle \leq 0.$$

(ii) *An operator $T$ with convex domain $D(T)$ is pseudomonotone, if and only if for every finite sequence $(x_i)_{i=1}^n \subset X$ and every $y = \Sigma_{i=1}^n \lambda_i x_i$, with $\Sigma_{i=1}^n \lambda_i = 1$ and $\lambda_i > 0$, the following implication holds:*

$$\exists i \in \{1, 2, \ldots, n\}, \quad \exists x_i^* \in T(x_i) : \langle x_i^*, y - x_i \rangle > 0 \Rightarrow$$
$$\exists j \in \{1, 2, \ldots, n\}, \quad \forall x_j^* \in T(x_j) : \langle x_j^*, y - x_j \rangle < 0.$$

In contrast to the class of monotone and pseudomonotone operators, the following definition (adapted to the quasimonotone case) leads to a different, more restrictive class of operators:

DEFINITION 15 An operator $T: X \rightarrow 2^{X^*}$ is called properly quasimonotone, if for every finite sequence $(x_i)_{i=1}^n \subset X$ and every $y = \Sigma_{i=1}^n \lambda_i x_i$, with $\Sigma_{i=1}^n \lambda_i = 1$ and $\lambda_i > 0$, there exists $i \in \{1, 2, \ldots, n\}$ such that

$$\forall x_i^* \in T(x_i) : \langle x_i^*, y - x_i \rangle \leq 0.$$

Choosing $n = 2$ and $y = (x_1 + x_2)/2$, we see that a properly quasimonotone operator is quasimonotone. In [12] it has been proved that every cyclically quasimonotone operator is properly quasimonotone. This has the following direct consequence:

COROLLARY 16 *Let $f$ be lsc and $\partial \subseteq \partial^\dagger$. Then $f$ is quasiconvex if and only if $\partial f$ is properly quasimonotone.*

It is also straightforward that proper quasimonotonicity and quasimonotonicity coincide whenever $X = R$. However, this is not true in general Banach spaces, as the following example from [12] shows.

*Example 17* Let $X = R^2$, $x_1 = (0, 1)$, $x_2 = (0, 0)$, $x_3 = (1, 0)$. We define $T: R^2 \rightarrow R^2$ by $T(x_1) = (-1, -1)$, $T(x_2) = (1, 0)$, $T(x_3) = (0, 1)$

and $T(x) = 0$ otherwise. Then the operator $T$ is quasimonotone without being properly quasimonotone.

We reproduce from [12] the following diagram that shows the relations between the various generalized monotonicity properties we considered:

$$\begin{array}{ccc}
\text{cyclically monotone} & \rightarrow & \text{monotone} \\
\downarrow & & \downarrow \\
\text{cyclically pseudomonotone} & \rightarrow & \text{pseudomonotone} \\
\downarrow & & \downarrow \\
\text{cyclically quasimonotone} & \rightarrow & \text{properly quasimonotone} \\
 & & \downarrow \\
 & & \text{quasimonotone}
\end{array}$$

According to the above diagram, any pseudomonotone ozperator is properly quasimonotone. Thus, proper quasimonotonicity appears to be only slightly stronger than quasimonotonicity. This can be also seen if we compare to semistrict quasimonotonicity; according to [14], a multivalued operator $T:X \rightarrow 2^{X^*}$ is called semistrictly quasimonotone if it is quasimonotone and for any distinct $x, y$ in its domain one has the implication:

$$\exists x^* \in Tx : \langle x^*, y - x\rangle > 0 \Rightarrow$$
$$\exists z \in \left(\frac{x+y}{2}, y\right), \exists z^* \in T(z) : \langle z^*, y - x\rangle > 0.$$

The denomination is justified by the fact that a locally Lipschitz function $f$ is semistrictly quasiconvex if and only if its Clarke-Rockafellar subdifferential is semistrictly quasimonotone [14].

It can be easily seen that a pseudomonotone operator is semistrictly quasimonotone. In the hierarchy of generalized monotonicity concepts [15], sorted from the strongest to the weakest, semistrict quasimonotonicity was just one step above quasimonotonicity. However, proper quasimonotonicity should be situated between these two concepts, since semistrict quasimonotonicity implies proper quasimonotonicity, as shown in [14].

Properly quasimonotone operators are closely related to the Variational Inequality Problem (VIP). Let $K$ be a closed, convex subset of $X$ and $T:K \rightarrow 2^{X^*}\backslash\{\emptyset\}$ be a multivalued operator. Most known proofs of existence of a solution for the VIP in $K$ use some form of generalized monotonicity of the operator $T$ (sometimes together with some additional assumptions) to ensure that the multifunction $G$ defined by

$$G(x) = \{y \in K : \forall x^* \in T(x) \text{ we have } \langle x^*, y - x\rangle \leq 0\}$$

is KKM, *i.e.*, has the following property: For each $x_1, x_2, \ldots, x_n \in K$ and $y \in co\{x_1, x_2, \ldots, x_n\}$, one has $y \in \cup_{i=1}^n G(x_i)$. However, it is obvious that $G$ is KKM if and only if $T$ is properly quasimonotone. Thus, it appears that the class of properly quasimonotone operators is just the right class for these proofs to go through. We refer the reader to [14] for details.

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# PARTIE IV

# Intégration d'opérateurs multivoques

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# Lower Subdifferentiability and Integration

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**Abstract.** We consider the question of integration of a multivalued operator $T$, that is the question of finding a function $f$ such that $T \subseteq \partial f$. If $\partial$ is the Fenchel–Moreau subdifferential, the above problem has been completely solved by Rockafellar, who introduced cyclic monotonicity as a necessary and sufficient condition. In this article we consider the case where $f$ is quasiconvex and $\partial$ is the lower subdifferential $\partial^{<}$. This leads to the introduction of a property that is reminiscent to cyclic monotonicity. We also consider the question of the density of the domains of subdifferential operators.

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**Key words:** quasiconvex function, lower subdifferential, integration.

## 1. Introduction

The integration of an operator $T: X \to X^*$, i.e., the question of finding a differentiable function $f$ such that $T = \nabla f$, has attracted much interest. When the operator $T$ is multivalued, this question is transformed into showing that for some function $f$ one has $T \subseteq \partial f$ (for some notion of subdifferential). The above problem has been solved by Rockafellar, in case one imposes that $f$ should be convex and takes $\partial$ to be the Fenchel–Moreau subdifferential of convex analysis:

$$\partial f(x) = \{x^* \in X^* : f(y) - f(x) \geqslant x^*(y - x),\ \forall y \in X\}. \tag{1}$$

This gave rise to the class of cyclically monotone operators. Every such operator $T$ is included in the subdifferential $\partial f_T$ of a l.s.c. convex function $f_T$ (and coincides with $\partial f_T$ if and only if $T$ is maximal). In particular the function $f_T$ turns out to be unique up to a constant [16].

The general question of integrating a non cyclically monotone multivalued operator $T: X \to 2^{X^*}$ has already been considered by several authors [3, 7, 15, 18], etc. In this article we relax the convexity requirement on $f$ to quasiconvexity, that is convexity of its sublevel sets. The class of quasiconvex functions is much larger than the class of convex functions and appears naturally in concrete problems. A first difficulty in the question of integration arises with the choice of a subdifferential. One line of research consists in using a subdifferential of local nature generalizing the derivative (see [4, 17], e.g.). In that case, characterizations of quasiconvexity have been established by means of the concept of quasimonotonicity for multivalued operators [1, 6, 11], e.g., and references therein). In this line of research, cyclic quasimonotonicity (defined in [5]) turned out to be an intrinsic property of the subdifferentials of quasiconvex functions. Thus an analogy with the convex case appears. However, it is far from obvious to find additional assumptions ensuring that a cyclically quasimonotone operator is included in the subdifferential of a quasiconvex function.

Here we depart from this track and we work with the lower subdifferential of Plastria [14] which is an adaptation to the quasiconvex case of the Fenchel–Moreau subdifferential (1). For any $x \in X$ with $f(x) < +\infty$, the lower subdifferential $\partial^< f(x)$ is given by:

$$\partial^< f(x) = \{x^* \in X^* : f(y) - f(x) \geqslant x^*(y - x),\ \forall y \in S^<_{f(x)}\}, \tag{2}$$

where $S^<_{f(x)} := \{x' \in X : f(x') < f(x)\}$ is the strict sublevel set. Relation (2) can also take the following form:

$$\partial^< f(x) = \left\{x^* \in X^* : f(y) \geqslant \min\left\{\begin{array}{c} f(x) \\ f(x) + x^*(y - x) \end{array}\right\}, \forall y \in X\right\}. \tag{3}$$

One easily observes that, as with the Fenchel–Moreau subdifferential, $\partial^<$ is not a local notion: two functions that coincide in a neighborhood of $x$, may not have the same lower subdifferential at this point. We also remark that for every $x^* \in \partial^< f(x)$, we have $\{\lambda x^* : \lambda \geqslant 1\} \subseteq \partial^< f(x)$, which shows that $\partial^< f(x)$ is not bounded. (In particular $\partial f$ and $\partial^< f$ are in general different even for convex functions.) However, under this notion, quasiconvex Lipschitz functions are characterized by the existence of a bounded selection for their lower subdifferential (see [14] for $X = \mathbb{R}^n$ and [8] for the general case). We extend these results in Section 4, while in Section 3 we consider the question of the density of the domain of the Fenchel–Moreau subdifferential of an arbitrary function $f$. Note that if the function $f$ is not convex, the Fenchel–Moreau subdifferential is often empty. As we show in Section 3, its nonemptiness in a dense subset of $X$ implies the convexity of $f$.

In Section 2 we review some results concerning cyclically monotone operators and Rockafellar's integration technique for the Fenchel–Moreau subdifferential. We note in particular that this integration requires a property that – a priori – seems to be weaker than cyclic monotonicity (CM), namely what we call 'cyclic monotonicity with respect to a certain point $x_0$' (CM($x_0$)). However, these properties turn out to be equivalent. This alternative description of cyclic monotonicity motivates the introduction, in Section 5, of a new class of operators, that is operators fulfilling a certain property ($L(x_0)$) with respect to some fixed point $x_0$. This property represents a pointwise version of cyclic monotonicity: indeed

$(L(x_0))$ is strictly weaker than cyclic monotonicity, while an operator $T$ is cyclically monotone if, and only if, $T$ satisfies $(L(x))$ for all $x \in \mathrm{dom}(T)$. We also show that the lower subdifferential $\partial^< f$ of any function $f$ restricted to the set $S^<_{f(x_0)} \cup \{x_0\}$ fulfills $(L(x_0))$. Moreover, any such operator $T$ is included in the lower subdifferential $\partial^< f$ of some quasiconvex l.s.c. function $f$.

In the last section we introduce the class of operators fulfilling another property – that we denote by $(R(x_0))$ – relative to a (fixed) point $x_0$. This property is strictly weaker than $(L(x_0))$. It is shown that if $T$ fulfills $(R(x))$ at every point of its domain, then it is monotone. The main result of Section 6 states that the operator $T$ defined by $T(x) = \partial^< f(x)$, if $x \neq x_0$ and $T(x_0) = \partial f(x_0)$ satisfies $(R(x_0))$, for any $f$ such that $\partial f(x_0) \neq \emptyset$. On the other hand, any operator of this class is always contained in the lower subdifferential of some quasiconvex l.s.c. function $f$. Thus we obtain a characterization of this class, which is similar to the one given for cyclic monotonicity by means of the Fenchel–Moreau subdifferential.

Let us point out that while lower semicontinuous convex functions are determined up to a constant by their Fenchel–Moreau subdifferentials, two continuous (even differentiable) quasiconvex functions having the same Plastria subdifferential may differ essentially. In fact, the Plastria subdifferential of a continuous quasiconvex function may even be empty, as shown by the example of the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^p$, where $p > 1$ is an odd integer. (More generally, $\partial^< f$ is empty whenever $\liminf_{\|x\| \to \infty} f(x)/\|x\| = -\infty$.)

Throughout this paper, we often use the following abbreviations: FM subdifferential for the Fenchel–Moreau subdifferential, l.s.c. for lower semicontinuous and CM operator for a cyclically monotone operator. Furthermore, $X$ denotes a Banach space with dual space $X^*$, $f$ a function on $X$ with values in $\mathbb{R} \cup \{+\infty\}$, and $T$ a multivalued operator defined on $X$ and taking its values in the subsets of $X^*$. For any $x \in X$ and any $x^* \in X^*$ we denote by $x^*(x)$ the value of the functional $x^*$ at the point $x$. We also use the standard notation: $B_\varepsilon(x)$ for the closed ball centered at $x$ with radius $\varepsilon > 0$, $\mathrm{dom}(f) := \{x \in X : f(x) \in \mathbb{R}\}$ for the domain of the function $f$, $S_{f(x)} := \{x' \in X : f(x') \leqslant f(x)\}$ and $S^<_{f(x)} = \{x' \in X : f(x') < f(x)\}$ for the sublevel and the strict sublevel sets of $f$ respectively and $\mathrm{dom}(T) := \{x \in X : T(x) \neq \emptyset\}$ for the domain of the multivalued operator $T$.

## 2. Integration of the Subdifferential of a Nonconvex Function

The properties we introduce and discuss in this article are defined by fixing a certain point $x_0$ as a base point. It is natural to ask whether this choice plays any role. In this section we shall see that this is not the case for the property of cyclic monotonicity.

DEFINITION 2.1. Let $T : X \to 2^{X^*}$ be a multivalued operator. The operator $T$ is called

(i) cyclically monotone with respect to a point $x_0 \in \mathrm{dom}(T)$ (or alternatively $T$ has the $(\mathrm{CM}(x_0))$ property), if for any $x_1, x_2, \ldots, x_n \in X$ and any $x_0^* \in T(x_0), x_1^* \in T(x_1), \ldots, x_n^* \in T(x_n)$ one has

$$x_n^*(x_0 - x_n) + \sum_{i=0}^{n-1} x_i^*(x_{i+1} - x_i) \leqslant 0,$$

(ii) cyclically monotone (CM), if it satisfies $(\mathrm{CM}(x))$ for every point $x$ of its domain.

It is clear that Definition 2.1(ii) coincides with the standard definition of cyclic monotonicity (see Definition 2.20 in [13]), while it obviously implies Definition 2.1(i). The following proposition shows that the converse is also true.

PROPOSITION 2.2. *Every operator satisfying* $(\mathrm{CM}(x_0))$ *is cyclically monotone.*

*Proof.* Suppose that $T$ satisfies $(\mathrm{CM}(x_0))$ and that for some $(z_i)_{i=1}^n \subset \mathrm{dom}(T)$ and $z_i^* \in T(z_i), i = 1, 2, \ldots, n$ we have $z_n^*(z_1 - z_n) + \sum_{i=1}^{n-1} z_i^*(z_{i+1} - z_i) = \alpha > 0$. For any $k \in \mathbb{N}$ and $i = 0, 1, 2, \ldots, k \cdot n$ we define $x_{i+1} = z_{i(\mathrm{mod}\, n)+1}$, $x_{i+1}^* = z_{i(\mathrm{mod}\, n)+1}^*$ (where for $i \geqslant 0$, we have $j = i \pmod n$ iff $i - j = pn$, for some $p \in \mathbb{N}$ and $0 \leqslant j < n$). Let $x_0^* \in T(x_0)$. Since $T$ satisfies $(\mathrm{CM}(x_0))$ we have:

$$x_{kn+1}^*(x_0 - x_{kn+1}) + \sum_{i=0}^{kn} x_i^*(x_{i+1} - x_i) \leqslant 0$$

which implies:

$$x_0^*(z_1 - x_0) + z_1^*(x_0 - z_1) + k\left\{z_n^*(z_1 - z_n) + \sum_{i=1}^{n-1} z_i^*(z_{i+1} - z_i)\right\} \leqslant 0.$$

Taking the limit as $k \to +\infty$ we obtain a contradiction. □

*Remark 2.3.* An operator $T$ can be cyclically monotone in a trivial way, if for instance $\mathrm{dom}(T) = \emptyset$ or if $\mathrm{dom}(T) = \{x_0\}$.

Let us observe that cyclic monotonicity of $\partial f$ is tied to the very definition of the Fenchel–Moreau subdifferential $\partial f$ and does not depend on the convexity of the function $f$. Indeed, if $f$ is any function and $T: X \to 2^{X^*}$ any operator satisfying $T \subseteq \partial f$, then for any $x_0, x_1, \ldots, x_n \in X$ and $x_i^* \in T(x_i)$ $(i = 0, 1, \ldots, n)$ relation (1) guarantees that $f(x_{i+1}) - f(x_i) \geqslant x_i^*(x_{i+1} - x_i)$. Setting $x_{n+1} := x_0$ and adding the previous inequalities yields $\sum_{i=0}^n x_i^*(x_{i+1} - x_i) \leqslant 0$. Let us state this observation as a lemma for further reference.

LEMMA 2.4. *For any function $f$, any operator $T$ satisfying $T \subseteq \partial f$ is cyclically monotone.*

The converse assertion dealing with the integration of cyclically monotone operators is more interesting. The proof can be found in [16] and essentially requires condition (CM($x_0$)).

THEOREM 2.5. *Let $T$ be a multivalued operator satisfying* (CM($x_0$)) *at some point $x_0$ of its domain. Then there exists a l.s.c. convex function $f_T$ such that $T \subseteq \partial f_T$.*

The l.s.c. convex function $f_T$ of the above theorem has been constructed in [16] (see also [13]) by the following formula, in which $c$ is a fixed constant:

$$f_T(x) = c + \sup\left\{ x_n^*(x - x_n) + \sum_{i=0}^{n-1} x_i^*(x_{i+1} - x_i) \right\}, \tag{4}$$

where the supremum is taken over all $n \in \mathbb{N} \setminus \{0\}$, all finite sequences $\{x_1, x_2, \ldots, x_n\}$ in $\mathrm{dom}(T)$ and all $x_i^* \in T(x_i)$, for $i = 0, 1, \ldots, n$.

Let us note here that (CM($x_0$)) ensures that $f_T$ is not identically equal to $+\infty$, since $f_T(x_0) = c$.

*Remark 2.6.* Combining Theorem 2.5 with Lemma 2.4 we obtain an alternative way to establish Proposition 2.2.

We also recall that the second conjugate $f^{**}$ of a proper function $f$ is given by:

$$f^{**}(x) = \sup_{x^* \in X^*} [x^*(x) - f^*(x^*)], \tag{5}$$

where

$$f^*(x^*) = \sup_{x \in X} [x^*(x) - f(x)]. \tag{6}$$

Since the subdifferential of any function $f$ is cyclically monotone, the l.s.c. convex function $f_T$ given in (4) is well defined when one takes $T = \partial f$ and $\partial f(x_0) \neq \emptyset$. If in particular $f$ is l.s.c. convex, the uniqueness of Rockafellar's integration ([16]) shows that for $c = f(x_0)$ one has $f_T = f$, so in particular $f_T = f^{**}$. If now $f$ is not convex, a natural question arises: is $f_T$ related to $f^{**}$? We provide below a positive answer in finite dimensions under a coercivity assumption on $f$. Let us first observe that (for $c = f(x_0)$) $f_T \leqslant f$ from which it follows $f_T \leqslant f^{**}$, since $f^{**}$ is the greatest l.s.c. convex function majorized by $f$.

PROPOSITION 2.7. *Let $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c.,* 1*-coercive function* (*i.e.*, $\lim_{\|x\| \to \infty} f(x)/\|x\| = +\infty$), *and let $T = \partial f$. Then for some constant c, the functions $f_T$ and $f^{**}$* (*defined in* (4) *and* (5) *respectively*) *coincide.*

*Proof.* From our assumptions it follows that $f$ attains its minimum at some point $x_0$, hence $0 \in \partial f(x_0)$. It follows that $f^{**}(x_0) = f(x_0)$. Taking $c = f(x_0)$ in (4), we conclude from (1) that $f_T \leqslant f$. Since $f_T$ is convex l.s.c., it follows that $f_T \leqslant f^{**}$.

Let us prove the reverse inequality. Since the function $f^{**}$ is l.s.c. and convex, it follows from Theorem B in [16] that:

$$f^{**}(x) = f(x_0) + \sup\left\{\sum_{i=0}^{n-1} x_i^*(x_{i+1} - x_i) + x_n^*(x - x_n)\right\}, \tag{7}$$

where the supremum is taken over all $n \in \mathbb{N}$, all finite sequences $\{x_1, x_2, \ldots, x_n\}$ in $\mathrm{dom}(\partial f^{**})$ and all choices $x_i^* \in \partial f^{**}(x_i)$, for $i = 0, 1, \ldots, n$.

Using the inequality $f^{**} \leqslant f$, for any $x \in \mathbb{R}^n$ one has:

$$f(x) = f^{**}(x) \Rightarrow \partial f^{**}(x) \subseteq \partial f(x). \tag{8}$$

In particular, since $f^{**}(x_0) = f(x_0)$, one observes that

$$\partial f^{**}(x_0) \subseteq \partial f(x_0). \tag{9}$$

Fix now $x \in X$ and consider any $M < f^{**}(x)$. For some $x_1, x_2, \ldots, x_n \in X$ and $x_i^* \in \partial f^{**}(x_i)$ one has

$$M - f(x_0) < x_0^*(x_1 - x_0) + x_1^*(x_2 - x_1) + \cdots + x_n^*(x - x_n). \tag{10}$$

Since the function $f$ is 1-coercive and is defined in a finite-dimensional space, using Theorem 3.6 of [2] we conclude that for $i \in \{1, 2, \ldots, n\}$, there exist $(y_i^j)_{j=1}^{k_i}$ in $X$, and $(\lambda_i^j)_{j=1}^{k_i}$in $(0, 1)$ with $\sum_{j=1}^{k_i} \lambda_i^j = 1$ such that

$$x_i^* \in \bigcap_{j=1,2,\ldots,k_i} \partial f(y_i^j) \tag{11}$$

and

$$x_i = \sum_{j=1}^{k_i} \lambda_i^j y_i^j. \tag{12}$$

CLAIM. *There exists some $y_1^{j_1}$ such that*

$$x_0^*(y_1^{j_1} - x_0) + x_1^*(x_2 - y_1^{j_1}) \geqslant x_0^*(x_1 - x_0) + x_1^*(x_2 - x_1). \tag{13}$$

*Proof.* If this were not the case, then for every $j$ we would have

$$x_0^*(y_1^j - x_0) + x_1^*(x_2 - y_1^j) < x_0^*(x_1 - x_0) + x_1^*(x_2 - x_1). \tag{14}$$

Multiplying both sides of (14) by $\lambda_1^j$ and adding the resulting inequalities for $j = 1, 2, \ldots, k_1$ we get a contradiction by using (12). □

Arguing in the same way as in the proof of the above claim, we can find some $y_2^{j_2}$ such that

$$x_1^*(y_2^{j_2} - y_1^{j_1}) + x_2^*(x_3 - y_2^{j_2}) \geqslant x_1^*(x_2 - y_1^{j_1}) + x_2^*(x_3 - x_2). \tag{15}$$

It follows that

$$\begin{aligned} &x_0^*(y_1^j - x_0) + x_1^*(y_2^{j_2} - y_1^{j_1}) + x_2^*(x_3 - y_2^{j_2}) \\ &\quad \geqslant x_0^*(x_1 - x_0) + x_1^*(x_2 - x_1) + x_2^*(x_3 - x_2). \end{aligned}$$

Proceeding like this, we inductively show that

$$M - f(x_0) < x_0^*(y_1^{j_1} - x_0) + x_1^*(y_2^{j_2} - y_1^{j_1}) + \cdots + x_n^*(x - y_n^{j_n}).$$

Note that from (9) we have $x_0^* \in \partial f(x_0)$, while from (11) we get $x_i^* \in \partial f(y_i^{j_i})$, for $i = 1, 2, \ldots, n$. Now (4) guarantees that $M < f_T(x)$. Since $M$ can be chosen to be arbitrarily close to $f^{**}(x)$, we conclude that $f_T(x) \geqslant f^{**}(x)$, hence equality holds. □

Let us remark that the above proof shows that $f_T = f^{**}$ whenever the l.s.c. function $f$ satisfies the following condition:

(C) For any $x \in \mathrm{dom}(\partial f^{**})$ and $x^* \in \partial f^{**}(x)$, there exist $(y_i)_{i=1}^k \subseteq X$ and $(\lambda_i)_{i=1}^k$ in $(0, 1)$ with $\sum_{i=1}^k \lambda_i = 1$, such that $x = \sum_{i=1}^k \lambda_i y_i$ and $x^* \in \bigcap_{j=1}^k \partial f(y_i)$.

The conclusion of Proposition 2.7 can be satisfied also by noncoercive functions (in infinite-dimensional spaces), as for instance by the function $f(x) = \min\{\|x\|, 1\}$.

COROLLARY 2.8 *Let $f$ and $g$ be two l.s.c. functions satisfying condition* (C). *If $\partial f = \partial g$, then $f^{**} = g^{**}$* (*up to a constant*).

*Proof.* Let $T = \partial f = \partial g$. Note that condition (C) yields $\mathrm{dom}(T) \neq \emptyset$. Let $x_0 \in \mathrm{dom}(T)$. The proof of Proposition 2.7 shows that $f^{**} = f_T$ when one takes $c = f(x_0)$ in (4) and that $g^{**} = f_T + g(x_0) - c$. □

## 3. Functions with a Dense Domain of Subdifferentiability

In the preceding section we considered operators that are (included in) the subdifferential of a nonconvex function. These operators are cyclically monotone, but this may happen in a trivial way, see Remark 2.3. The example of the function $f(x) = \min\{\|x\|, 1\}$ (also $f(x) = \sqrt{\|x\|}$) shows that one may have $f_T = f^{**}$ even if $\partial f$ is a singleton. However this relation is more likely to be satisfied when the domain $\mathrm{dom}(\partial f)$ is large. In this section, we shall consider the question of the density of the domain of such operators. The following proposition shows that for l.s.c. functions that do not take the value $+\infty$, the density of $\partial f$ is equivalent to the convexity of the function.

PROPOSITION 3.1. *Let* $f: X \to \mathbb{R}$ *(i.e.,* $\text{dom}(f) = X$*) be l.s.c. and such that* $\text{dom}(\partial f)$ *is dense in* $X$*. Then* $f$ *is convex and locally Lipschitz.*

*In particular the operator* $\partial f$ *is maximal monotone and locally bounded.*

*Proof.* We first show that $f$ is convex. Since $\text{dom}(\partial f)$ is nonempty, we conclude that $f^{**} > -\infty$, which together with $f \geqslant f^{**}$ shows that $X = \text{dom}(f) \subseteq \text{dom}(f^{**})$. It follows that the l.s.c. convex function $f^{**}$ is continuous.

We now show that the functions $f$ and $f^{**}$ coincide. One observes that $f(x) = f^{**}(x)$, for every $x \in \text{dom}(\partial f)$. Take now any $x$ in $X$. Our assumption implies the existence of a sequence $(x_n)_n$ in $\text{dom}(\partial f)$ such that $(x_n) \to x$. Since $f^{**}(x_n) = f(x_n)$, for $n \in \mathbb{N}$, $f$ is l.s.c. and $f^{**}$ is continuous we get:

$$f^{**}(x) = \liminf_n f^{**}(x_n) = \liminf_n f(x_n) \geqslant f(x) \geqslant f^{**}(x).$$

Thus $f = f^{**}$. For the last assertion see Theorem 2.25 and Theorem 2.28 in [13], e.g. □

We do not know if the assumption $\text{dom}(f) = X$ in the above proposition can be omitted. The following corollary shows that this assumption is not necessary if $X = \mathbb{R}^n$. In this case it becomes part of the conclusions.

COROLLARY 3.2. *Let* $f: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be l.s.c. and such that* $\text{dom}(\partial f)$ *is dense in* $\mathbb{R}^n$*. Then* $\text{dom}(f) = \mathbb{R}^n$ *and the function* $f$ *is convex and locally Lipschitz.*

*Proof.* We have $\text{dom}(\partial f) \subseteq \text{dom}(f^{**})$, so $\text{dom}(f^{**})$ is also dense in $\mathbb{R}^n$. Since $\text{dom}(f^{**})$ is convex, it follows that $\text{dom}(f^{**}) = \mathbb{R}^n$, hence $f^{**}$ is continuous.

Arguing as in the last part of the proof of Proposition 3.1 we conclude again that $f$ is convex and continuous. □

However the following example shows that the lower semicontinuity assumption cannot be dropped, even in the case $X = \mathbb{R}$.

EXAMPLE. Consider the indicator function $i_D$ of any dense subset $D$ of $\mathbb{R}$:

$$i_D(x) = \begin{cases} 0 & \text{if } x \in D, \\ +\infty & \text{if } x \notin D. \end{cases}$$

We note that this function is l.s.c. on its domain, without being l.s.c. in the whole space (unless $D = \mathbb{R}$). Moreover, for every $x \in D$, we have $\partial i_D(x) = \{0\}$, hence $D \subseteq \text{dom}(\partial i_D)$. However, the function $i_D$ is not convex.

Let us now give an infinite-dimensional version of Corollary 3.2 by means of an additional assumption on the operator $\partial f$. We shall say that an operator $T: X \to 2^{X^*}$ has a (*locally*) *bounded selection on its domain*, if for every $x_0 \in X$ there exists $M > 0$ and $\rho > 0$ such that:

$$\forall z \in \text{dom}(T) \cap B_\rho(x_0), \ \exists z^* \in T(z) : \ \|z^*\| \leqslant M. \tag{16}$$

LEMMA 3.3. *Let* $f: X \to \mathbb{R} \cup \{+\infty\}$ *be a l.s.c. function such that* $\mathrm{dom}(\partial f)$ *is dense in* $X$. *If* $\partial f$ *has a (locally) bounded selection on* $\mathrm{dom}(\partial f)$*, then* $\mathrm{dom}(f) = X$ *and* $f$ *is (locally) Lipschitz.*

*Proof.* Let us first assume that $\partial f$ has a locally bounded selection on $\mathrm{dom}(\partial f)$ and let $\rho > 0$ and $M > 0$ be as in (16). We show that the function $f$ is Lipschitzian on the interior $\mathrm{int}\, B_\rho(x_0)$ of $B_\rho(x_0)$ with constant at most $M$. Indeed take any $x, y \in \mathrm{int}\, B_\rho(x_0)$. Since $\mathrm{dom}(\partial f)$ is dense on $X$, there exists a sequence $(x_n)_{n\in N}$ in $\mathrm{dom}(\partial f) \cap B_\rho(x_0)$ and $x_n^* \in \partial f(x_n)$, with $\|x_n^*\| \leqslant M$, such that $(x_n) \to x$. From (1) we conclude that $f(x_n) \leqslant f(y) + x_n^*(x_n - y)$. Since $f$ is l.s.c., taking the limit as $n \to +\infty$ we get

$$f(x) \leqslant f(y) + M\|x - y\|. \tag{17}$$

Since (17) holds for all $y$ in $\mathrm{int}\, B_\rho(x_0)$, choosing $y$ in $\mathrm{dom}(f)$ we conclude that $f$ is finite at $x$. Since $x$ is arbitrary in $\mathrm{int}\, B_\rho(x_0)$, we conclude that $\mathrm{int}\, B_\rho(x_0) \subseteq \mathrm{dom}(f)$. It now follows easily that $f$ is Lipschitz on $\mathrm{int} B_\rho(x_0)$.

If now $\partial f$ has a bounded selection on $\mathrm{dom}(\partial f)$, taking $\rho = +\infty$ we conclude that $f$ is Lipschitz. □

We now state the following corollary.

COROLLARY 3.4. *Let* $f: X \to \mathbb{R} \cup \{+\infty\}$ *be a l.s.c. function. The following statements are equivalent:*

(i) $\mathrm{dom}(\partial f)$ *is dense in* $X$ *and* $\partial f$ *has a (locally) bounded selection on* $\mathrm{dom}(\partial f)$.
(ii) $\mathrm{dom}(\partial f) = X$ *and* $\partial f$ *is (locally) bounded.*
(iii) $\mathrm{dom}(f) = X$ *and* $f$ *is convex and (locally) Lipschitz.*

*Proof.* The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are obvious. The implication (i) $\Rightarrow$ (iii) follows from Lemma 3.3 and Proposition 3.1. □

## 4. Lower Subdifferentials with a Dense Domain

In this section we endeavor to complete results of the literature concerning quasiconvex functions and their lower subdifferentials, in order to reveal analogies with the characterization of Corollary 3.4. We recall that a function $f: X \to \mathbb{R} \cup \{+\infty\}$ is called *quasiconvex*, if its sublevel sets $S_\lambda(f) := \{x \in X : f(x) \leqslant \lambda\}$ are convex for $\lambda \in \mathbb{R}$, or equivalently, if for any $x, y \in X$ and $t \in [0, 1]$ the following inequality holds:

$$f(tx + (1 - t)y) \leqslant \max\{f(x), f(y)\}.$$

We first state the following lemma concerning the lower subdifferential $\partial^<$ (defined in (2) or (3)). We omit its proof, since it is similar to the proof of Lemma 3.3.

LEMMA 4.1. *Let* $f: X \to \mathbb{R} \cup \{+\infty\}$ *be a l.s.c. function such that* $\mathrm{dom}(\partial^{<} f)$ *is dense on* $X$. *If the operator* $\partial^{<} f$ *has a* (*locally*) *bounded selection on* $\mathrm{dom}(\partial^{<} f)$, *then* $\mathrm{dom}(f) = X$ *and* $f$ *is* (*locally*) *Lipschitz.*

The theorem that follows is analogous to Corollary 3.4.

THEOREM 4.2. *Let* $f: X \to \mathbb{R} \cup \{+\infty\}$ *be a l.s.c. function. The following assertions are equivalent:*

(i) $\mathrm{dom}(\partial^{<} f)$ *is dense on* $X$ *and* $\partial^{<} f$ *has a bounded selection on* $\mathrm{dom}(\partial^{<} f)$.
(ii) $\partial^{<} f$ *has a bounded selection on* $X$.
(iii) $f$ *is quasiconvex, Lipschitz and* $\mathrm{dom}(f) = X$.

*Proof.* The equivalence (ii) $\Leftrightarrow$ (iii) was proved in [9] (see Corollary 3.3). Implication (ii) $\Rightarrow$ (i) is obvious. For (i) $\Rightarrow$ (iii) we first apply Lemma 4.1 to conclude that $f$ is Lipschitz. In particular the sublevel sets $S_\lambda$ of $f$ have nonempty interior, whenever $\lambda > \inf f$. It now follows from Proposition 3.1(i) of [10] that $f$ is quasiconvex. □

The following result extends Theorem 4.2 in a non-Lipschitzian case and is comparable to Corollary 3.4. However the implication (iii) $\Rightarrow$ (ii) does not hold in general, as shown by the example below.

PROPOSITION 4.3. *Let* $f: X \to \mathbb{R} \cup \{+\infty\}$ *be a l.s.c. function. Among the following statements one has* (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii).

(i) $\mathrm{dom}(\partial^{<} f)$ *is dense and* $\partial^{<} f$ *has a locally bounded selection on* $\mathrm{dom}(\partial^{<} f)$.
(ii) $\partial^{<} f$ *has a locally bounded selection on* $X$.
(iii) $\mathrm{dom}(f) = X$ *and* $f$ *is quasiconvex and locally Lipschitz.*

*If the restrictions of* $f$ *to its sublevel sets are Lipschitzian, then the above statements are equivalent.*

*Proof.* Implication (ii) $\Rightarrow$ (i) is obvious. If (i) holds, then using Lemma 4.1 we conclude that $\mathrm{dom} f = X$ and $f$ is locally Lipschitz. From Proposition 3.1(i) of [10] it now follows that $f$ is quasiconvex, hence (iii) holds.

Let us now assume that $f$ is quasiconvex, continuous, $\mathrm{dom}(f) = X$ and for any $\lambda \in \mathbb{R}$ the restriction of $f$ to $S_\lambda := \{x \in X : f(x) \leqslant \lambda\}$ is a Lipschitz function of constant $k$, for some $k > 0$. We show that $\partial^{<} f$ has a bounded selection on $S_\lambda$.

Indeed, consider any $x_0 \in S_\lambda$. If $f(x_0) = \inf f$, then $0 \in \partial^{<} f(x_0)$. Hence we may suppose that $f(x_0) > \inf f$. Since $f$ is continuous, the closed convex set $S_{f(x_0)}$ has a nonempty interior. Separating $\mathrm{int}\, S_{f(x_0)}$ from $\{x_0\}$, we obtain a functional $z^* \in X^*$, with $\|z^*\| = 1$ such that $z^*(x) < z^*(x_0)$, for all $x \in \mathrm{int}\, S_{f(x_0)}$. It is easily seen that $x_0$ is minimizer of $f$ on the half space $\{y \in X : z^*(y) \geqslant z^*(x_0)\}$. Set $x_0^* = k' z^*$ with $k' > k$.

CLAIM. $x_0^* \in \partial^{<} f(x_0)$.

*Proof.* Suppose that $x_0^* \notin \partial^< f(x_0)$. It follows from (2) that for some $x \in S^<_{f(x_0)}$ we have $f(x_0) - f(x) > x_0^*(x_0 - x)$. Given any $\varepsilon > 0$, we may find $y \in X$ such that $x_0^*(y) = x_0^*(x_0)$ and $x_0^*(y - x) + \varepsilon \geqslant \|x_0^*\|\|y - x\| = k'\|y - x\|$. Since $f$ is continuous, we can find some $x'$ in the segment $[x, y]$ such that $f(x') = f(x_0)$. We easily get that $x_0^*(x' - x) + \varepsilon \geqslant k'\|x' - x\|$. Since $f(x_0) - f(x) > x_0^*(x_0 - x) = x_0^*(y - x) > x_0^*(x' - x)$, it follows that $f(x') - f(x) > k'\|x' - x\| - \varepsilon$. Since $\varepsilon$ is arbitrary, we have contradicted the fact that $f$ is Lipschitz on $S_{f(x_0)}$ with constant $k$. □

Since $x_0$ is arbitrary in $S_\lambda$ (and since $\lambda$ is arbitrary), we have shown that $\mathrm{dom}(\partial^< f) = X$. Moreover, the continuity assumption of (iii) ensures that for any $x \in X$ and $\lambda > f(x)$ there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subset S_\lambda$. If $k$ is the Lipschitz constant of $f$ on $S_\lambda$, the previous claim asserts that $\partial^< f$ has a selection on $B_\varepsilon(x)$ which is (norm) bounded by any $k' > k$. □

*Remark.* The claim of the preceding proof relies heavily on techniques employed in [14] (see also Corollary 4.20 in [8] or Proposition 6.2 in [12]) in order to prove the equivalence (ii) $\Leftrightarrow$ (iii) in Theorem 4.2 if $X = \mathbb{R}^n$. In finite dimensions it has been shown in Corollary 4.20 of [8] that, if condition (iii) of Proposition 4.3 holds and $f$ is inf-compact (that is for all $\lambda \in \mathbb{R}$, the set $S_\lambda$ is compact), then $f$ is everywhere lower subdifferentiable, that is $\mathrm{dom}(\partial^< f) = \mathbb{R}^n$. Note that the assumptions $f$ is inf-compact and $\mathrm{dom}(f) = X$ imply that the space $X$ can be written as a countable union of compact sets, hence it is finite-dimensional. On the other hand, an easy compactness argument shows that if condition (iii) holds and $f$ is inf-compact, then the restriction of $f$ to the sublevel sets is a Lipschitz function. Hence Proposition 4.3 can be seen as an extension of Corollary 4.20 in [8] to infinite dimensions, which also establishes the existence of a locally bounded selection.

One cannot expect a characterization similar to Theorem 4.2. The following example shows that, without additional assumptions, a locally Lipschitz quasiconvex function $f$ may have its subdifferential $\partial^< f$ everywhere empty.

EXAMPLE. Let $X = \mathbb{R}$ and consider the quasiconvex function $f\colon \mathbb{R} \to \mathbb{R}$, with

$$f(x) = \begin{cases} -x^2 & \text{if } x < 0, \\ x & \text{if } x \geqslant 0. \end{cases}$$

It is easy to see that $f$ is locally Lipschitz, but $\partial^< f(x) = \emptyset$, for all $x \in \mathbb{R}$.

## 5. Integration by Means of the Lower Subdifferential

In this section we consider again the problem of integrating a multivalued operator, by relaxing this time the assumption on $f$ (to be quasiconvex instead of being convex) and by taking $\partial$ to be the lower subdifferential $\partial^{<}$. We replace accordingly cyclic monotonicity with a certain point-based property that we call $(L(x_0))$. This property yields the construction of a l.s.c. quasiconvex function $g_T$ in a way reminiscent to the construction of the l.s.c. convex function $f_T$ in (4) by means of Definition 2.1(i). We show that a cyclically monotone operator fulfills $(L(x))$ at any point $x \in \mathrm{dom}(T)$. Conversely, if an operator satisfies $(L(x))$ at every point of its domain, then it is cyclically monotone (see Proposition 5.2). Roughly speaking, property $(L(x_0))$ is to be understood as a pointwise version of cyclic monotonicity.

DEFINITION 5.1. An operator $T\colon X \to 2^{X^*}$ is said to have property $(L(x_0))$ with respect to some $x_0 \in \mathrm{dom}(T)$, if for any $n \geqslant 1$, any $x_1, x_2, \ldots, x_n \in \mathrm{dom}(T)$ and any $x_i^* \in T(x_i)$ for $i = 0, 1, \ldots, n$, one has:

$$\min \left\{ \begin{array}{c} x_0^*(x_1 - x_0) \\ x_1^*(x_2 - x_1) + x_0^*(x_1 - x_0) \\ \cdots \\ x_n^*(x_0 - x_n) + \sum_{i=0}^{n-1} x_i^*(x_{i+1} - x_i) \end{array} \right\} \leqslant 0.$$

It follows easily that if $T$ is cyclically monotone (see Definition 2.1(ii)), then it satisfies $(L(x))$ at every point of its domain. The following proposition shows that the converse is also true:

PROPOSITION 5.2. *If $T$ satisfies $(L(x))$ for every $x \in \mathrm{dom}(T)$, then $T$ is cyclically monotone.*

*Proof.* Suppose that $T$ is not cyclically monotone. Then there exist $n \geqslant 2$ and $x_0, x_1, \ldots, x_{n-1}$ in $X$ and $x_0^* \in T(x_0)$, $x_1^* \in T(x_1)$,..., $x_{n-1}^* \in T(x_{n-1})$ such that (setting $x_n = x_0$)

$$\sum_{i=0}^{n-1} x_i^*(x_{i+1} - x_i) > 0. \tag{18}$$

For $i = 0, 1, \ldots, n-1$ and for $j = i \pmod n$ (i.e., $j = nm + i$ for some $m \in \mathbb{N}$) we set $\beta_j = x_i^*(x_{i+1} - x_i)$, so that (18) can be rewritten:

$$\sum_{j=0}^{n-1} \beta_j > 0. \tag{19}$$

Thus, there exists some $h_1 \in \{0, 1, \ldots, n-1\}$ such that $\beta_{h_1} > 0$. Since the operator $T$ satisfies $L(x_{h_1})$, there exists some $k \in \{h_1 + 1, h_1 + 2, \ldots, h_1 + n\}$ such that

$$\sum_{j=h_1}^{k} \beta_j \leqslant 0. \tag{20}$$

Note that the fact that $k \neq h_1 + n$ is ensured by (18). Taking now $k$ to be the largest integer in $\{h_1 + 1, h_1 + 2, \ldots, h_1 + n - 1\}$ such that (20) is satisfied, we conclude that $\beta_{k+1} > 0$. Setting now $h_2 = k+1$ and proceeding like this, we define inductively a strictly increasing sequence $(h_q)_{q=1}^{\infty}$ such that for any $q \geqslant 1$ we have $\beta_{h_q} > 0$ and

$$\sum_{i=h_q}^{h_{q+1}-1} \beta_i \leqslant 0. \tag{21}$$

Since the sequence $(h_q \pmod n))_{q\in N}$ has an accumulating point, we can find $p > q \geqslant 1$ such that $h_p = h_q + mn$, for some $m \in \mathbb{N}$ (i.e., $h_p = h_q \pmod n$). We thus obtain the following equality:

$$\sum_{i=h_q}^{h_{q+1}-1} \beta_i + \sum_{i=h_{q+1}}^{h_{q+2}-1} \beta_i + \cdots + \sum_{i=h_{p-1}}^{h_p-1} \beta_i = \sum_{i=h_q}^{h_p-1} \beta_i = m \sum_{i=0}^{n-1} \beta_i$$

which is not possible in view of (19) and (21). $\square$

*Remark.* Considering for instance the operator $T:\mathbb{R} \to 2^{\mathbb{R}}$ given by $T(0) = \{0\}$ and $T(x) = [-1, 1]$, if $x \neq 0$, it is easy to see that $T$ satisfies property $(L(x_0))$ for $x_0 = 0$, without being CM.

Motivated by (4) we consider the following function $g_T: X \to \mathbb{R} \cup \{+\infty\}$:

$$g_T(x) = c + \sup\min \left\{ \begin{array}{c} x_0^*(x_1 - x_0) \\ x_1^*(x_2 - x_1) + x_0^*(x_1 - x_0) \\ \cdots \\ x_n^*(x - x_n) + \sum_{i=0}^{n-1} x_i^*(x_{i+1} - x_i) \end{array} \right\}, \tag{22}$$

where $c$ is an arbitrary constant and the supremum is taken over all $n \in \mathbb{N}$, all finite sequences $(x_i)_{i=1}^n \in \mathrm{dom}(T)$ and all $x_i^* \in T(x_i^*)$, for $i = 0, 1, \ldots, n$. Note that the choice $n = 0$ in the above supremum yields $g_T(x) \geqslant \sup_{x_0^* \in T(x_0)}\{x_0^*(x - x_0)\} + c$. In particular $g_T(x) > -\infty$, for all $x \in X$.

Since $g_T$ is represented as a supremum of a family of subaffine continuous functions (i.e., of functions of the form $x \to \min\{c,\ x^*(x) + b\}$, where $b, c \in \mathbb{R}$), it follows that it is quasiconvex and lower semicontinuous. Comparing (4) and (22) one notes that $g_T(x) \leqslant f_T(x)$, for every $x \in X$.

The following theorem is analogous to Theorem 2.5:

THEOREM 5.3. *If T fulfills* $(L(x_0))$ *then there exists a l.s.c. quasiconvex function* $g$ *such that* $T(x_0) \subseteq \partial g(x_0)$ *and for all* $x \in X$, $T(x) \subseteq \partial^{<} g(x)$.

*Proof.* Set $g = g_T$. Since $T$ fulfills $(L(x_0))$, it follows (by taking $n = 1$ and $x_1 = x_0$) that $g_T(x_0) = c$, hence as observed before, for any $x \in X$ and any $x_0^* \in T(x_0)$ we have

$$x_0^*(x - x_0) + g_T(x_0) \leqslant g_T(x)$$

which shows that $x_0^* \in \partial g_T(x_0)$.

Let $x^* \in T(x)$. For any $M < g_T(x)$, there exist $n \geqslant 0$ and (for $n > 0$) $x_1, x_2, \ldots, x_n \in X$, $x_0^* \in T(x_0), x_1^* \in T(x_1), \ldots, x_n^* \in T(x_n)$ such that

$$M \; < \; c + \min \left\{ \begin{array}{c} x_0^*(x_1 - x_0) \\ x_1^*(x_2 - x_1) + x_0^*(x_1 - x_0) \\ \cdots \\ x_n^*(x - x_n) + \sum_{i=0}^{n-1} x_i^*(x_{i+1} - x_i) \end{array} \right\}. \tag{23}$$

In particular, setting $x_{n+1} := x$ (and considering separately the cases $n = 0$ and $n > 0$), one gets $M < \sum_{i=0}^{n} x_i^*(x_{i+1} - x_i) + c$. For any $y \in X$, and adding to both sides of this inequality the quantity $x^*(y - x)$ we obtain:

$$M + x^*(y - x) < \sum_{i=0}^{n} x_i^*(x_{i+1} - x_i) + x^*(y - x) + c. \tag{24}$$

Combining (23) and (24) and taking the minimum we obtain:

$$\min\{M, M + x^*(y - x)\} \leqslant c + \min \left\{ \begin{array}{l} x_0^*(x_1 - x_0) \\ x_1^*(x_2 - x_1) + x_0^*(x_1 - x_0) \\ \cdots \\ x_n^*(x - x_n) + \sum_{i=0}^{n-1} x_i^*(x_{i+1} - x_i) \\ x^*(y - x) + \sum_{i=0}^{n} x_i^*(x_{i+1} - x_i) \end{array} \right\}$$

(with the convention $x_{n+1} := x$). As the right-hand side of the preceding inequality is always less than or equal to $g_T(y)$ and since $M$ can be arbitrarily close to $g_T(x)$, using (3) we conclude that $x^* \in \partial^< g_T(x)$. This finishes the proof. □

*Remarks.* (1) If one omits the inclusion $T(x_0) \subseteq \partial g(x_0)$ in the above statement (i.e., replaces it by $T(x_0) \subseteq \partial^< g(x_0)$), then the remaining conclusion holds trivially, since one can take for $g$ the constant function.

(2) If the operator $T$ of Theorem 5.3 has a (locally) bounded selection at least in a dense subset of $X$, then the function $g$ (of Theorem 5.3) will be (locally) Lipschitz. This is an immediate consequence of Theorem 4.2 (resp. Proposition 4.3).

We finally state the following 'converse' to Theorem 5.3.

PROPOSITION 5.4. *For any function $f$ and any $x_0 \in \mathrm{dom}(f)$, the operator $T$: $S^<_{f(x_0)} \cup \{x_0\} \to 2^{X^*}$ given by $T(x) = \partial^< f(x)$ fulfills $(L(x_0))$.*

*Proof.* The result follows from the fact that for any $x \in S^<_{f(x_0)}$ and any $x_0^* \in T(x_0)$ one has $x_0^*(x - x_0) \leqslant 0$. □

Note that Proposition 5.4 is similar to Lemma 2.4, the difference being the domain of the operator ($S^<_{f(x_0)} \cup \{x_0\}$ instead of the whole space $X$).

Property $(L(x_0))$, introduced in the present section, is a logical step from cyclic monotonicity and the FM subdifferential to the lower subdifferential. Theorem 5.3 and Proposition 5.4 almost characterizes this property. However, given a function $f$ with $\partial f(x_0) \neq \emptyset$, Proposition 5.4 (unlike Lemma 2.4) does not describe the behavior of the operator

$$T(x) = \begin{cases} \partial^{<} f(x) & \text{if } x \neq x_0, \\ \partial f(x_0) & \text{if } x = x_0, \end{cases} \tag{25}$$

on the whole space, but only on the strict level set $S^{<}_{f(x_0)}$. This is clearly shown by the following example:

EXAMPLE. Let $f: \mathbb{R} \to \mathbb{R}$ be given by:

$$f(x) = \begin{cases} -1 & \text{if } x \leqslant -1, \\ x & \text{if } x > -1. \end{cases}$$

Then the operator $T$ defined in (25) with $x_0 = 0$ is given as follows:

$$T(x) = \begin{cases} \{1\} & \text{if } x = 0, \\ [1, +\infty) & \text{if } x \in (-1, 0) \cup (0, +\infty), \\ \mathbb{R} & \text{if } x \leqslant -1. \end{cases}$$

It is easy to see – considering the points $x_0 = 0$, $x_1 = 1$ and $x_2 = 3/2$ – that $T$ fails to satisfy $L(0)$.

## 6. Characterization of Operators which are contained in the Lower Subdifferential of a Function

In this section we introduce the property $(R(x_0))$ aiming at describing the above operator $T$ (see (25)) in the whole space. Although this property is weaker than $(L(x_0))$, we show that operators fulfilling $(R(x_0))$ can still be 'integrated' (in the sense of Theorem 5.3). This leads to a situation similar to Lemma 2.4 and Theorem 2.5. We also show that any operator satisfying $(R(x))$ at every point of its domain, is monotone.

DEFINITION 6.1. An operator $T: X \to 2^{X^*}$ is said to have property $(R(x_0))$ with respect to some $x_0 \in \operatorname{dom}(T)$, if for any $n \geqslant 1$, for any $x_1, x_2, \ldots, x_n \in \operatorname{dom}(T)$ and any $x_i^* \in T(x_i^*)$ for $i = 0, 1, \ldots, n$, one has:

$$x_0^*(x_1 - x_0) + \sum_{i=1}^{n-1} \{x_i^*(x_{i+1} - x_i)\}^- + \{x_n^*(x_0 - x_n)\}^- \leqslant 0, \tag{26}$$

where $\{x_i^*(x_{i+1} - x_i)\}^- := \min\{x_i^*(x_{i+1} - x_i),\ 0\}$.

Definition 6.1 is in the same spirit as Definition 5.1 and Definition 2.1(i). In particular every operator that satisfies $(L(x_0))$ also satisfies $(R(x_0))$. The following example shows that the converse is not true:

EXAMPLE. Let $T: \mathbb{R} \to 2^{\mathbb{R}}$ be such that $T(0) = \{1\}$, $T(1) = \{2\}$, $T(2) = \{1\}$ and $T(x) = \emptyset$ elsewhere. One can verify that $T$ has property $(R(x_0))$ for $x_0 = 0$, without satisfying $(L(x_0))$.

In this example one may observe that the operator $T$ does not satisfy $(R(x))$ at every point of its domain (it fails at the point $x_0 = 1$). The following proposition (together with the fact that for one-dimensional spaces cyclic monotonicity and monotonicity coincide ([5], e.g.)) gives an explanation for this.

PROPOSITION 6.2. *If an operator $T$ fulfills $(R(x))$ at every point of its domain, then $T$ is monotone.*

*Proof.* Take any $x, y \in X$, $x^* \in T(x)$, $y^* \in T(y)$ and assume that

$$x^*(y-x) + y^*(x-y) > 0. \tag{27}$$

Interchanging the roles of $x$ and $y$, we may suppose that $y^*(x-y) > 0$. Then taking $n = 1$, $x_0 = x$ and $x_n = y$, relation (26) yields that $x^*(y-x) \leqslant 0$. Taking now $n = 1$, $x_0 = y$ and $x_n = x$, relation (26) leads to a contradiction with (27). □

COROLLARY 6.3. *If $X = \mathbb{R}$, then $T$ fulfills $(R(x))$ for all $x \in \mathrm{dom}(T)$ if, and only if, $T$ is cyclically monotone.*

The following theorem characterizes the class of operators that satisfy property $(R(x_0))$.

THEOREM 6.4. *The operator $T$ satisfies $(R(x_0))$ for some $x_0 \in \mathrm{dom}(T)$ if, and only if, there exists a l.s.c. quasiconvex function $h_T$ such that $T(x_0) \subseteq \partial h_T(x_0)$ and $T(x) \subseteq \partial^< h_T(x)$, for all $x \in X$.*

*Proof.* (a) Let us first assume that $T$ satisfies $(R(x_0))$ at some point $x_0$ of its domain. We consider the following function $h_T: X \to \mathbb{R} \cup \{+\infty\}$ given by

$$h_T(x) = c + \sup\left\{ x_0^*(x_1 - x_0) + \sum_{i=1}^{n} \{x_i^*(x_{i+1} - x_i)\}^- \right\}, \tag{28}$$

where $x_{n+1} := x$, $c$ is an arbitrary constant and the supremun is taken over all $n \in \mathbb{N}$, all choices $x_1, x_2, \ldots, x_n \in \mathrm{dom}(T)$ and all $x_i^* \in T(x_i^*)$ for $i = 0, 1, \ldots, n$. We make here the convention that the choice $n = 0$ in the above supremum is acceptable and corresponds to the term $\sup_{x_0^* \in T(x_0)} x_0^*(x - x_0) + c$.

It is easy to see that $h_T$ is l.s.c. and quasiconvex. From Definition 6.1 above, we conclude that $h_T(x_0) \leqslant c$, and in fact $h_T(x_0) = c$. It follows directly from (28) that for every $x \in X$ we have

$$h_T(x) \geqslant \sup_{x_0^* \in T(x_0)} x_0^*(x - x_0) + c = \sup_{x_0^* \in T(x_0)} x_0^*(x - x_0) + h_T(x_0)$$

which in view of (1) ensures that $T(x_0) \subseteq \partial h_T(x_0)$.

Let now $x \in X$ and $x^* \in T(x)$. For $M < h_T(x)$, (28) shows that there exist $n \in \mathbb{N}$, $x_1, x_2, \ldots, x_{n+1} := x \in X$ and $x_0^* \in T(x_0)$, $x_1^* \in T(x_1), \ldots, x_n^* \in T(x_n)$ such that

$$c + x_0^*(x_1 - x_0) + \sum_{i=1}^{n-1}\{x_i^*(x_{i+1} - x_i)\}^- + \{x_n^*(x - x_n)\}^- > M. \tag{29}$$

(If $n = 0$, then we have $c + x_0^*(x - x_0) > M$.) For any $y \in X$, setting $x_{n+1} := x$, adding to both sides of (29) the quantity $\{x^*(y-x)\}^-$ (and considering successively the cases $n = 0$ and $n > 0$), we obtain

$$\begin{aligned} &c + x_0^*(x_1 - x_0) + \sum_{i=1}^{n}\{x_i^*(x_{i+1} - x_i)\}^- + \{x^*(y - x)\}^- \\ &\quad > M + \{x^*(y - x)\}^-. \end{aligned} \tag{30}$$

We note that the left side of (30) is always less than or equal to $h_T(y)$. Since $M$ can be chosen arbitrarily close to $h_T(x)$, we conclude from (30) that:

$$h_T(y) \geqslant \min\left\{\begin{array}{c} h_T(x) \\ x^*(y - x) + h_T(x) \end{array}\right\}. \tag{31}$$

It now follows from (3) that $x^* \in \partial^< h_T(x)$. We conclude that for every $x \in X$, $T(x) \subseteq \partial^< h_T(x)$.

(b) Given any function $f$ with $\partial f(x_0) \neq \emptyset$ we consider the multivalued operator

$$T(x) = \begin{cases} \partial^< f(x), & x \neq x_0, \\ \partial f(x_0), & x = x_0. \end{cases} \tag{32}$$

For any $x_0^* \in T(x_0)$ and any $x_1 \in \mathrm{dom}(T)$ we have:

$$f(x_1) - f(x_0) \geqslant x_0^*(x_1 - x_0). \tag{33}$$

Furthermore, for any $x_i \in \mathrm{dom}(T)$, $x_i^* \in T(x_i)$ and any $x_{i+1} \in X$, we conclude from (32) and (3) that

$$f(x_{i+1}) - f(x_i) \geqslant \min\{x_i^*(x_{i+1} - x_i),\ 0\}. \tag{34}$$

Considering any finite cycle $\{x_0, x_1, \ldots, x_n, x_{n+1} := x_0\}$ in $\mathrm{dom}(T)$ and any choice $x_i^* \in T(x_i)$, for $i = 0, 1, \ldots, n$, we conclude from (33) and (34) that:

$$x_0^*(x_1 - x_0) + \sum_{i=1}^{n}\{x_i^*(x_{i+1} - x_i)\}^- \leqslant 0 \tag{35}$$

which shows (see Definition 6.1) that $T$ satisfies $(R(x_0))$.

The observation that property $(R(x_0))$ is inherited by smaller operators (in the sense of the inclusion of graphs) finishes the proof. □

The above theorem gives a characterization of the class of operators that satisfy $(R(x_0))$. The situation is analogous to the one corresponding to the class of cyclically monotone operators as described by Lemma 2.4 and Theorem 2.5.

*Remarks.* (1) Since property $(L(x_0))$ entails $(R(x_0))$, Theorem 5.3 can be deduced as a consequence of the 'only if' part of Theorem 6.4. Let us also note that, as was the case in Theorem 5.3, the inclusion $T(x_0) \subseteq \partial h_T(x_0)$ is an essential part of Theorem 6.4.

(2) Using Theorem 4.2 or Proposition 4.3, we may conclude that the quasiconvex function $h_T$ constructed in the above proof is (locally) Lipschitz whenever the operator $T$ has a (locally) bounded selection in a dense subset of $X$.

(3) If there exists $x_0 \in \mathrm{dom}(T)$ such that $T(x_0) = \{0\}$, then the above construction leads to the constant function $h_T = c$. Let us observe that this situation cannot occur if $T$ is given by (32) unless $\partial f(x_0) = \{0\}$.

(4) One may wonder whether the analogy between $(\mathrm{CM}(x_0))$ (cyclically monotone) and $(R(x_0))$ operators can go any further. Namely, starting from an arbitrary function $f$ with $\partial f(x_0) \neq \emptyset$, one may define an operator $T$ of the class $(R(x_0))$ (resp. of the class $(\mathrm{CM}(x_0))$) via relation (25) (resp. $T = \partial f$) and subsequently consider the l.s.c. quasiconvex function $h_T$ (resp. the l.s.c. convex function $f_T$) given by the formula (28) (resp. (4)). In both cases we have:

$$x_0^*(x - x_0) \leqslant h_T(x) \leqslant f_T(x) \leqslant f(x). \tag{36}$$

It is easily seen that if $f$ is affine, then the functions $h_T$, $f_T$ and $f$ coincide (modulo the constant $f(x_0)$). It is also known that if $f$ is convex and l.s.c., then $f_T$ and $f$ coincide [16]. However in general the function $h_T$ does not coincide with $f$ and in particular – unlike the convex case – the operator $T$ defined in (25) does not uniquely determine the function $f$. A comparison of (4), (22) and (28) yields $h_T \leqslant g_T \leqslant f_T$. In the following example we show that if $T$ is defined by (25), the functions $h_T$ and $g_T$ are in general strictly majorized by $f$.

EXAMPLE. Consider the function $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x+1| - 1$. Then for $x_0 = 0$, the operator $T$ in (32) is given as follows:

$$T(x) = \begin{cases} [1, +\infty) & \text{if } x \in (-1, 0) \cup (0, +\infty), \\ \{1\} & \text{if } x = 0, \\ \mathbb{R} & \text{if } x = -1, \\ (-\infty, -1] & \text{if } x < -1, \end{cases}$$

hence the constructions (22) and (28) lead to functions $g_T$ and $h_T$:

$$g_T(x) = h_T(x) = \begin{cases} x & \text{if } x > -1, \\ -1 & \text{if } x \leqslant -1. \end{cases}$$

*Remark.* As pointed out by the referee, the results of this paragraph and the integration procedure of Rockafellar ([16]) can both be seen as particular cases of the following scheme:

Consider a general function $b: X \times X \times X^* \to \mathbb{R}$. Then for any function $f: X \to \mathbb{R} \cup \{+\infty\}$ let us define the $b$-subdifferential $\partial^b f: X \to 2^{X^*}$ by

$$\partial^b f(x) = \{x^* \in X^* : f(y) \geqslant f(x) + b(x, y, x^*), \text{ for all } y \in X\}. \tag{37}$$

Further, given an operator $T: X \to 2^{X^*}$ and a point $x_0$ in $\mathrm{dom}(T)$, define the $b(x_0)$-property as follows: For any $x_1, x_2, \dots, x_n \in X$ and any $x_0^* \in T(x_0), x_1^* \in T(x_1), \dots, x_n^* \in T(x_n)$

$$\sum_{i=0}^{n} b(x_i, x_{i+1}, x_i^*) \leqslant 0, \tag{38}$$

where the convention $x_{n+1} = x_0$ is used. Then if $T$ has this property, adapting the procedure of Rockafellar (in [16]) we can construct a function $f_T$ in such a way that $T \subseteq \partial^b f_T$. The function $f_T$, being a supremum of functions of the form $b(x, y, x^*)$, will enjoy a certain property based on $b(\cdot, \cdot, \cdot)$, that we call $b$-convexity. In the light of this general scheme, the conclusions of Theorem 2.5 and Theorem 6.4 may read in a unified way as follows:

$$T \text{ has } b(x_0) \Leftrightarrow T \subseteq \partial^b\, f_T, \qquad \text{for some } b\text{-convex function } f_T.$$

Note that Theorem 2.5 corresponds to the case $b(x, y, x^*) = x^*(y - x)$, where one recovers in (37) the definition of the Fenchel–Moreau subdifferential and in (38) the definition of cyclic monotonicity (see Definition 2.1(i)). In this case, $b$-convexity is equivalent to convexity plus lower semicontinuity. On the other hand, Theorem 6.4 corresponds to the choice

$$b(x, y, x^*) = \begin{cases} x^*(y - x) & \text{if } x = x_0, \\ \min\{x^*(y - x), 0\} & \text{if } x \neq x_0, \end{cases}$$

where (38) is the considered $R(x_0)$ property, and $b$-convexity is nothing else than lower semicontinuity and quasiconvexity.

QUESTION. The class of operators fulfilling $(R(x))$ at every point of their domain is located between monotone and cyclically monotone operators (see Propositions 5.2, 6.2 and comments after Definition 6.1). However we do not know which of these inclusions is strict.

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# INTEGRATION OF MULTIVALUED OPERATORS AND CYCLIC SUBMONOTONICITY

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Abstract. We introduce a notion of cyclic submonotonicity for multivalued operators from a Banach space $X$ to its dual. We show that if the Clarke subdifferential of a locally Lipschitz function is strictly submonotone on an open subset $U$ of $X$, then it is also maximal cyclically submonotone on $U$, and, conversely, that every maximal cyclically submonotone operator on $U$ is the Clarke subdifferential of a locally Lipschitz function, which is unique up to a constant if $U$ is connected. In finite dimensions these functions are exactly the lower $C^1$ functions considered by Spingarn and Rockafellar.

## 1. Introduction

We deal with the integration of a multivalued operator considered as the inverse process of taking the subdifferential of a function. This important question has been tackled by several authors: see for instance [20] (for the Fenchel-Moreau subdifferential of a convex function), [12], [19], [17] (for the Clarke subdifferential in finite dimensions), [4], [26] (for the Clarke subdifferential in infinite dimensions), [1] (for the moderate subdifferential of Michel-Penot in finite dimensions) and [25], [27] (for various subdifferentials of a lower semicontinuous function). The first mentioned result concerns the case of monotone operators: in [20] Rockafellar shows that any cyclically monotone operator $T$ is included in the subdifferential of a lower semicontinuous convex function, with equality if $T$ is maximal cyclically monotone. Janin [12] introduces a concept of cyclic submonotonicity in finite-dimensional Euclidean spaces and uses it to integrate (in the preceding sense) locally bounded operators satisfying that condition into locally Lipschitz functions. In a different context, a concept of cyclicity has also been used by Qi in [19], where the author characterizes operators that coincide with a subdifferential of some locally Lipschitz function, using the Lebesgue measure and (implicitly) the Rademacher theorem. Elaborating upon these ideas, Borwein and Moors [3] introduce and study the class $S_e(X)$ of *essentially smooth* (locally Lipschitz) functions, that is, functions $f$ whose Clarke subdifferential $\partial^C f$ is single-valued in the complement of a Haar null set. One of the main features of this class stems from the fact that for every $f \in S_e(X)$, the problem of finding a locally Lipschitz function $g$ such that $\partial^C g \subseteq \partial^C f$ has a unique

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solution modulo a constant (i.e., $g = f + c$). In [4], Borwein, Moors and Shao extend the results of Qi [19] to separable Banach spaces, using line integrals and Christensen's generalization of the Rademacher theorem via Haar null sets. Their result is further improved by Wang [26], who characterizes the class of integrable locally bounded operators in separable Banach spaces. These operators are called in [19] and [26] "cyclically normal".

In another line of research, Spingarn [24] showed that in finite dimensions, *lower* $C^1$ functions (i.e., functions arising as maxima of compactly indexed families of $C^1$ functions) are characterized by the fact that their Clarke subdifferentials are *strictly submonotone* operators with nonempty values. This last notion was extended to infinite dimensions by Georgiev in [10], [11] (see the definition of *directional strict submonotonicity* in Section 2). Functions with such subdifferentials (herein called *subsmooth*) are always regular (in the sense of Clarke [6]) and semi-smooth (in the sense of Mifflin [14]), see [24] and [11]. In particular, subsmooth functions have "small" (namely *minimal w*-cusco*) Clarke subdifferentials. Let us recall that, in general, Lipschitz functions have "generically" very large Clarke subdifferentials ([26]). In finite dimensions, as the notions of strict submonotonicity and directional strict submonotonicity coincide [11], a function $f$ is subsmooth if, and only if, it is lower $C^1$.

Our main results rely on a notion of cyclic submonotonicity introduced here; in finite dimensions it coincides with the definition of Janin [12]. Using this concept, we show that if $U$ is an open subset of a Banach space $X$, then

- the subdifferential of every subsmooth function (defined on $U$) is maximal cyclically submonotone (on $U$);
- if $U$ is connected and $f$, $g$ are subsmooth functions on $U$ such that $\partial^C f = \partial^C g$, then $f = g + c$ for some constant $c \in \mathbb{R}$; and
- every maximal cyclically submonotone operator on $U$ is the subdifferential of a subsmooth function defined on $U$ (unique up to a constant if $U$ is connected).

A specific feature of our approach is that it does not depend on results from measure theory and is valid beyond the class of separable spaces.

*Notation.* Let us now fix our notation. We denote by $(X, \|.\|)$ a Banach space, by $S_X$ its unit sphere and by $(X^*, \|.\|)$ its dual space. We also denote by $B_r(x)$ (resp. $B_r[x]$) the open (resp. closed) ball with center $x$ and radius $r$, and by $B_r^*(x)$ (resp. $B_r^*[x]$) the same objects in $X^*$. Let $2^{X^*}$ be the set of all subsets of $X^*$, and $\mathbb{R}$ (resp. $\mathbb{N}$) be the set of all real (resp. nonnegative integer) numbers. For any $k \in \mathbb{N}$, we set $\mathbb{N}_k := \{1, 2, ..., k\}$. For any $x, y$ in $X$, we denote by $[x, y] := \{x_t = tx + (1-t)y,\ 0 \le t \le 1\}$ the closed segment with endpoints $x, y$. For any subset $K$ of $X$ and any $\delta > 0$, we consider the $\delta$- (open) neighborhood $B_\delta(K)$ of $K$ defined by $B_\delta(K) := \{x \in X : \exists y \in K, \|x - y\| < \delta\}$. We also set $\operatorname{diam}(K) := \sup\{\|x - y\| : x, y \in K\}$ for the diameter of the set $K$. We denote by $\operatorname{cone}(K)$ the cone generated by $K$, i.e., $\operatorname{cone}(K) := \{\lambda x : \lambda \ge 0,\ x \in K\}$. For any subset $A$ of $X^*$, we denote by $\overline{co}^{w^*}(A)$ the $w^*$-closed convex hull of $A$. Finally, throughout the paper we shall assume that *all functions $f$ are locally Lipschitz* and we shall denote by $\operatorname{dom}(f)$ their domain.

## 2. Preliminaries

Given a multivalued operator $T : X \rightrightarrows X^*$, we denote by $\mathrm{dom}(T)$ the set $\{x \in X : T(x) \neq \emptyset\}$ and by $\mathrm{Gr}(T) := \{(x, x^*) \in X \times X^* : x^* \in T(x)\}$ (or simply $T$ if no confusion may arise) the graph of $T$. We also define the operators $\overline{co}^{w^*}(T)$ by

$$\overline{co}^{w^*}(T)(x) := \overline{co}^{w^*}(T(x)) \tag{1}$$

for all $x \in X$ and $\overline{T} : X \to 2^{X^*}$ by

$$x^* \in \overline{T}(x) \Longleftrightarrow \begin{cases} \exists \{x_i\}_i \text{ in } X,\ \exists \{x_i^*\}_i \text{ in } X^*\text{: } x_i^* \in T(x_i) \\ \\ x = \lim\limits_i x_i, \qquad x^* = w^*\text{-}\lim\limits_i x_i^* \end{cases} \tag{2}$$

where $\{x_i\}_i$ and $\{x_i^*\}_i$ denote, respectively, nets in $X$ and in $X^*$. Note that $\mathrm{Gr}(\overline{T}) = \overline{\mathrm{Gr}(T)}^{\|\cdot\| \times w^*}$.

The operator $T : X \rightrightarrows X^*$ is said to be *locally bounded* at $x \in X$ if there exist $M > 0$ and a neighborhood $B_r(x)$ of $x$ such that $u \in B_r(x)$ and $u^* \in T(u)$ imply $\|u^*\| < M$. Then $T$ is called locally bounded on a subset $U$ of $X$ if $T$ is locally bounded at all $x \in U$. (Note that $U$ is not necessarily a subset of $\mathrm{dom}(T)$.)

Furthermore, the operator $T$ is said to be $w^*$-upper-semicontinuous at $x \in X$ if for every $w^*$-open set $W \supset T(x)$, there exists an open ball $B_r(x)$ of $x$ such that $T(u) \subseteq W$ for every $u \in B_r(x)$. Let us note that if $T$ is locally bounded on $U$, then $\overline{T}$ (given by relation (2)) is $w^*$-upper-semicontinuous at every $x \in \mathrm{dom}(\overline{T}) \cap U$.

**$w^*$-cusco mappings.** A multivalued mapping $T : X \rightrightarrows X^*$ is said to be *$w^*$-cusco* on $U$ ([2], [5], e.g.), if it is $w^*$-upper semicontinuous with nonempty $w^*$-compact convex values on $U$. A $w^*$-cusco mapping on $U$ that does not strictly contain any other $w^*$-cusco mapping with domain in $U$ is called *minimal $w^*$-cusco* on $U$.

Given an operator $S$, we can consider $w^*$-cusco mappings $T$ that are minimal under the property of containing $S$. In the important case of the following proposition, one can give a complete description of the minimal (in fact least) element of the family of $w^*$-cusco mappings containing $S$.

**Proposition 1.** *Let $S$ be a densely defined locally bounded operator on an open subset $U$ of $X$ with values in $X^*$. Then the family of $w^*$-cusco mappings containing $S$ has a least element $T$ given by the formula*

$$T(x) = \overline{co}^{w^*} \bigcap_{\varepsilon > 0} \overline{\{S(x') : x' \in B_\varepsilon(x) \cap \mathrm{dom}(S)\}}^{w^*}. \tag{3}$$

*Proof.* In [2, Proposition 1.3] (see also [15, Proposition 1.2]), the following formula for the operator $T$ is given:

$$T(x) = \bigcap_{\varepsilon > 0} \overline{co}^{w^*} \{S(x') : x' \in B_\varepsilon(x) \cap \mathrm{dom}(S)\}.$$

In order to justify (3), let us set

$$R(x) := \bigcap_{\varepsilon > 0} \overline{\{S(x') : x' \in B_\varepsilon(x) \cap \mathrm{dom}(S)\}}^{w^*}.$$

Since $S$ is locally bounded on $U$, it is easily seen that $R(x) = \overline{S}(x)$ (given by relation (2)) and that $R$ is the smallest $w^*$-upper-semicontinuous multivalued mapping containing $S$. Thus $R(x) \subseteq T(x)$ and $\overline{co}^{w^*}(R(x)) \subseteq T(x)$ for each $x \in U$.

Since $\overline{co}^{w^*}(R(x))$ is $w^*$-cusco (see [5, Proposition 2.7], e.g.) and $T$ is the minimal $w^*$-cusco containing $S$, we get $\overline{co}^{w^*}(R) = T$. □

The preceding proposition has an interesting (and immediate) consequence on the representation of the Clarke subdifferential in certain Banach spaces. We recall that the *Clarke generalized derivative* of a locally Lipschitz function $f$ at a point $x \in \mathrm{dom}(f)$ is defined for all $u \in X$ as follows:

$$f^o(x;u) = \limsup_{(y,t)\to(x,0^+)} \frac{f(y+tu)-f(y)}{t},$$

and the *Clarke subdifferential* of $f$ at $x \in \mathrm{dom}(f)$ by

$$\partial^C f(x) = \{x^* \in X^* : \langle x^*, u\rangle \leq f^o(x,u), \forall u \in X\}. \tag{4}$$

For all $x \in \mathrm{dom}(f)$, we have $\partial^C f(x) \neq \emptyset$.

Let us also recall the definitions of other usual subdifferentials that will occur in the sequel:

- the *Fréchet subdifferential* $\partial^F f(x)$

$$\partial^F f(x) := \{x^* \in X^* : f(y) \geq f(x) + \langle x^*, y-x\rangle + o(y-x), \forall y \in X\},$$

where $o : X \to R$ is some real-valued function satisfying $\lim_{u\to 0} \frac{o(u)}{\|u\|} = 0$;

- the *Hadamard subdifferential* $\partial^H f(x)$

$$\partial^H f(x) = \{x^* \in X^* : \langle x^*, u\rangle \leq \liminf_{(w,t)\to(u,0^+)} \frac{f(x+tw)-f(x)}{t}, \forall u \in X\}.$$

Let us note that if $f$ is locally Lipschitz, then for all $u \in X$,

(5)

$$df(x,u) := \liminf_{(w,t)\to(u,0^+)} \frac{f(x+tw)-f(x)}{t} = \liminf_{t\searrow 0^+} \frac{f(x+tu)-f(x)}{t} := f'(x,u),$$

so that the Hadamard derivative coincides with the Gâteaux derivative of $f$ when they exist.

Let us now recall that in every Asplund space, the Clarke subdifferential $\partial^C f$ of a locally Lipschitz function $f$ is given by the following formula of Preiss ([18, Remark 2.3]):

$$\partial^C f(x) = \bigcap_{\varepsilon>0} \overline{co}^{w^*}\{D^F f(x') : x' \in B_\varepsilon(x) \cap \mathrm{dom}(D^F f)\}, \tag{6}$$

while, if $X$ has a Gâteaux smooth renorming,

$$\partial^C f(x) = \bigcap_{\varepsilon>0} \overline{co}^{w^*}\{D^H f(x') : x' \in B_\varepsilon(x) \cap \mathrm{dom}(D^H f)\}, \tag{7}$$

where $D^F f(x)$ (resp. $D^H f(x)$) denotes the Fréchet (resp. Hadamard) derivative of $f$ at $x$ and $\mathrm{dom}(D^F f)$ (resp. $\mathrm{dom}(D^H f)$) is the domain of $D^F f$ (resp. $D^H f$).

Since $\partial^C f$ is a $w^*$-cusco mapping ([3]), combining Proposition 1 with formulas (6) and (7), we obtain in view of [18, Theorem 2.4] the following corollary.

**Corollary 2.** *For every (locally Lipschitz) function $f$ on $X$ we have:*

(i) *if $X$ is an Asplund space, then*

$$\partial^C f(x) = \overline{co}^{w^*} \bigcap_{\varepsilon>0} \overline{\{D^F f(x') : x' \in B_\varepsilon(x) \cap \mathrm{dom}(D^F f)\}}^{w^*}; \tag{8}$$

(ii) *if $X$ has an equivalent Gâteaux differentiable norm, then*

$$\partial^C f(x) = \overline{co}^{w^*} \bigcap_{\varepsilon>0} \overline{\{D^H f(x') : x' \in B_\varepsilon(x) \cap \mathrm{dom}(D^H f)\}}^{w^*}. \tag{9}$$

**Submonotone and strictly submonotone mappings.** In 1981, J. Spingarn [24] introduced the notion of a *strictly submonotone* mapping in a finite-dimensional space. His definition is naturally extended to infinite dimensions as follows: a multivalued mapping $T{:}X \rightrightarrows X^*$ is said to be *strictly submonotone* (for short, *s-submonotone*) at $x \in X$ provided that for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\frac{\langle x_1^* - x_2^*, x_1 - x_2\rangle}{\|x_1 - x_2\|} \geq -\varepsilon \tag{10}$$

whenever $x_i \in B_\delta(x),\ x_i^* \in T(x_i),\ i = 1, 2$, and $x_1 \neq x_2$.

The operator $T{:}X \rightrightarrows X$ is called *submonotone* at $x$, if (10) holds under the additional assumption $x_2 = x$. (Note that $T$ is submonotone at every $x \notin \mathrm{dom}(T)$ and s-submonotone at every $x \notin \overline{\mathrm{dom}}(T)$.)

Appropriate directional versions of these notions have been introduced in [10] (see also [11] and [16]): an operator $T : X \rightrightarrows X^*$ is called *directionally strictly submonotone*[1] (for short, *ds-submonotone*) at $x$, if for every $e \in S_X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\frac{\langle x_1^* - x_2^*, x_1 - x_2\rangle}{\|x_1 - x_2\|} \geq -\varepsilon \tag{11}$$

whenever $x_i \in B_\delta(x),\ x_i^* \in T(x_i),\ i = 1, 2,\ x_1 \neq x_2$ and $\|\frac{x_1 - x_2}{\|x_1 - x_2\|} - e\| < \delta$.

The operator $T : X \rightrightarrows X^*$ is called *directionally submonotone* (for short, *d-submonotone*) at $x$, if (11) holds under the additional assumption $x_2 = x$.

It is easily seen that if (11) holds uniformly for all directions $e \in S_X$, then $T$ is s-submonotone at $x$. Similarly, if (11) holds uniformly for $x_2 = x$, then $T$ is submonotone at $x$. If $X = \mathbb{R}^n$, the compactness of the unit sphere in $\mathbb{R}^n$ entails that an operator $T$ is ds-submonotone (resp. d-submonotone) if, and only if, it is s-submonotone (resp. submonotone).

Given a nonempty subset $U$ of $X$, we say that $T$ is s-submonotone (resp. submonotone, ds-submonotone, d-submonotone) on $U$, if $T$ has the corresponding property at every $x \in U$.

Let us recall from [11, Theorem 2.4] that every ds-submonotone operator $T$ on $X$ is *locally bounded* on $\mathrm{int}\,\mathrm{dom}(T)$. The definition of ds-submonotonicity (relation (11)) is reminiscent of monotonicity and can be considered as a limiting variant of it. It can also be considered as a mild continuity condition, since any continuous function $g : U \to X^*$ can be seen as a (single-valued) s-submonotone operator on $U$. Thus, every monotone operator is s-submonotone, while the converse is not true. The class of s-submonotone operators is stable under addition and is relatively large.

## 3. A sufficient condition for integration

In this section we give sufficient conditions for integrating multivalued operators. We first need some terminology. Given a segment $[x, y]$, a finite sequence $\{x_i\}_{i=1}^k$

[1] "strictly submonotone" according to the terminology of [10], [11].

of $[x, y]$ is called a *subdivision of the segment* $[x, y]$, if $x_1 = x$, $x_k = y$ and

$$\sum_{i=1}^{k-1} \|x_{i+1} - x_i\| = \|x - y\|. \tag{12}$$

A polygonal path $[w_h]_{h=1}^m$ is a union of consecutive segments; it is said to be closed if $w_m = w_1$. A finite sequence $\{x_i\}_{i=1}^n$ is called a *subdivision of the path* $[w_h]_{h=1}^m$ if there exists an increasing sequence $1 = k_1 < k_2 < ... < k_m = n$ such that for $1 \le h \le m-1$, $\{x_i\}_{i=k_h}^{k_{h+1}}$ is a subdivision of the segment $[w_h, w_{h+1}]$.

The following definition is a reformulation in infinite dimensions of a property introduced by Janin [12] for the class of bounded operators defined on compact subsets of $\mathbb{R}^n$.

**Definition 3.** An operator $T : X \rightrightarrows X^*$ is called *radially cyclically submonotone on a subset* $U$ of its domain if for any closed polygonal path $[w_h]_{h=1}^m \subseteq U$ and any $\varepsilon > 0$, there exists $\delta > 0$, such that for any subdivision $\{x_i\}_{i=1}^n$ of $[w_h]_{h=1}^m$ satisfying $\|x_{i+1} - x_i\| < \delta$ (for $i = 1, 2, ..., n$) and any $x_i^* \in T(x_i)$ one has

$$\sum_{i=1}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle \le \varepsilon. \tag{13}$$

The proof of the following result borrows ideas from [12], mainly in steps 1 and 4.

**Theorem 4.** *Let $T : X \rightrightarrows X^*$ be locally bounded and radially cyclically submonotone on an open subset $U$ of* $\mathrm{dom}(T)$. *Suppose that $T$ is d-submonotone, or more generally, that for any $x \in U$, $x^* \in T(x)$, $u \in X$ one has*

$$\liminf_{t \to 0^+} \sup_{y^* \in T(x+tu)} \langle y^* - x^*, u \rangle \ge 0. \tag{14}$$

*Then there exists a locally Lipschitz function $f : U \to \mathbb{R}$ such that $T \subseteq \partial^H f$ on $U$. If, in addition, $T$ is submonotone, then $T \subseteq \partial^F f$ on $U$.*

*Proof.* **Case 1:** Let us first suppose that $U$ is connected. Then let $V$ be the set of $(x, y) \in U \times U$ such that $[x, y] \subseteq U$. Given $(x, y) \in V$ and $\sigma > 0$, let us denote by $S_\sigma(x, y)$ the set of subdivisions $\{x_i\}_{i=1}^n$ of the segment $[x, y]$ such that $x_1 := x$, $x_n := y$, and $\|x_{i+1} - x_i\| < \sigma$ for $i = 1, 2, ..., n-1$.

We consider the function $g(\cdot, \cdot) : V \to \mathbb{R} \cup \{+\infty\}$ given by

$$g(x, y) := \inf_{\sigma > 0} \sup \left\{ \sum_{i=1}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle : \ \{x_i\}_{i=1}^n \in S_\sigma(x, y),\ x_i^* \in T(x_i) \right\}. \tag{15}$$

Since $T$ is locally bounded, a compactness argument shows that for any $(x, y) \in V$, there exist $k > 0$ and $\rho > 0$ such that for all $(x', y') \in B_\rho(x) \times B_\rho(y)$, we have $(x', y') \in V$ and

$$|\, g(x', y') \,| \le \ k\|x' - y'\|. \tag{16}$$

Let us now fix some $x_0$ in $U$ and define $f : U \to \mathbb{R} \cup \{+\infty\}$ as follows:

$$f(x) := \sup \left\{ \sum_{h=1}^{m-1} g(w_h, w_{h+1}) \right\}, \tag{17}$$

where the supremum is taken over all $m \geq 2$ and all polygonal paths $[w_h]_{h=1}^m$ with $w_1 := x_0$ and $w_m := x$ such that $[w_h, w_{h+1}] \subseteq U$ for all $h = 1, 2, ..., m-1$. (Note that this family of paths is nonempty, since $U$ is open and connected.)

*Step 1:* The domain of $f$ is nonempty.

We shall show, in particular, that $f(x_0) = 0$. Since $f(x_0) \geq g(x_0, x_0) = 0$, it suffices to show that $f(x_0) \leq 0$. To this end, let us suppose that $f(x_0) > 0$ and take any $0 < \varepsilon < f(x_0)$. By (17) we infer that for some closed polygonal path $[w_h]_{h=1}^m$ (with $w_1 = w_m = x_0$) we have $\sum_{h=1}^{m-1} g(w_h, w_{h+1}) > \varepsilon$. Then according to (15), for any $\delta > 0$ we can find a subdivision $\{x_i\}_{i=1}^n$ of the path $[w_h]_{h=1}^n$ (where $x_1 = x_n = x_0$) and $\{x_i^*\}_{i=1}^n$ in $X^*$ such that $x_i^* \in T(x_i)$, $\|x_{i+1} - x_i\| < \delta$ for all $i \geq 1$, and $\sum_{i=1}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle > \varepsilon$. Since $T$ is radially cyclically submonotone, we get a contradiction.

*Step 2:* $f(y) \geq f(x) + g(x, y), \ \forall (x, y) \in V$.

Take any $r < f(x)$ and choose a polygonal path $[w_h]_{h=1}^m$ in $U$ with $w_1 = x_0$ and $w_m = x$ such that $\sum_{h=1}^{m-1} g(w_h, w_{h+1}) > r$. Set $w_{m+1} = y$. It follows from (17) that

$$f(y) \geq \sum_{h=1}^{m} g(w_h, w_{h+1}) > r + g(w_m, w_{m+1}) = r + g(x, y).$$

Since $r < f(x)$ is arbitrarily close to $f(x)$, the proof is complete.

*Step 3:* $f(\bar{x}) < +\infty$ for all $\bar{x} \in U$, and $f$ is locally Lipschitz on $U$.

Take any $\bar{x} \in U$ and choose a polygonal path $[w_h]_{h=1}^m$ in $U$ with $w_1 = \bar{x}$ and $w_m = x_0$. It follows from Step 2 that for $y = x_0$ and for $x = w_{m-1}$, we have

$$0 = f(x_0) \geq f(w_{m-1}) + g(w_{m-1}, x_0),$$

which shows that $f(w_{m-1})$ is finite. Taking now $y = w_{m-1}$ and $x = w_{m-2}$, we conclude that $f(w_{m-2})$ is finite. Proceeding like this, we finally conclude that $f(\bar{x}) = f(w_1) < +\infty$. Now to show that $f$ is locally Lipschitz, given $\bar{x} \in U$ we take $x = y = \bar{x}$ and $\rho > 0$ such that for any $x', y' \in B_\rho(\bar{x})$ we have $(x', y') \in V$ and the estimate in (16). It follows from Step 2 that

$$|\, f(y') - f(x')\,| \leq \max\{-g(x', y'), -g(y', x')\},$$

which yields that $f$ is locally Lipschitz on $U$.

*Step 4:*

$$T(x) \subseteq \partial^H f(x) \quad \forall x \in U. \tag{18}$$

Fix $x \in U$ and $x^* \in T(x)$. Let $r > 0$ be such that $B_r(x) \subseteq U$. For every $u \in S_X$, we have by Step 2 that

$$\frac{f(x + tu) - f(x)}{t} \geq \frac{g(x, x + tu)}{t}, \text{ for all } t \in \,]0,\ r[.$$

Since $f$ is locally Lipschitz, it suffices to show that for any $u \in S_X$ and any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\inf_{0<t<\delta} \frac{g(x, x + tu)}{t} \geq \langle x^*, u \rangle - \varepsilon. \tag{19}$$

From (14), we can associate to any $\varepsilon > 0$ some $\delta > 0$ (depending on $u$) such that for all $s \in ]0, \delta[$ we can find $y^* \in T(x + su)$ satisfying

$$\langle y^*, u\rangle \geq \langle x^*, u\rangle - \varepsilon. \tag{20}$$

Fix $0 < t < \delta$. Given $\sigma > 0$ and any subdivision $\{t_i\}_{i=1}^n$ of $[0, t]$ such that

$$\sup_i (t_{i+1} - t_i) < \sigma,$$

we can find $x_i^* \in T(x + t_i u)$ such that

$$\langle x_i^*, u\rangle \geq \langle x^*, u\rangle - \varepsilon.$$

Setting $x_i := x + t_i u$, we observe that $\{x_i\}_{i=1}^n \in S_\sigma(x, x + tu)$ and

$$\begin{aligned}\sum_{i=1}^{n-1} \langle x_i^*, x_{i+1} - x_i\rangle &= \sum_{i=1}^{n-1} \langle x_i^*, (t_{i+1} - t_i)u\rangle \\ &\geq (\langle x^*, u\rangle - \varepsilon) \sum_{i=1}^{n-1} (t_{i+1} - t_i) = \langle x^*, u\rangle t - \varepsilon t.\end{aligned}$$

Since $\sigma > 0$ can be taken arbitrarily small, we obtain

$$g(x, x + tu) \geq \langle x^*, tu\rangle - t\varepsilon,$$

and relation (19) follows for the $\delta$ introduced above.

Note that, if we assume in addition that $T$ is submonotone, then the above $\delta$ (in Step 4) does not depend on the direction $u \in S_X$; hence (19) yields $x^* \in \partial^F f(x)$.

**Case 2** (general case): Let $U$ be an arbitrary nonempty open set. Then $U$ can be written as a disjoint union of open connected sets $U_i$. Applying the result of Case 1 for each $i$, we obtain a locally Lipschitz function $f_i$ on $U_i$ with $T(x) \subseteq \partial^H f(x)$ (resp. $T(x) \subseteq \partial^F f(x)$, if $T$ is submonotone), for all $x \in U_i$. Define $f : U \to \mathbb{R}$ by $f(x) = f_{i(x)}(x)$, where $i(x)$ is the unique index such that $x \in U_{i(x)}$. It follows that $f$ is locally Lipschitz and $T \subseteq \partial^H f$ (resp. $T \subseteq \partial^F f$). □

## 4. Cyclic submonotonicity

In Theorem 4 we obtained a sufficient condition ensuring that an operator $T$ is included in the subdifferential $\partial^C f$ of a locally Lipschitz function $f$. In this section we reinforce Definition 3 (by using a notion of approximate subdivisions of closed polygonal paths) to ensure the coincidence of $T$ with the subdifferential $\partial^C f$. This leads to a notion of cyclic submonotonicity, which turns out (in Section 5) to be a necessary and sufficient condition for the integration process described in Theorem 4. Its relation with radial cyclic submonotonicity is given in Proposition 17.

Let us first give the definition of a $\delta$-subdivision of a closed polygonal path.

**Definition 5.** Given $\delta > 0$ and a closed polygonal path $[w_h]_{h=1}^m$, we say that $\{x_i\}_{i=1}^n$ is a $\delta$ *-subdivision* of $[w_h]_{h=1}^m$ if $x_n = x_1$ and

(i) $\{x_i\}_{i=1}^n \subseteq B_\delta([w_h]_{h=1}^m)$,

(ii) $\|x_{i+1} - x_i\| < \delta$, for $i \in \mathbb{N}_{n-1}$, and

(iii) there exists a finite sequence $\{k_h\}_{h=1}^m$ with $1 = k_1 < k_2 < ... < k_m := n$ such that for $1 \leq h \leq m - 1$ we have

$$k_h \leq i < k_{h+1} \implies \|\frac{x_{i+1} - x_i}{\|x_{i+1} - x_i\|} - \frac{w_{h+1} - w_h}{\|w_{h+1} - x_h\|}\| < \delta.$$

We are now ready to give the following definition.

**Definition 6.** An operator $T$ is called *cyclically submonotone,* if for any closed polygonal path $[w_h]_{h=1}^m$ and any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\delta$-subdivisions $\{x_i\}_{i=1}^n$ of $[w_h]_{h=1}^m$ and all $x_i^* \in T(x_i)$, one has

$$\sum_{i=1}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle \leq \varepsilon \sum_{i=1}^{n-1} \|x_{i+1} - x_i\|. \tag{21}$$

If $U$ is an open subset of $X$, an operator $T$ is said to be *cyclically submonotone on* $U$ if (21) holds for closed polygonal paths and $\delta$-subdivisions in $U$. Furthermore, a cyclically submonotone operator $T$ on $U$ is called *maximal cyclically submonotone* on $U$, if there is no cyclically submonotone operator $S \neq T$ such that $T(x) \subseteq S(x)$ for all $x \in U$.

Let us note that, as follows from (12), the length of a subdivision of a path is always equal to the length of the initial path. On the contrary, the definition of a $\delta$-subdivision is more general, since no direct constraint on its length is imposed. This flexibility in Definition 5 enables us to show that, unlike the case of radial cyclic submonotonicity, every cyclically submonotone operator is also ds-submonotone.

**Proposition 7.** *Every cyclically submonotone operator on $U$ is ds-submonotone on $U$. Consequently, if $U \subseteq \operatorname{int}\operatorname{dom}(T)$, then $T$ is also locally bounded on $U$.*

*Proof.* Let $x_0 \in U$, $\varepsilon > 0$ and $e \in S_X$. Since $U$ is open, there exists $\lambda > 0$ such that $[x_0, x_0 + \lambda e] \subseteq U$. Let $m = 3$, $w_1 = x_0 = w_3$ and $w_2 = x_0 + \lambda e$. For $\varepsilon > 0$ and for the path $[w_h]_{h=1}^3$, take $\delta > 0$ as in Definition 6, and set $\delta' = \delta/2$. Then if $x_1, x_2 \in B_{\delta'}(x_0)$ are such that $x_1 \neq x_2$ and $\|\frac{x_1 - x_2}{\|x_1 - x_2\|} - e\| < \delta'$, we can easily check that for $x_3 = x_1$, $\{x_i\}_{i=1}^3$ is a $\delta$-subdivision of the path $[w_h]_{h=1}^3$. So relation (21) yields (11), and $T$ is ds-submonotone at $x_0$. Since $x_0$ is arbitrary in $U$, it follows that $T$ is ds-submonotone on $U$. The last assertion follows from [11, Theorem 2.4]. ☐

*Remarks.* **1.** Every cyclically submonotone operator is radially cyclically submonotone on every open subset $U$ of its domain. Indeed, if (21) is true and $\{x_i\}_{i=1}^n$ is a subdivision of $[w_h]_{h=1}^m$ in $U$, then $\sum_{i=1}^{n-1} \|x_{i+1} - x_i\| = \sum_{i=1}^{n-1} \|w_{i+1} - w_i\|$. It follows that (21) yields (13) for $\varepsilon' = \varepsilon(\sum_{i=1}^{n-1} \|w_{i+1} - w_i\|)^{-1}$.

**2.** It is obvious that every cyclically monotone operator is cyclically submonotone. On the other hand, an operator can even be strongly monotone, without being cyclically submonotone, as one can see from the example (also used in [9] for a similar purpose) of the operator $T : \mathbb{R}^2 \to \mathbb{R}^2$, with

$$T(x, y) = \langle \frac{x}{2} - y, x + \frac{y}{2} \rangle.$$

**3.** Cyclic submonotonicity is a separably determined property, i.e., an operator $T : X \rightrightarrows X^*$ is cyclically submonotone on $U$ if, and only if, for every separable closed subspace $Y$ of $X$, the operator $T\,|_Y : Y \rightrightarrows Y^*$ (defined for all $y_1, y_2 \in Y$ by $\langle T|_Y(y_1), y_2 \rangle := \langle T(y_1), y_2 \rangle$) is cyclically submonotone on $U \cap Y$.

We now show that if $T$ is locally bounded and cyclically submonotone (resp. ds-submonotone) on $U$, then so is the $w^*$-cusco generated by $T$. Let us recall that if $U$ is an open subset of $\operatorname{dom}(T)$, the local boundedness assumption on $T$ becomes superfluous, since it follows from its ds-submonotonicity (see Proposition 7).

**Proposition 8.** *If $T$ is locally bounded and cyclically submonotone (resp. ds-submonotone) on $U$, then the operators $\overline{T}$ and $\overline{co}^{w^*}(T)$ are also cyclically submonotone (resp. ds-submonotone) on $U$. In particular, the $w^*$-cusco generated by $T$ is cyclically submonotone (resp. ds-submonotone) on $U$.*

*Proof.* It is easily seen from (21) that $\overline{co}^{w^*}(T)$ is cyclically submonotone. In order to prove that $\overline{T}$ is cyclically submonotone, let us consider a closed polygonal path $[w_h]_{h=1}^m \subseteq U$ and $\varepsilon > 0$. Let us take $\delta > 0$ guaranteed by Definition 6 for the operator $T$, and let us consider any $\delta$-subdivision $\{x_i\}_{i=1}^n$ of $[w_h]_{h=1}^m$ in $U$ and $x_i^* \in \overline{T}(x_i)$. Then there exist nets $(x_i(\lambda))_{\lambda\in\Lambda}$ and $(x_i^*(\lambda))_{\lambda\in\Lambda}$ such that $x_i^*(\lambda) \in T(x_i(\lambda))$, $(x_i(\lambda)) \xrightarrow{\|\cdot\|} x_i$ and $(x_i^*(\lambda)) \xrightarrow{w^*} x_i^*$. Since $\{x_i\}_{i=1}^n$ is a finite sequence, there exists $\lambda_0 \in \Lambda$ such that $\{x_i(\lambda)\}_{i=1}^n$ is a $\delta$-subdivision of $[w_h]_{h=1}^m$ for all $\lambda \succeq \lambda_0$ (where $\succeq$ is the preorder relation of $\Lambda$). It follows from (21) that

$$\sum_{i=1}^n \langle x_i^*(\lambda), x_{i+1}(\lambda) - x_i(\lambda)\rangle \le \varepsilon \sum_{i=1}^n \|x_{i+1}(\lambda) - x_i(\lambda)\|.$$

Taking limits on both sides, we obtain

$$\sum_{i=1}^n \langle x_i^*, x_{i+1} - x_i\rangle \le \varepsilon \sum_{i=1}^n \|x_{i+1} - x_i\|.$$

This shows that $\overline{T}$ is cyclically submonotone; hence so is the operator $\overline{co}^{w^*}(\overline{T})$. Since now $T$ is locally bounded on $U$, we have

$$\overline{T}(x) := \bigcap_{\varepsilon>0} \overline{\{T(x') : x' \in B_\varepsilon(x) \cap \mathrm{dom}(T)\}}^{w^*}$$

for all $x \in U$. Using Proposition 1, we conclude that $\overline{co}^{w^*}(\overline{T})$ is the minimal $w^*$-cusco operator containing $T$. This finishes the proof. The assertions concerning ds-submonotonicity can be proved likewise. □

The following proposition reveals an important feature of cyclic submonotonicity.

**Proposition 9.** *Let $U$ be an open subset of* $\mathrm{dom}(T)$*, and $T$ a cyclically submonotone operator on $U$. The following statements are equivalent:*

(i) *$T$ is $w^*$-cusco on $U$;*
(ii) *$T$ is minimal $w^*$-cusco on $U$;*
(iii) *$T$ is maximal ds-submonotone on $U$;*
(iv) *$T$ is maximal cyclically submonotone on $U$.*

*Proof.* Implication (ii)⇒(i) is obvious. Assume now that (i) holds. Using Proposition 7, we conclude that $T$ is ds-submonotone on $U$. Since $T$ is $w^*$-cusco, from [11, Lemma 3.2] it follows that $T$ is maximal ds-submonotone on $U$. Hence (i)⇒(iii).

(iii)⇒(iv): Let $S$ be a cyclically submonotone operator whose graph contains the graph of $T$. Then $S$ is ds-submonotone (see Proposition 7); hence it coincides with $T$.

(iv)⇒(ii): Since $T$ is locally bounded on $U \subseteq \mathrm{int}\,\mathrm{dom}(T)$, Proposition 8 guarantees that $T$ is $w^*$-cusco. Assume that there exists $S \subseteq T$ such that $S$ is $w^*$-cusco. Obviously $S$ will also be cyclically submonotone. Since (i)⇒(iv), $S$ is maximal cyclically submonotone, whence $S = T$. □

*Remark* 10. We recall from [23] that if $X$ is Asplund (resp. $X$ has a Gâteaux differentiable norm), then every minimal $w^*$-cusco operator is single-valued and $(\|.\| - \|.\|)$ upper semicontinuous (resp. $(\|.\| - w^*)$ upper semicontinuous) at every point of a $G_\delta$ dense set.

**Corollary 11.** *Let $T$ be a cyclically submonotone operator on an open subset $U$. Then $\overline{co}^{w^*}(T)$ is maximal cyclically submonotone on $U$.*

*Proof.* Since $T$ is locally bounded, the operator $S := \overline{co}^{w^*}(\overline{T})$ is $w^*$-cusco. By Proposition 8, $S$ is also cyclically monotone. The conclusion follows from Proposition 9. □

Let $Z$ be a closed subspace of $X$ and define the multivalued operator $S : Z \rightrightarrows Z^*$ as follows:

$$S(z) = \{z^* \in Z^* : \exists x^* \in T(z) \text{ such that } z^* = x^*|_Z\} \tag{22}$$

(where $x^*|_Z$ denotes the restriction of the functional $x^*$ to $Z$).

**Lemma 12.** (i) *If $T$ is locally bounded and $w^*$-cusco on an open subset $U$, then $S$ is $\sigma(Z^*, Z)$-cusco on $U \cap Z$, where $\sigma(Z^*, Z)$ denotes the $w^*$-topology on $Z^*$.*

(ii) *If $T$ is maximal cyclically submonotone on $U$ and if $U \subseteq \operatorname{intdom}(T)$, then $S$ is maximal cyclically submonotone on $U \cap Z$.*

*Proof.* Assertion (i) follows easily since $S$ has a $\sigma(Z^*, Z)$-closed graph on $U \cap Z$. To show (ii), let us observe (from Definition 6) that if $T$ is cyclically submonotone on $U$, then $S$ is cyclically submonotone on $U \cap Z$. By Propositions 7 and 9 we have that $T$ is locally bounded and $w^*$-cusco. It follows by (i) that $S$ is $\sigma(Z^*, Z)$-cusco on $U \cap Z$; so the proof finishes by a new application of Proposition 9 (i)→(iv). □

## 5. Main results

Throughout this section $U$ will always denote a nonempty open subset of $X$. Let us give the following definition.

**Definition 13.** A locally Lipschitz function $f : U \to \mathbb{R}$ is called *subsmooth* if $\partial^C f$ is ds-submonotone on $U$.

Every subsmooth function $f$ is regular (see [24], [11, Theorem 4.1]), a locally Lipschitz function $f$ being called *regular* ([6]) if $f'(x, d) = f^o(x, d)$ for all $d \in X$. It follows that

$$\partial^C f = \partial^H f. \tag{23}$$

In the particular case where $X$ has some regularity, subsmoothness is characterized as follows.

**Proposition 14.** *Let $X$ be an Asplund space (resp. $X$ has a Gâteaux differentiable renorming). Then a locally Lipschitz function $f : U \to \mathbb{R}$ is subsmooth if, and only if, $\partial^F f$ (resp. $\partial^H f$) is ds-submonotone on $U$.*

*Proof.* The "necessity" part is obvious. To show the "sufficiency" part, let $T = \partial^F f$ (resp. $\partial^H f$). Since $f$ is locally Lipschitz and $T$ is included in $\partial^C f$, it follows that $T$ is locally bounded. Using Corollary 2 and Proposition 8, we obtain that $\partial^C f$ is ds-submonotone, hence that $f$ is subsmooth. (Note that this implies that $\operatorname{dom}(\partial^H f) = U$.) □

It follows from Definition 13 and the comments after the definition of ds-submonotonicity (in Section 2) that every convex or continuously differentiable (i.e., $C^1$) function is subsmooth. One of the main results in [24] is the following characterization of subsmooth functions in finite dimensions: *a (locally Lipschitz) function $f$ on $U \subseteq \mathbb{R}^n$ is subsmooth if, and only if, it is lower $C^1$*, a function $f$ being called *lower $C^1$* if for each $x_0 \in U$, there exist a neighborhood $V$ of $x_0$, a compact set $S$ and a jointly continuous function $g : V \times S \to \mathbb{R}$ such that, for all $x \in V$, $f(x) = \max_{s \in S} g(x,s)$ and $D_x g$ (exists and) is jointly continuous. In the last section, we will give some typical examples of subsmooth functions in infinite dimensions.

We now state the main results of the paper.

**Theorem A.** *For a locally Lipschitz function $f : U \to \mathbb{R}$, the following are equivalent:*
(i) *$f$ is subsmooth;*
(ii) *$\partial^C f$ is maximal cyclically submonotone on $U$.*

**Theorem B.** *Let $U$ be an open connected subset of $X$, and $f_1$, $f_2$ two subsmooth (or, more generally, regular) functions on $U$ such that $\partial^C f_1 = \partial^C f_2$. Then $f_1 = f_2 + c$ for some $c \in \mathbb{R}$.*

**Theorem C.** *If $T : X \rightrightarrows X^*$ is a multivalued operator and $U$ an open subset of* $\mathrm{dom}(T)$*, then $T$ is maximal cyclically submonotone on $U$ if, and only if, $T = \partial^C f = \partial^H f$ for some subsmooth function $f : U \to \mathbb{R}$, which is unique (up to a constant) on every connected subset of $U$. If, in addition, $T$ is submonotone, then $T = \partial^F f$.*

*Proof of Theorem A.* The implication (ii)⇒(i) is clear in view of Proposition 7 and Definition 13.

For the implication (i)⇒(ii), set $T := \partial^C f$. Since $T$ is $w^*$-cusco, in view of Proposition 9 (i)⇒(iv), it clearly suffices to show that $T$ is cyclically submonotone. To this end, consider any closed polygonal path $[w_h]_{h=1}^m \subseteq U$ and any $\varepsilon > 0$. Set $C = [w_h]_{h=1}^m$ and

$$e_h = \frac{w_{h+1} - w_h}{\|w_{h+1} - w_h\|}$$

for $h \in \mathbb{N}_{m-1} := \{1, 2, ..., m-1\}$.

Since $T$ is ds-submonotone, it follows that for every $x \in C$ and $h \in \mathbb{N}_{m-1}$, there exists $\alpha(x,h) > 0$ such that

$$\frac{\langle x_1^* - x_2^*, x_2 - x_1 \rangle}{\|x_1 - x_2\|} < \varepsilon \tag{24}$$

whenever $x_1 \neq x_2$ with $\|x_i - x\| < \alpha(x,h)$ , $x_i^* \in T(x_i)$ $(i = 1,2)$ and

$$\left\| \frac{x_1 - x_2}{\|x_1 - x_2\|} - e_h \right\| < \alpha(x,h).$$

Set $\beta(x) := \min_{h \in \mathbb{N}_m} \alpha(x,h)$, and note that (24) holds for all $x_1 \neq x_2$ such that $x_i \in B_{\beta(x)}(x)$ $(i = 1,2)$ and $\frac{x_1 - x_2}{\|x_1 - x_2\|} \in \bigcup_{h \in \mathbb{N}_{m-1}} B_{\beta(x)}(e_h)$.

Let $\delta > 0$ be a Lebesgue number of the open covering $\left(B_{\beta(x)}(x)\right)_{x \in C}$ of the compact set $C$, i.e.,

$$\forall w \in C, \exists x \in C : B_\delta(w) \subseteq B_{\beta(x)}(x). \tag{25}$$

Let us consider any $\delta$-subdivision $\{x_i\}_{i=1}^n$ of $C = [w_h]_{h=1}^m$. Since $f$ is locally Lipschitz, using Lebourg's mean value theorem ([13]) on every segment $[x_i, x_{i+1}]$ (for $i \in \mathbb{N}_{n-1}$), we infer the existence of $z_i \in ]x_i, x_{i+1}[$ and $z_i^* \in T(z_i)$ such that

$$f(x_{i+1}) - f(x_i) = \langle z_i^*, x_{i+1} - x_i \rangle. \tag{26}$$

Adding the above equalities, we have

$$\sum_{i=1}^{n-1} \langle z_i^*, x_{i+1} - x_i \rangle = 0,$$

which yields

$$\sum_{i=1}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle = \sum_{i=1}^{n-1} \langle x_i^* - z_i^*, x_{i+1} - x_i \rangle. \tag{27}$$

Since $\{x_i\}_{i=1}^n$ is a $\delta$-subdivision of $C$, it follows from (24) and (25) that for every $i \in \mathbb{N}_{n-1}$,

$$\langle x_i^* - z_i^*, \frac{x_{i+1} - x_i}{\|x_{i+1} - x_i\|} \rangle = \langle x_i^* - z_i^*, \frac{z_i - x_i}{\|z_i - x_i\|} \rangle < \varepsilon,$$

which, combined with (27), yields

$$\sum_{i=1}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle = \varepsilon \sum_{i=1}^{n-1} \|x_{i+1} - x_i\|.$$

This finishes the proof. □

Before proceeding to the proof of Theorem B, we need the following easy result.

**Lemma 15.** *Let $f : U \to \mathbb{R}$ be a locally Lipschitz function, let $Z$ be a closed linear subspace of $X$, and consider the function $g : Z \cap U \to \mathbb{R}$ defined by*

$$g(z) = f(z) \text{ for all } z \in U \cap Z. \tag{28}$$

*If $S : Z \rightrightarrows Z^*$ is as in (22) for $T = \partial^H f$, then we have:*

(i) *$S(z) \subseteq \partial^H g(z)$ for all $z \in U \cap Z$.*

(ii) *If for some $x_0 \in U \cap Z$ we have $\partial^H f(x_0) = \partial^C f(x_0)$, then $S(x_0) = \partial^H g(x_0) = \partial^C g(x_0)$.*

*Proof.* (i) Let $z \in U \cap Z$ and $z^* \in S(z)$. Then $z^* = x^*|_Z$ for some $x^* \in \partial^H f(z)$. Since $f$ is locally Lipschitz, by (5) we conclude that $g'(z, u) = f'(z; u) \geq \langle x^*, u \rangle$, for all $u \in Z$. It follows that $z^* \in \partial^H g(z)$.

(ii) Suppose now that for some $x_0 \in U \cap Z$ we have $\partial^H f(x_0) = \partial^C f(x_0)$. By (i) we have $S(x_0) \subseteq \partial^H g(x_0) \subseteq \partial^C g(x_0)$. Let us show that $\partial^C g(x_0) \subseteq S(x_0)$. Indeed, let $z_0^* \in \partial^C g(x_0)$. Then from (4) we have

$$\langle z_0^*, u \rangle \leq g^o(x_0; u) \leq f^o(x_0; u), \text{ for all } u \in Z.$$

Using the Hahn-Banach theorem, we conclude the existence of some $x^* \in X^*$ such that $x^*|_Z = z^*$ and $\langle x^*, . \rangle \leq f^o(x_0; .)$, so that $x^* \in \partial^C f(x_0) = \partial^H f(x_0)$. It follows that $z^* \in S(x_0)$. □

*Proof of Theorem B.* Suppose that $f_1$, $f_2$ are two subsmooth functions on $U$ such that $\partial^C f_1 = \partial^C f_2$. Without loss of generality, we suppose that $0 \in U$. For any

$x \in X$, let us set $Z = \text{span}[x]$, $g_i = f_i \mid_Z$ (the restriction of $f_i$ to $Z$, $i = 1, 2$) and $T = \partial^H f_1$. By (23) we have

$$T = \partial^H f_1 = \partial^C f_1 = \partial^C f_2 = \partial^H f_2,$$

and by Lemma 15(ii),

$$\partial^H g_1 = \partial^C g_1 = \partial^C g_2 = \partial^H g_2. \tag{29}$$

Since $Z$ is a one-dimensional space (in fact, separable would suffice), it follows from [3, Theorem 5.12] (see also [5, Section 4.2]) that the regular functions $g_1$, $g_2$ are essentially smooth.

**Case 1:** Suppose that $U$ is convex.

Then the set $U \cap Z$ is connected; so relation (29) yields $g_1 = g_2 + c$ for some $c \in \mathbb{R}$ (see [5, Proposition 4.12] or [3, Proposition 5.9]). Since $g_i = f_i|_Z$, we obtain

$$f_1(x) - f_2(x) = c = f_1(0) - f_2(0).$$

Since $x$ is arbitrarily chosen, we obtain $f_1 = f_2 + c$ on $U$.

**Case 2** (general): Since $U$ is open and $X$ is locally convex, from case 1 we conclude that $f_1 - f_2$ is locally constant on $U$. Since now $U$ is connected, it follows that $f_1 - f_2$ is constant on $U$. The proof is complete. □

Let us now proceed to the proof of Theorem C. We shall need the following lemma.

**Lemma 16.** *Let $f : U \to \mathbb{R}$ be locally Lipschitz, $x_0 \in X$, and let $Y$ be a separable subspace of $X$. Then there exists a separable subspace $Z$ of $X$ containing $Y$ and $x_0$ such that for the function $g : Z \cap U \to \mathbb{R}$ given by $g = f|_{U \cap Z}$ (as in (28)) we have*

$$g^o(x_0; u) = f^o(x_0; u) \text{ for all } u \in Z, \tag{30}$$

*and consequently*

$$\partial^C g(x_0) = \{z^* \in Z^* : z^* = x^*|_Z, z^* \in \partial^C f(x_0)\}. \tag{31}$$

*Proof.* Let $Y_0 = \text{span}[Y, x_0]$ be the closed linear space generated by $Y$ and $\{x_0\}$, and let $D_0$ be a countable dense subset of $Y_0$. Then for every $d \in D_0$, there exist $\{x_n\}_{n \geq 1}$ in $X$ and $\{t_n\}_{n \geq 1}$ in $]0, 1[$ such that $(x_n) \longrightarrow x_0$, $(t_n) \to 0^+$ and

$$f^o(x_0; d) < \frac{f(x_n + t_n d) - f(x_n)}{t_n} + \frac{1}{n}. \tag{32}$$

Set $A_0(d) = \{x_n : n \in \mathbb{N}\}$ and $A_0 = \bigcup_{d \in D_0} A_0(d)$. Consider the separable space $Y_1 = \text{span}[Y_0, D_0]$, let $D_1$ be a countable dense subset of $Y_1$ and define (using (32)) $A_1(d)$ for all $d$ in $D_1$ as above and $A_1 = \bigcup_{d \in D_1} A(d)$. Proceeding like this, we obtain an increasing sequence of closed separable subspaces $Y_n$ of $X$ and a sequence $(D_n)$ of countable subsets such that $D_n$ is dense in $Y_n$. Set

$$Z = \bigcup_n Y_n$$

and $g = f|_Z$. Then for any $u \in Z$ and $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and $d \in D_n$ such that $\|u - d\| < \varepsilon$. Using (32), we conclude easily that $g^o(x_0; d) = f^o(x_0; d)$. Since the functions $u \mapsto g^o(x_0; u)$ and $u \mapsto f^o(x_0; u)$ are Lipschitz, (30) follows. Relation (31) is now an easy consequence of (30) and the Hahn-Banach theorem. □

*Proof of Theorem C.* The sufficiency part following from Theorem A, we only have to show the necessity part.

To this end, let us suppose that $T$ is maximal cyclically submonotone on $U$. In particular, $T$ is a locally bounded ds-submonotone and radially cyclically submonotone operator on $U$ (see Proposition 7 and Remark 1). It follows by Theorem 4 that $T(x) \subseteq \partial^H f(x)$ for all $x \in U$, where $f$ is given by (17). Let us show that $T(x) = \partial^C f(x)$ for all $x \in U$.

Suppose that the contrary holds. Then for some $x_0 \in X$ and $x_0^* \in \partial f^C(x_0)$, we have $x_0^* \notin T(x_0)$. By Proposition 9, $T(x_0)$ is a nonempty $w^*$-closed convex set; hence there exists $u \in X$ such that

$$\langle x_0^*, u\rangle > \sup_{x^* \in T(x_0)} \langle x^*, u\rangle. \tag{33}$$

Set $Y = \operatorname{span}[x_0, u]$, and consider the separable subspace $Z$ of $X$ given by Lemma 16 and $g = f|_Z$. Let $S : Z \rightrightarrows Z^*$ be as in (22), i.e., for every $z \in U \cap Z$,

$$S(z) = \{z^* \in Z^* : z^* = x^*|_Z \text{ for some } x^* \in T(z)\}.$$

Then, by Lemma 12(i), $S$ is $\sigma(Z^*, Z)$-cusco on $U \cap Z$, and, by Lemma 15(i),

$$S \subseteq \partial^H g.$$

Since $S$ has nonempty values on $U \cap Z$, the above relation yields that $S(x) = \{D^H g(x)\}$ for all points $x$ for which the Hadamard derivative $D^H g(x)$ exists. Since $Z$ is a separable Banach space, it admits a Gâteaux smooth renorming. It follows that the Clarke subdifferential $\partial^C g$ is given by (7) and is the smallest $\sigma(Z^*, Z)$-cusco mapping whose graph contains the graph of the Hadamard derivative $D^H g$. Since $S$ is $\sigma(Z^*, Z)$-cusco, it follows that $\operatorname{Gr}(\partial^C g) \subseteq \operatorname{Gr}(S)$, for all $x \in U$. Since $\operatorname{Gr}(\partial^H g) \subseteq \operatorname{Gr}(\partial^C g)$, we conclude that

$$S = \partial^H g = \partial^C g \text{ on } U \cap Z.$$

In particular, $S(x_0) = \partial^C g(x_0)$, and, using the conclusion of Lemma 16,

$$S(x_0) = \{z^* \in Z^* : z^* = x^*|_Z,\ x^* \in \partial^C f(x_0)\}.$$

Let $z_0^* := x_0^*|_Z \in S(x_0) = \partial^C g(x_0)$. Since $u \in Z$, it follows that

$$\langle x_0^*, u\rangle = \langle z_0^*, u\rangle \leq g^o(x_0; u) = f^o(x_0; u).$$

This yields a contradiction to (33), since

$$g^o(x_0; u) = \sup_{z^* \in \partial^C g(x_0)} \langle z^*, u\rangle = \sup_{x^* \in S(x_0)} \langle x^*, u\rangle = \sup_{x^* \in T(x_0)} \langle x^*, u\rangle.$$

Hence we have shown that $T = \partial^C f$ on $U$. It follows from Definition 13 that $f$ is subsmooth, and by Theorem B that it is unique (modulo a constant) in every connected subset of $U$.

If moreover $T$ is submonotone, then, using Theorem 4 again, we infer that $T(x) \subseteq \partial^F f(x)$, for all $x \in U$. Hence $\operatorname{Gr}(T) \subseteq \operatorname{Gr}(\partial^F f) \subseteq \operatorname{Gr}(\partial^C f)$ and $T = \partial^C f$ (on $U$), whence $T = \partial^F f = \partial^C f$ on $U$. □

An inspection of the above proof yields the following result.

**Proposition 17.** *Suppose that $T$ is a locally bounded $w^*$-cusco operator on an open subset $U$ of $X$. Then $T$ is (maximal) cyclically submonotone on $U$ if, and only if, $T$ is radially cyclically submonotone and ds-submonotone on $U$.*

*Proof.* If $T$ is cyclically submonotone, then from Proposition 7 and Remark 1, it follows that $T$ is radially cyclically submonotone and ds-submonotone on $U$. Conversely, if $T$ is radially cyclically submonotone and ds-submonotone on $U$, then by Theorem 4 we infer that $T \subseteq \partial^H f$ on $U$ for some locally Lipschitz function $f$.

Since now $T$ is locally bounded and $w^*$-cusco on $U$, by Lemma 12 (i), for every closed subspace $Z$ of $X$, the operator $S$ given in (22) is $\sigma(Z^*, Z)$-cusco on $U \cap Z$. Thus, repeating the arguments of the above proof, we obtain that $T = \partial^C f$ on $U$ and that $f$ is subsmooth. It follows from Theorem A that $T$ is (maximal) cyclically submonotone on $U$. □

## 6. Examples of subsmooth functions

Apart from the classes of convex, continuous, or $C^1$ functions (or of sums of such functions), typical examples of subsmooth functions include certain types of marginal functions, as for instance the class of lower $C^1$ functions introduced in [24] (and also considered in [21] and [16]). Let us note that subdifferentiability properties of marginal functions have been studied by many authors; see for instance [3], [7], [8] and [11].

In the sequel let $A$ be an arbitrary nonempty set and $U$ an open subset of $X$. We consider the marginal function $f : U \to \mathbb{R}$ defined for every $x \in U$ by

$$f(x) = \sup_{\alpha \in A} g(x, \alpha), \tag{34}$$

where $g : U \times A \to \mathbb{R}$ is such that $g(\cdot, \alpha)$ is a regular locally Lipschitz function and $f(x) < +\infty$ for every $x \in U$. Let us also make the following assumptions.

(i) For every $x_0 \in U$, there exists $\delta > 0$ such that the set

$$\bigcup \{\partial^C g(x, \alpha) : \alpha \in A, x \in B_\delta(x_0), g(x, \alpha) \geq f(x_0) - \delta\}$$

is norm bounded.

(ii) For every $x \in X$ and $e \in S_X$ there is an $\varepsilon > 0$ such that for every $\gamma > 0$ there exists $\delta > 0$ such that

$$f'(y, a; e) - f'(x, a; e) < \gamma$$

whenever $\| x - y \| < \delta$ and $g(x, \alpha) \geq f(x) - \varepsilon$.

(iii) For every $x_0 \in U$, $e \in S_X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$g'(x, a; e) - \frac{g(x + te, a) - g(x, a)}{t} < \varepsilon \tag{35}$$

for all $x \in B_\delta(x_0)$, all $t \in \,]0, \ \delta[$ and all $a \in A$ with $g(x, a) \geq f(x_0) - \delta$.

The following result is an easy consequence of results established in [11].

**Theorem 18.** *If $f$ is a marginal function* (*relation* (34)) *satisfying conditions* (i)-(iii), *then $f$ is subsmooth on $U$ and $\partial^H f = \partial^C f$ is maximal cyclically submonotone on $U$.*

*Proof.* By [11, Theorem 5.4 (a)], $f$ is locally Lipschitz, and by [11, Lemma 5.3] and [11, Theorem 5.2 (d)], $f$ is regular; therefore $\partial^H f = \partial^C f$. Now again by [11, Theorem 5.4 (a)], $\partial^C f$ is ds-submonotone in $U$, i.e., $f$ is subsmooth. Theorem A finishes the proof. □

*Remarks.* **1.** It is easily seen that the above class of functions contains the class of lower $C^1$ functions. Combining this with Theorem 18 and Spingarn's characterization of lower $C^1$ functions ([24, Theorem 3.9]), we conclude that in finite dimensions a function $f$ is lower $C^1$ if, and only if, $f$ is given by (34) and satisfies conditions (i)-(iii).

**2.** If, in addition to the assumptions of Theorem 18, the choice of $\delta$ in (35) does not depend on $e$, then by [11, Theorem 5.4 (b)] we conclude that $\partial^C f$ is s-submonotone, and by Theorem C, that $\partial^F f = \partial^C f$.

Now let $A$ be an arbitrary nonempty subset of the Banach space $X$. Let us define the *distance function* by

$$d_A(x) := \inf_{\alpha \in A} \| x - \alpha \| \qquad (x \in X).$$

The following proposition provides another typical example of subsmooth functions.

**Proposition 19.** *Suppose that the norm of $X$ is uniformly Gâteaux (resp. uniformly Fréchet) differentiable. For any nonempty closed set $A$ of $X$, let us consider the function*

$$f(x) = -d_A(x).$$

(i) *Then $f$ is subsmooth, hence regular, on $X \setminus A$.*

(ii) *$\partial^C f = \partial^H f$ (resp. $\partial^C f = \partial^F f$) is maximal cyclically submonotone and $\partial^C d_A(.)$ is minimal $w^*$-cusco on $X \setminus A$.*

*Proof.* The assertions follow from [11, Theorem 5.6 (a),(b)], Theorem A and Remark 2. □

Let us now consider another important class of examples of subsmooth functions. We shall say that a function $f : U \to \mathbb{R}$ is *amenable* ([22, Definition 10.23]) if for any $x_0 \in U$, there exist an open neighborhood $V$ of $x_0$, a Banach space $Y$, a continuously differentiable function $F : V \to Y$ and a proper lower semicontinuous convex function $g : Y \to \mathbb{R} \cup \{+\infty\}$ such that

$$f(x) = g(F(x)) \text{ for all } x \in V \tag{36}$$

and

$$\mathbb{R}_+(\operatorname{dom} g - F(x_0)) + F'(x_0)(X) = Y. \tag{37}$$

For the sake of simplicity, and since in this paper we limit our study to locally Lipschitz functions, we further consider the subclass $\mathcal{A}(U)$ of amenable functions $f$ such that in the decomposition (36) we have $F(V) \subseteq \operatorname{int}\operatorname{dom}(g)$. Obviously, every function in $\mathcal{A}(U)$ is locally Lipschitz and condition (37) is satisfied.

**Proposition 20.** *If $f \in \mathcal{A}(U)$, then $\partial^C f$ is s-submonotone (hence, in particular, $f$ is subsmooth).*

*Proof.* Let $f$ be in $\mathcal{A}(U)$. With no loss of generality, we may assume that $V = U$, so that $f = g \circ F$ with $g$ and $F$ as in (36). Since $g$ is regular on $F(U)$ (because it is convex and continuous on $\operatorname{int}\operatorname{dom}(g)$), applying [6, Theorem 2.3.10] we conclude that $f$ is also regular, that is, $\partial^C f = \partial^H f$. Now set $T = \partial^C f = \partial^H f$. We shall show that $T$ is s-submonotone.

To this end, let $x_0 \in U$, $x_1, x_2 \in U$ and $x_i^* \in \partial f(x_i)$, $i = 1, 2$. Then there exist $y_i^* \in \partial g(F(x_i))$ $(i = 1, 2)$ such that $x_i^* = y_i^* \circ F'(x_i)$, where $F'(x)$ denotes the Fréchet derivative of $F$ at $x$. It follows that

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle = \langle y_1^*, F'(x_1)(x_1 - x_2)\rangle - \langle y_2^*, F'(x_2)(x_1 - x_2)\rangle. \tag{38}$$

Since $F$ is continuously differentiable, there exist $\delta > 0$ and a function $r : U \times U \to Y$ such that

$$F(v) - F(u) = F'(u)(v - u) + r(u, v) \tag{39}$$

for all $u, v \in B_\delta(x_0)$, and

$$\lim_{\substack{u,v \to x_0 \\ u \neq v}} \frac{\|r(u, v)\|}{\|u - v\|} = 0. \tag{40}$$

Combining (38) with (39), thanks to the monotonicity of $\partial g$ we obtain

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq \langle y_1^*, r(x_1, x_2)\rangle + \langle y_2^*, r(x_2, x_1)\rangle,$$

which yields, when $x_1 \neq x_2$,

$$\frac{\langle x_1^* - x_2^*, x_1 - x_2 \rangle}{\|x_1 - x_2\|} \geq \langle y_1^*, \frac{r(x_1, x_2)}{\|x_1 - x_2\|}\rangle + \langle y_2^*, \frac{r(x_2, x_1)}{\|x_1 - x_2\|}\rangle.$$

The result now follows from (40) and the local boundedness of $\partial g$ near $F(x_0)$. □

*Remark.* Since every strictly Gâteaux differentiable function $F : U \to Y$ is locally Lipschitz ([6, Proposition 2.2.1]), a slight modification of the above proof suffices to establish that $\partial^C f$ is ds-submonotone on $U$, whenever $F$ is strictly Gâteaux differentiable and $g$ is locally Lipschitz with $\partial^C g$ s-submonotone on an open set containing $F(U)$.

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# PARTIE V

# Optimisation multi-critère

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**Article [16]**

"Connectedness of the Efficient Set for Three Objective Quasiconcave Maximization Problems"
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# Connectedness of the Efficient Set for Three-Objective Quasiconcave Maximization Problems[1]

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**Abstract.** For three-objective maximization problems involving continuous, semistrictly quasiconcave functions over a compact convex set, it is shown that the set of efficient solutions is connected. With that, an open problem stated by Choo, Schaible, and Chew in 1985 is solved.

**Key Words.** Multi-objective maximization, semistrictly quasiconcave functions, efficient solution set, connectedness.

## 1. Introduction

Consider the multi-objective maximization problem

$$\max_{x \in S} F(x) = (f_1(x), \ldots, f_n(x)), \tag{1}$$

where $S \subseteq R^k$ is nonempty, compact, and convex and the functions $f_i : S \to R$ are continuous. It is well known that the set of efficient (Pareto optimal) solutions $E$ is connected if all the functions $f_i$ are concave (Ref. 1). This topological property of $E$ is algorithmically important.

Several authors have tried to relax the concavity assumption without giving up connectedness of the efficient set, usually working with particular kinds of quasiconcave functions (e.g., Refs. 2–7).

If $n=1$, then $E$ coincides with the set of optimal solutions of (1) which is convex, hence connected, as long as $f_1$ is quasiconcave. However for $n=2$, quasiconcavity of $f_1$ and $f_2$ does not guarantee connectedness of $E$ in general (Ref. 3). It turns out that semistrict quasiconcavity [previously called

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strict quasiconcavity (Ref. 8)] is sufficient to ensure connectedness of $E$ (Ref. 3).

The case $n=3$ is treated in Ref. 4. With proof techniques different from those in Ref. 3, it is shown that the closure of the efficient frontier $F(E)$ is connected if $f_1, f_2, f_3$ are semistrictly quasiconcave. However, connectedness of $E$ could not be established under these assumptions, nor a counterexample be given in Ref. 4. It has remained an open problem in the literature until now (Ref. 7). Only partial answers for particular kinds of generalized concave functions could be proved, using various approaches (e.g., Ref. 2, Refs. 5–7, and the references therein). These preliminary results seem to point toward a positive answer to the conjecture that $E$ is connected if $f_1, \ldots, f_n$ are semistrictly quasiconcave for any number of functions.

In this paper, we will show that the conjecture is true for $n=3$; i.e., $E$ is connected if $f_1, f_2, f_3$ are semistrictly quasiconcave. Though the approach differs from the one in Ref. 4, the paper borrows heavily from the work in Ref. 4 in addition to that in Ref. 3.

## 2. Notation and Relevant Results

Let $x=(x_1, x_2, \ldots, x_k)$ and $y=(y_1, y_2, \ldots, y_k)$ be two vectors in $R^k$. We denote by $[x, y]$ the line segment $\{tx+(1-t)y: t\in[0, 1]\}$, and we define the open line segment $(x, y)$ analogously. We write

$$x\leq y, \qquad \text{if } x_i\leq y_i, \text{ for all } i=1, 2, \ldots, k.$$

We also write

$$x\lneq y, \qquad \text{if } x\leq y \text{ and } x\neq y.$$

Let $S\subset R^k$ be convex. A function $f: S\to R$ is called semistrictly quasiconcave, if $x, y\in S$ and $f(x)<f(y)$ imply $f(x)<f(z)$ for all $z\in(x, y)$. An upper semicontinuous, semistrictly quasiconcave function is quasiconcave (Ref. 8).

Given a vector-valued function $F=(f_1, f_2, \ldots, f_n)$ on $S$, a point $\bar{x}\in S$ is called efficient with respect to $F$, if $x\in S$ and $F(\bar{x})\leq F(x)$ imply $F(\bar{x})=F(x)$. The set of all efficient points is denoted by $E$. Its image $F(E)$ is called the efficient frontier. The connectedness of $E$ is related to the connectedness of $F(E)$ through the following theorem (Ref. 4, Theorem 6).

**Theorem 2.1.** If $S$ is compact and convex and $f_1, f_2, \ldots, f_n$ are continuous and quasiconcave, then $E$ is connected if and only if $F(E)$ is connected.

Throughout the paper, we shall assume that $S$ is nonempty, compact, and convex and $f_i$ are continuous and semistrictly quasiconcave. In the special case $n=2$, the efficient frontier is the graph of a continuous, strictly decreasing function. To see this, we define

$$\bar{f}_i = \max\{f_i(x): x \in S\}, \qquad i=1,2,$$

$$\underline{f}_1 = \max\{f_1(x): f_2(x) = \bar{f}_2\},$$

$$\underline{f}_2 = \max\{f_2(x): f_1(x) = \bar{f}_1\}.$$

The following lemma is a combination of Lemmas 1 and 2 in Ref. 3.

**Lemma 2.1.** Let $n=2$. If $x \in E$, then $\underline{f}_i \le f_i(x) \le \bar{f}_i$, $i=1,2$. Conversely, if for $i=1$ or $i=2$ we have $\underline{f}_i \le t \le \bar{f}_i$, then there exists $x \in E$ such that $f_i(x) = t$.

We now have the following theorem.

**Theorem 2.2.** Let $n=2$. Then, there exists a continuous, strictly decreasing function $g$ from $[\underline{f}_1, \bar{f}_1]$ onto $[\underline{f}_2, \bar{f}_2]$ such that

$$F(E) = \{(t, g(t)): t \in [\underline{f}_1, \bar{f}_1]\}. \tag{2}$$

**Proof.** By the previous lemma, for any $(t_1, t_2) \in F(E)$, we have $t_1 \in [\underline{f}_1, \bar{f}_1]$. Conversely, for any $t_1 \in [\underline{f}_1, \bar{f}_1]$, we may find $t_2 \in [\underline{f}_2, \bar{f}_2]$ such that $(t_1, t_2) \in F(E)$. This $t_2$ is uniquely determined by $t_1$; indeed, if we had $(t_1, t_2') \in F(E)$, with say $t_2 < t_2'$, then we would have $(t_1, t_2) \lneq (t_1, t_2')$, which contradicts $(t_1, t_2) \in F(E)$, Setting $t_2 = g(t_1)$, we see that (2) holds. Using again the above argument, we infer that $g$ is strictly decreasing. Applying again Lemma 2.1, this time to the second coordinate, we infer that $g$ is a map onto $[\underline{f}_2, \bar{f}_2]$. Hence, $g$ is continuous. □

Theorem 2.2 shows that, in the case $n=2$, $F(E)$ is pathwise connected and closed. Since $E$ is the inverse image of $F(E)$, $E$ is also closed, and by Theorem 2.1 it is also connected; see also Ref. 3, Theorems 2 and 3.

In what follows, we shall restrict ourselves to the case $n=3$. Besides the set $E$ of the efficient points of $S$ with respect to the vector-valued function $F=(f_1, f_2, f_3)$, we also define the vector-valued functions $F^i$ by

$$F^1 = (f_2, f_3), \qquad F^2 = (f_1, f_3), \qquad F^3 = (f_1, f_2),$$

and the corresponding sets of efficient points $E^i$.

Furthermore, for any $t \in R$, we define the sets

$$S_t^i = \{x \in S : f_i(x) \geq t\}, \qquad i = 1, 2, 3.$$

Since $f_i$ are continuous and semistrictly quasiconcave, $S_t^i$ are closed and convex. We denote by $E_t^i$ the set of the efficient points of $S_t^i$ with respect to $F^i$. Obviously, $E^i$ and $E_t^i$ are closed and connected subsets of $S$.

The proof of the following theorem is contained in the proof of Lemma 3 in Ref. 4. We refer the reader to Ref. 4 for the definition of lower semicontinuity of multifunctions.

**Theorem 2.3.** The multifunctions $t \to F(E_t^i)$, $i = 1, 2, 3$, are lower semicontinuous.

Finally, we need the following lemma.

**Lemma 2.2.** Suppose that $x \in E_t^i$ and $x \notin E^i$. Then, $x \in E$ and $f_i(x) = t$.

**Proof.** The fact that $x \in E$ is the content of Theorem 7 in Ref. 4. To prove that $f_i(x) = t$, we may suppose that $i = 1$. Since $x \in E_t^1$, we have $f_1(x) \geq t$. Let us suppose that $f_1(x) > t$. Since $x \notin E^1$, there exists $y \in S$ such that $F^1(x) \lneqq F^1(y)$. By semistrict quasiconcavity (and quasiconcavity) of $f_2$ and $f_3$, we have

$$F^1(x) \lneqq F^1(z), \qquad \text{for any } z \in (x, y).$$

Since $f_1(x) > t$, taking $z \in (x, y)$ close enough to $x$, we would have $f_1(z) > t$, i.e., $z \in S_t^1$, which contradicts the fact that $x \in E_t^1$. □

## 3. Main Result

The sets $E^i$, $i = 1, 2, 3$, are closed subsets of $S$, hence compact. We define

$$t_i = \max\{f_i(x) : x \in E^i\}. \tag{3}$$

The outline of the proof is as follows. We first show that the sets $F(E \cap S_{t_i}^i)$, $i = 1, 2, 3$, are connected. Then we prove that any two of them have a nonempty intersection, hence their union is connected. Finally, we show that any other point in $F(E)$ can be joined to a point in these sets by a continuous curve lying inside $F(E)$.

We begin with a useful lemma.

**Lemma 3.1.** For all $i = 1, 2, 3$, we have $E \cap S_{t_i}^i = \bigcup_{t \geq t_i} E_t^i$.

**Proof.** We may suppose that $i=1$. Let $x \in E \cap S^1_{t_1}$. Then, $f_1(x) \geq t_1$. Obviously, since $x \in E$, $x$ is an efficient point of $S^1_{f_1(x)}$ with respect to $F^1$. Hence,

$$x \in \bigcup_{t \geq t_1} E^1_t.$$

Conversely, let

$$x \in \bigcup_{t \geq t_1} E^1_t, \qquad \text{i.e., } x \in E^1_t, \text{ for some } t \geq t_1.$$

If $x \notin E^1$, then $x \in E$, as follows directly from Lemma 2.2. If on the other hand $x \in E^1$, then for any $y \in S$ with $F(x) \leq F(y)$, we have in particular $F^1(x) \leq F^1(y)$, hence $F^1(x) = F^1(y)$. It follows that $y \in E^1$, hence $f_1(y) \leq t_1 \leq f_1(x)$, which shows that $F(x) = F(y)$, i.e., $x \in E$. □

The next lemma is an easy application of previous results.

**Lemma 3.2.** For all $i = 1, 2, 3$, the set $F(E \cap S^i_{t_i})$ is connected.

**Proof.** By the previous lemma, we have

$$F(E \cap S^i_{t_i}) = F\left(\bigcup_{t \geq t_i} E^i_t\right).$$

Since the sets $E^i_t$ are connected, so are their images $F(E^i_t)$ under the continuous function $F$. By Theorem 2.3, the multifunctions $t \rightarrow F(E^i_t)$ are lower semicontinuous, so applying Corollary 2.1 of Ref. 4, we conclude that the set

$$\bigcup_{t \geq t_i} F(E^i_t) = F\left(\bigcup_{t \geq t_i} E^i_t\right) = F(E \cap S^i_{t_i})$$

is connected for all $i = 1, 2, 3$. □

We proceed by showing the following lemma.

**Lemma 3.3.** For $i \neq j$, we have $E \cap S^i_{t_i} \cap S^j_{t_j} \neq \varnothing$.

**Proof.** We may suppose that $i=1$ and $j=2$. By the definition of $t_1$, there exists $x \in E^1$ such that $f_1(x) = t_1$. Obviously, $x \in E \cap S^1_{t_1}$. Likewise, there exists $y \in E^2$ such that $f_2(y) = t_2$ and $y \in E \cap S^2_{t_2}$. If $x \in S^2_{t_2}$, then $E \cap S^1_{t_1} \cap S^2_{t_2} \neq \varnothing$. Otherwise, $f_2(x) < t_2 = f_2(y)$. Since $x \in E^1$, we have $f_3(x) > f_3(y)$. This, together with the fact that $y \in E^2$, implies that $f_1(x) < f_1(y)$.

Since $f_1(x)=t_1$, we conclude that $y\in S^1_{t_1}$, and so again

$$E\cap S^1_{t_1}\cap S^2_{t_2}\neq\varnothing. \qquad \square$$

We finally state our last crucial lemma.

**Lemma 3.4.** Let $x\in E$ be such that $f_i(x)<t_i$, $i=1,2,3$. Then, there exists a point $y\in E\cap S^3_{t_3}$ and a continuous curve with values in $F(E)$, joining $F(x)$ and $F(y)$.

**Proof.** Let $F(x)=(a,b,c)$. We set further

$$\tilde{f}_1=\max\{f_1(x'): x'\in S^3_{t_3}\},$$

$$\tilde{f}_2=\max\{f_2(x'): x'\in S^3_{t_3}\},$$

$$\underset{\sim}{f}_1=\max\{f_1(x'): x'\in S^3_{t_3} \text{ and } f_2(x')=\tilde{f}_2\}.$$

By Lemma 3.3, we have $S^3_{t_3}\cap S^1_{t_1}\neq\varnothing$. For any $x'\in S^3_{t_3}\cap S^1_{t_1}$, we obviously get $t_1\leq f_1(x')\leq\tilde{f}_1$. Since $a=f_1(x)<t_1$, we deduce that $a<\tilde{f}_1$. Similarly, from the fact that $S^3_{t_3}\cap S^2_{t_2}\neq\varnothing$, we deduce that $b<\tilde{f}_2$.

Let $z\in S^3_{t_3}$ be such that $f_2(z)=\tilde{f}_2$ and $f_1(z)=\underset{\sim}{f}_1$. We then have $f_2(x)<f_2(z)$ and $f_3(x)<t_3\leq f_3(z)$. Since $x\in E$, we get $f_1(x)>f_1(z)=\underset{\sim}{f}_1$. Hence,

$$\underset{\sim}{f}_1<a<\tilde{f}_1.$$

Applying now Lemma 2.1 to the set $S^3_{t_3}$ and the function $F^3$, we infer that there exists $y\in E^3_{t_3}$, such that $f_1(y)=a$. So, we have $f_1(y)=f_1(x)$ and $f_3(y)\geq t_3>f_3(x)$; recalling that $x\in E$, we get $f_2(y)<f_2(x)$. Consequently, $F^3(y)\gneqq F^3(x)$, so $y\notin E^3$. Then Lemma 2.2 implies that $y\in E$ and $f_3(y)=t_3$.

We now show that $F(x)$ and $F(y)$ can be joined by a continuous curve lying in $F(E)$. We note that $x,y\in E\cap S^1_a$, hence in particular $x,y\in E^1_a$. Applying Theorem 2.2 to the set $S^1_a$ and the vector-valued function $F^1$, we infer that $F^1(x), F^1(y)$ belong to the graph of a continuous strictly decreasing function. Since $f_2(y)<f_2(x)$, this means that there exists a continuous strictly decreasing function $g$, defined on $[b',b]$, where $b'=f_2(y)$, such that $(t,g(t))\in F^1(E^1_a)$, for every $t\in[b',b]$, with $F^1(y)=(b',g(b'))$ and $F^1(x)=(b,g(b))$. For each $b'<t<b$, let $x_t\in E^1_a$ be such that $F^1(x_t)=(t,g(t))$. Then,

$$f_2(x_t)=t<b=f_2(x)\leq\tilde{f}_2=f_2(z).$$

Similarly, we have

$$f_3(x_t)=g(t)<g(b')=f_3(y)=t_3\leq f_3(z).$$

Hence, $F^1(x_t)<F^1(z)$, i.e., $x_t\notin E^1$. Invoking now Lemma 2.2, we get $x_t\in E$ and $f_1(x_t)=a$. Consequently, the curve $[b',b]\ni t\to F(x_t)=(a,t,g(t))$ is

continuous, lies in $F(E)$, and joins $F(x)$ to an element of $F(E \cap S^3_{t_3})$, as asserted. □

**Theorem 3.1.** The set $E$ is connected.

**Proof.** By Lemmas 3.2 and 3.3, the set $F(E \cap (S^1_{t_1} \cup S^2_{t_2} \cup S^3_{t_3}))$ is connected. Furthermore, by Lemma 3.4, for any $x \in E \backslash (S^1_{t_1} \cup S^2_{t_2} \cup S^3_{t_3})$, there exists a continuous curve, lying in $F(E)$ and joining $F(x)$ to a point in $F(E \cap (S^1_{t_1} \cup S^2_{t_2} \cup S^3_{t_3}))$. Hence, $F(E)$ is connected. Finally, recalling Theorem 2.1, we infer that $E$ is connected. □

## 4. Conclusions

The paper solves an open problem in the literature of generalized concave multi-objective maximization (Ref. 4). It is shown that in the three-objective case, semistrict quasiconcavity of all three functions guarantees the connectedness of the efficient set $E$. In Ref. 4, it could only be shown that the closure of the efficient frontier $F(E)$ is connected.

In the case of two functions, semistrict quasiconcavity guarantees connectedness of $E$ as well; it is conjectured that, for any number of functions, semistrict quasiconcavity ensures connectedness of $E$. Unfortunately, some of the lemmas in this paper are not easily extendable to more than three functions. A new approach seems to be needed to deal with the general case.

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# Arrow-Barankin-Blackwell Theorems and Related Results in Cone Duality: A Survey

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**Abstract.** We attempt a brief survey on the cone duality and on the density theorems of Arrow-Barankin & Blackwell's type. Concerning the latter aspect we show the equivalence of two recent and ostensibly different results. We follow a unified approach which provides in particular a simple way of surveying these results and their proofs.

**Key words:** Vector optimization, efficient points, positive functionals, denting points.

## 1 Introduction

In 1953, Arrow, Barankin and Blackwell stated an interesting density result in multicriteria optimization (see [1]) concerning the approximation of the Pareto efficient points of a compact convex subset of $R^n$ by points that are maximizers of some strictly positive functional on this set. This theorem was extended to cover more general notions of efficiency that are defined via an abstract cone, see [2], [19] and was subsequently generalized to an infinite dimensional setting, involving either weakly or norm compact sets.

In this article we endeavour a survey on these density results of Arrow, Barankin and Blackwell's type. Our aim is to survey the state of the art and to set in detail the relations among ostensibly different results. To this end, we shall adopt a unified approach available nowadays and, in doing so, we shall slightly improve some norm approximation results concerning weakly compact subsets of a Banach space. Finally we shall show the equivalence of a recent result of Gong [16] with a well-known earlier one of Petschke [32].

## 2 Notation

Throughout this paper, $X$ will always be a Banach space and $X^*$ its (topological) dual. However for most of what follows this is not essential and one can also consider a more general setting (for instance that of a locally convex space). In the sequel, we shall focus our interest in the norm and the weak topology of $X$, which will be denoted respectively by $\|\cdot\|$-topology and $w$-topology.

If $\varepsilon > 0$ and $x \in X$, we denote by $B_\varepsilon(x)$ the closed ball centered at $x$ with radious $\varepsilon$. For any $x, y \in X$, we define by $[x, y]$ the closed segment $\{tx + (1-t)y : 0 \le t \le 1\}$, while the segments $(x, y)$, $(x, y]$ and $[x, y)$ are defined analogously. For any subset $A \subseteq X$, we denote by $int(A)$ the norm interior of the set $A$, by $cl(A)$ (resp. $w - cl(A)$) the norm (resp. the weak) closure of $A$ and by $co(A)$ its convex hull. It is well known that for convex subsets of $X$ the norm and the weak topological closures coincide (see [7] e.g.).

Let now $K$ be a nonempty subset of $X$. A point $x_0 \in K$ is said to belong to the *algebraic interior* $\ell gint(K)$ of the set $K$, if for every $y \in X$, the intersection of the set $K$ with the line joining $x_0$ and $y$, contains an open interval around the point $x_0$. It is easily seen that $int(K) \subseteq a\ell gint(K)$. Moreover if $x_0 \in a\ell gint(K)$, then one has $\bigcup_{\lambda>0} \lambda(K - \{x_0\}) = X$. If $K$ is closed and convex, then using Baire's theorem one can deduce from the latter relation that $intK \neq \emptyset$ and $int(K) = a\ell gint(K)$.

We further recall the definition of a *quasi-relative interior* point, see [5, Def. 2.3], or *inner* point, according to the terminology used in [18].

**Definition 1.** Let $K$ be a nonempty closed convex subset of $X$ and let $x_0 \in K$. The point $x_0$ is called a quasi-relative interior (or inner) point of the set $K$, if the set $cl(\bigcup_{\lambda>0} \lambda(K - \{x_0\}))$ is a subspace of $X$.

We shall keep the simple term 'inner point' in order to refer to this notion. We further denote by $innK$ the set of all inner points of $K$. The following proposition (see [5, Prop. 2.16]) reveals an interesting and characteristic property of these points. This property was actually used as the definition of inner points in [18].

**Proposition 1.** *Let $K$ be a nonempty closed convex subset of $X$. Then $x_0 \in innK$ if and only if $x_0$ is a nonsupport point of $K$, in the sense that the following implication is true for every $x^* \in X^*$:*

$$(x^*, x - x_0) \le 0, \forall x \in K \Longrightarrow (x^*, x - x_0) = 0, \forall x \in K \tag{1}$$

It is easy to see that $intK \subseteq a\ell gintK \subseteq innK$. If $K$ is closed and convex, each of the previous inequalities becomes equality whenever the smaller set is nonempty. We further recall from [18, Prop. 2.1] the following proposition:

**Proposition 2.** *If $K$ is a (nonempty, closed, convex and) separable subset of $X$, then $innK \neq \emptyset$.*

Recently, inner points met important applications in variational inequality problems, see [18], [9] and [23]. In the following paragraph we shall see that this concept fits naturally also in the cone duality.

## 3 Order relations in Banach spaces

A nonempty subset $C$ of a Banach space $X$ is called a *cone,* if for every $x \in C$ the whole semiline $\{\lambda x : \lambda > 0\}$ is contained in $C$. A cone $C$ is called *pointed*, if it does not contain whole lines, or equivalently if 0 is an extreme point of $C$. We recall here that a point $x_0$ is said to be an *extreme* point for the set $A$, if $x_0 \in A$ and $x_0$ is not contained in any open segment $(x, y)$ lying in $A$. In the sequel we shall assume that the cone $C$ is always closed, convex and pointed.

It is well known (see for instance [31] or [22]) that the cone $C$ induces a partial order relation $\preceq$ on $X$ by means of the following formula:

$$x \preceq y \Leftrightarrow y - x \in C \tag{2}$$

Setting $x = 0$ in the above formula (2) we see that the cone $C$ itself corresponds to the set of nonnegative elements.

Let further $A$ be a nonempty subset of $X$. The set $A$ inherits naturally from $X$ the aforementioned order relation $\preceq$. Consequently one can consider the set $Max(A, C)$ of *maximal* (or *efficient*) points of $A$ (with respect to the cone $C$) as follows:

$$Max(A, C) = \{x_0 \in A : \{x_0\} = A \cap (x_0 + C)\} \tag{3}$$

The *dual cone* $C^*$ of $C$ is defined by

$$C^* = \{f \in X^* : f(x) \geq 0, \forall x \in C\} \tag{4}$$

The dual cone $C^*$ corresponds to the set of all *positive functionals.* It is easily seen that $C^*$ is always a nonempty closed convex cone of $X^*$.
We further denote by

$$C^\sharp = \{f \in Y^* : f(x) > 0, \forall x \in C, x \neq 0\} \tag{5}$$

the set of all *strictly positive functionals.* This set is also a cone; however in some cases it may be empty (see the example that follows Proposition 3). In fact one can show (see [5] e.g.) that $C^\sharp$ actually coincides with the set of inner points $innC^*$ of the closed convex set $C^*$, so its nonemptiness is assured if the space $X^*$ is separable (see Proposition 2 above). The importance of the strictly positive functionals stems from the fact that they are closely related to the notion of a *cone base.* The definition of the latter is recalled below:

**Definition 2.** A closed convex subset $V$ of $C$ is said to be a (cone) base, if for every $x \in C$, $x \neq 0$, there exist unique $\lambda > 0$, $b \in V$ such that $y = \lambda b$.

The existence of a cone base for a given cone $C$ is in fact equivalent to the nonemptiness of the set $C^\sharp = innC^*$, see also [22]. Indeed, if $C^\sharp \neq \emptyset$, then for any $f \in C^\sharp$ the set $\{x \in C : f(x) = 1\}$ defines a cone base on $C$. Conversely, if the cone $C$ has a base $V$, then separating $V$ from 0 (by the Hahn-Banach theorem), one immediately obtains a functional $f \in C^\sharp$.

It follows directly from Proposition 2 that if $X$ is a separable Asplund space (i.e. $X^*$ is separable), then every cone has a base. This result can be refined even further, as shows the following proposition in [5, Th. 2.19].

**Proposition 3.** *Assume that $X$ is a separable Banach space. Then every (closed, convex pointed) cone $C$ on $X$ has a base.*

The separability assumption is indispensable in the statement of Proposition 3. Indeed, without this assumption nice cones may not have a base, as shows the following example taken from [18].

**Example:**
Let $I$ be any uncountable set and $Y = \ell^2(I)$ be the Hilbert space of all square integrable (with respect to the counting measure) functions $f : I \to R$. Consider the cone $C$ of all non-negative real valued functions of $Y$. One easily sees that $C^* = C$. However this cone has no inner points, hence $C$ has no base.

We further consider the interior $intC^*$ of the cone $C^*$, which is a (possibly empty) convex cone. One obviously has $intC^* \subseteq innC^* = C^\sharp$, the equality holding whenever $intC^* \neq \emptyset$. In particular, the latter is equivalent with the existence of a bounded base for the cone $C$, as states the following proposition, see [22].

**Proposition 4.** *Let $C$ be a closed, convex, pointed cone of $Y$. The following are equivalent:*
*(i) The dual cone $C^*$ has a non-empty interior $intC^*$.*
*(ii) The cone $C$ has a bounded base $V$.*

However it is possible to have $intC^* = \emptyset$ and $innC^* \neq \emptyset$. In fact this is very often the case. To enlighten further the above situation we present below some standard examples of Banach spaces possessing a natural ordering structure.

**Examples:**
**1.** Let $X = R^n = X^*$, and $C = C^* = R^n_+$. In this case the cone has a bounded base, defined for instance by the strictly positive linear form $y = (1, 1, ..., 1) \in R^n$.

**2.** Let $X = \ell^1(N)$ be the space of all absolutely summable sequences and $C = \ell^1_+(N)$ be the corresponding cone of all nonnegative sequences of $\ell^1(N)$. One can easily see that the dual cone $C^*$ (which is the set $\ell^\infty(N)_+$ of all nonnegative bounded sequences of the dual space $X = \ell^\infty(N)$) has a nonempty interior, which coincides with the set of all positive bounded sequences. We conclude from Proposition 4 that $C$ has a bounded base.

**3.** Let $X = \ell^p(N)$, $X^* = \ell^q(N)$ where $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p, q < +\infty$. Consider the cone $C = \ell^p(N)_+$. It follows from Proposition 2 (or Proposition 3) that the dual cone $C^* = \ell^q(N)_+$ has inner points, hence $C$ has a base. However since $C^*$ has an empty interior, every base of $C$ is unbounded.

**4.** Let $X$ be the space $c_0(N)$ of all null sequences and let $C = c_0(N)_+$ be the cone of all nonnegative null sequences. Then $X^* = \ell^1(N)$ and $C^* = \ell^1(N)_+$. As in the previous case we conclude that the cone $c_0(N)_+$ has a base, but not a bounded base.

**5.** Let $X$ be the space $C([0,1])$ of the real continuous functions equipped with the topology of the uniform convergence. Since $X$ is separable, applying Proposition 3 we conclude that the cone $C([0,1])_+$ of the nonnegative valued functions has a base. However in this case $X^*$ coincides with the set $BV([0,1])$ of all regural Borel (signed) measures on $[0,1]$ and $C^*$ with the set $BV([0,1])_+$ of all regural Borel positive measures. Since the latter set has an empty interior, we conclude that the cone $C([0,1])_+$ has no bounded base.

From the previous examples it becomes clear that the existence of a cone base is a natural assumption in vector optimization, which is always fulfilled if $X$ is separable. On the other hand this is not the case for the assumption of the boundedness of the base: Among the classical Banach spaces, this condition is fulfilled only in $\ell^1(N)$ (or in general in $L^1(\mu)$) and in the finite dimensional spaces. We summarize below our main conclusions from the above discussion:

**Proposition 5.** *Let $C$ be a closed, convex, pointed cone of $X$. Then*
*(i) $C$ has a base iff $innC^* \neq \emptyset$*
*(ii) $C$ has a bounded base iff $intC^* \neq \emptyset$*
*(iii) If $X$ is separable, then $innC^* \neq \emptyset$*

## 4 Positive (or proper efficient) points. Arrow-Barankin-Blackwell Theorem.

In the sequel we shall deal with a closed, convex pointed cone $C$ with a base $V$ in a Banach space $X$. In this case one has $C^\sharp = innC^* \neq \emptyset$, hence for any subset $A \subseteq X$ one can define the set of positive points $Pos(A, C)$ of $A$ as follows:

$$Pos(A,C) = \{x_0 \in A : \exists f \in innC^*, f(x_0) = \sup f(A)\} \qquad (6)$$

where $\sup f(A)$ denotes the supremum of the functional $f$ on the set $A$. We mention here that also other (more restrictive) notions of efficiency have been defined in the literature, as for instance the notion of "superefficiency" introduced in [6], see also [17] for a survey.

It is straightforward from relations (3), (6) and the definition of $C^\sharp$ (relation (5)) that $Pos(A,C) \subseteq Max(A,C)$. However simple examples even in two-dimensionsal spaces certify that in general this inclusion is strict.

In the special case $X = R^n$ and $C = R^n_+$ relations (3) and (6) have a certain interpretation in Economics in terms of the Pareto efficient commodity bundles and the supporting system of prices. This has motivated Arrow, Barankin and Blackwell in 1953 to show the following density result [1] (see also [30] for an alternative approach).

**Theorem 1.** *Let $A$ be a compact convex subset of $R^n$ and $C = R^n_+$. Then $Pos(A,C)$ is dense in $Max(A,C)$.*

In [19] and independently in [2] the preceding theorem has been extended to cover the case of more general cones $C$ in $R^n$. Theorem 1 was also generalized to an infinite dimensional setting. The particular case of $\ell^\infty(N)$ has a certain significance in Economics involving models with an infinite horizon production, and has been studied in [33], [27], [29] and [12]. However the statement of Theorem 1 itself as a density result had an independent interest and generated pure mathematical extensions to arbitrary Banach spaces. Many authors have worked in this direction, see for instance [34], [4], [21], [10] etc.

In infinite dimensions there are two topologies that enter naturally into consideration, the weak and the norm topology. The result that follows was originally proved in [15]. Nowadays an easy and direct proof of it is available, that uses the technique of 'dilating cones' (see [20]). This technique is now classical and has already been repeated several times in density results of this kind in [35], [16], [28] and [13]; see also [14] for a more general approach in a locally convex setting. However we give here a sketch of this proof, since it will help the presentation of the forthcoming density results.

In the following statement one can consider $\Im$ to be either the norm or the weak topology of $X$.

**Theorem 2.** *Let $X$ be a Banach space, $X^*$ its dual and $\Im$ any topology of the dual system $(X, X^*)$. Let $A$ be a $\Im$-compact, convex subset of $X$ and $C$ a closed, convex, pointed cone with a base $V$. Then*

$$Max(A,C) \subseteq \overline{Pos(A,C)}^{\Im} \qquad (7)$$

*Proof.* (Sketch) Let $x_0 \in Max(A, C)$, i.e. $\{x_0\} = A \cap (x_0 + C)$.
We first observe that $C = \overline{cone}(V)$, where $\overline{cone}(V)$ denotes the closed cone generating by $V$. Moreover, it is no loss of generality to assume that the distance $d(0, V)$ of the cone base $V$ from 0, is greater than $1/2$.

*Step 1:* For every $n \geq 2$, consider the (closed, convex, pointed, based) cone $C_n = \overline{cone}(V + B_{\frac{1}{n}}(0))$. Then we obviously have $C = \bigcap_{n\geq 2} C_n$.
Note that in general $x_0$ does not remain a maximal point of $A$ for the larger cone $C_n$.

*Step 2:* For each $n \geq 2$, choose a maximal (with respect to $C_n$) point $x_n \in Max(A, C_n)$, such that $x_n \in A_n := (x_0 + C_n) \cap A$. This is always possible (see for instance [26, Cor. 3.6]), since the set $A_n$ is $\Im$-compact. Since the relation $C = \bigcap_{n\geq 2} C_n$ implies that $\{x_0\} = \bigcap_{n\geq 2} A_n$, we easily conclude that $x_n \to x_0$ in the $\Im$-topology.

*Step 3:* Since $\{x_n\} = A \cap (x_n + C_n)$ and the cone $C_n$ has a nonempty interior, there exists a functional $x^* \in C_n^*$ that supports the set $A$ at the point $x_n$. The proof now finishes by the observation that $x^*$ is actually a strictly positive functional for the original cone $C$. □

A careful investigation of the previous proof leads easily to the forthcoming corollary. We will first need the following definition.

**Definition 3.** We say that $x_0 \in A$ is a *point of continuity* of the set $A$, if the identity mapping $id : (A, w) \to (A, \|\|)$ is continuous at $x_0$.

The proof of the following corollary is straighforward. However this result will be useful in the sequel. Let us recall from the proof of Theorem 2 that for $n \geq 2$, $C_n := \overline{cone}(V + B_{\frac{1}{n}}(0))$ and $A_n := (x_0 + C_n) \cap A$.

**Corollary 1.** *Let $A$ be a $w$-compact, convex subset of $X$. Assume that $x_0 \in Max(A, C)$ and that for some $n_0 \geq 2$, $x_0$ is a point of continuity of the set $A_{n_0}$. Then $x_0 \in \overline{Pos(A, C)}^{\|\cdot\|}$.*

*Proof.* Repeating the proof of Theorem 2 we produce a sequence $(x_n)_n \subset Pos(A, C)$ that is weakly converging to $x_0$. We note that this sequence is eventually contained in $A_{n_0}$, hence in view of Definition 3, it is actually norm converging to $x_0$. □

Theorem 2 expresses simultaneously two different density results, one for the norm and one for the weak topology. However in the first case, the norm compactness assumption imposed on the convex set $A$ is very restrictive in infinite dimensions. On the other hand the approximation result that we obtain in the second case is rather weak. It is desirable to obtain a strong approximation result involving weakly compact subsets of $X$, as for example does (in a local way) Corollary 1. To this end, Jahn [21] was the first to derive a norm approximation result for weakly compact subsets, by assuming that the cone $C$ was of a 'Bishop-Phelps type'. Subsequently Petschke [32] (see also [15] for a different approach) refined Jahn's proof to conclude the same result, using - more general - any cone having a bounded base. We state below Petsche's result [32].

**Theorem 3.** *Let $A$ be a $w$-compact convex subset of $X$ and assume that $C$ has a bounded base. Then*

$$Max(A, C) \subseteq \overline{Pos(A, C)}^{\|\cdot\|} \tag{8}$$

However, as we have already discussed in the previous section, the assumption of a bounded based cone is unpleasant. Recently Gong [16] tried to deal with this inconvenience by relaxing this assumption to an apparently weaker one. Before we proceed to this result, we shall need the following definition.

**Definition 4.** Let $A$ be a closed convex subset of $X$ and $x_0 \in A$.
(i) $x_0$ is called a *denting* point of $A$, if for every $\varepsilon > 0$, we have $x_0 \notin \overline{co}(A \setminus B_\varepsilon(x_0))$, where $\overline{co}(A \setminus B_\varepsilon(x_0))$ denotes the closed convex hull of the set $(A \setminus B_\varepsilon(x_0))$.
(ii) $x_0$ is called a *strongly exposed* point of $A$ by the functional $x^* \in X^*$, if for every sequence $(x_n)_n \subset A$, the relation $x^*(x_n) \to x^*(x_0)$ implies the norm convergence of the sequence $(x_n)_n$ to $x_0$.

It follows easily from Definitions 3 and 4 that every denting point of $A$ is a point of continuity for this set. Moreover every strongly exposed point of $A$ is denting. It is worthmentioning that these last two notions coincide if $A = C$ and $x_0 = 0$, since in that case they are both equivalent to the boundedness of the cone base. This is the content of the following proposition in [22] (see also [16] for the equivalence of (ii) and (iii)).

**Proposition 6.** *The following statements are equivalent:*
*(i)* $0$ *is a strongly exposed point of the cone* $C$.
*(ii)* $0$ *is a denting point of* $C$.
*(iii)* $C$ *has a bounded base*

We are now ready to state Gong's density result, see [16].

**Theorem 4.** *Let $A$ be a $w$-compact convex subset of $X$. Assume that one of the following two conditions is fulfilled.*
*(i) Every maximal point of $A$ is denting.*
*(ii) $0$ is a point of continuity of the cone $C$, i.e.*

$$\forall \varepsilon > 0, \quad 0 \notin \overline{C \setminus B(0, \varepsilon)}^{w} \tag{9}$$

*Then the following approximation result holds:*

$$Max(A, C) \subseteq \overline{Pos(A, C)}^{\|\cdot\|} \tag{10}$$

Condition (i) of Theorem 4 is satisfied if for example $A$ is taken to be the unit ball of $\ell^p$, for $1 < p < +\infty$, see [16]. In the next section we shall see that this condition can be remplaced by a weaker one that would only require that every maximal point of $A$ is a point of continuity. However even this latter condition remains undesirable, since it imposes an a priori assumption on the set of maximal points of $A$.

On the other hand, in view of Proposition 6 and of Definition 4(i) and its subsequent comments, it follows that condition (9) holds trivially whenever the cone $C$ has a bounded base. In that sense the result of Theorem 4(ii) appears to be more general than the one in Theorem 3. In [16], the author queries (and states it as an open question) whether Theorem 4(ii) is indeed a real extension of Theorem 3. In next section we shall answer this question to the negative, by means of a characterization of the denting points of the closed convex subsets of a Banach space.

## 5 Equivalence of Petscke's and Gong's theorems.

In this section we show that if 0 is a point of continuity of a pointed cone $C$, then it is also a denting point of $C$. Consequently, it will follow that Theorems 3 and 4(ii) are equivalent.

Let $K$ be a closed convex subset of $X$ and $x_0 \in K$. As already partially seen in the previous section, every denting point is both an extreme and a point of continuity of $K$. In [24] (see also [25]) it has been proved that these two properties actually characterize denting points, in case of a closed convex and bounded subset $K$. The following proposition extends this result to the class of all closed convex subsets of $X$.

**Proposition 7.** *Let $x_0$ be a point of a closed convex subset $K$ of a Banach space. Then $x_0$ is denting if and only if $x_0$ is an extreme point and a point of continuity.*

*Proof.* Let us assume that $x_0$ is both an extreme and a point of continuity of the set $K$. Take any $R > 0$ and consider the set $K_R = \{x \in K : \|x - x_0\| \leq R\}$. Since $K_R \subseteq K$ and $x_0 \in K_R$, it follows easily that $x_0$ remains an extreme point and a point of continuity for the set $K_R$. Since the latter set is bounded, it follows from [25] that $x_0$ is a denting point of it. The following claim finishes the proof.

*Claim:* $x_0$ remains a denting point for the set $K$.

[Indeed, take any $\varepsilon > 0$. With no loss of generality we can assume that $R > \varepsilon$. Since $x_0$ is a denting point of the set $K_R$, we have $x_0 \notin \overline{co}(K_R \setminus B_\varepsilon(x_0))$, hence there exist $x^* \in X^*$ and $\alpha \in R$ such that $x^*(x_0) < \alpha < x^*(x')$, $\forall x' \in \overline{co}(K_R \setminus B_\varepsilon(x_0))$. Set $W = \{x \in X : x^*(x) < \alpha\}$ and observe that since $W$ is a half-space and $K$ is convex, we have $W \cap K \subset B_\varepsilon(x_0) \cap K$. Note now that $W \cap K$ is a neighborhood of $x_0$ for the (relative) weak topology of $K$. It now follows that $\overline{co}(K \setminus B_\varepsilon(x_0)) \subseteq K \setminus W$, hence in particular $x_0 \notin \overline{co}(K \setminus B_\varepsilon(x_0))$. The claim is proved.]

□

**Remark:** It is interesting to observe that the previous result has the following interesting restatement:

$$\forall\varepsilon > 0, x \notin \overline{co}\,(K\backslash B(x,\varepsilon)) \Leftrightarrow \forall\varepsilon > 0, x \notin co(K\backslash B(x,\varepsilon)) \text{ and } x \notin \overline{K\backslash B(x,\varepsilon)}^{w}$$

i.e. the convex and the weak topological hull of the set $(K\backslash B(x,\varepsilon))$ can be considered separately.

In the special case of a closed, convex pointed cone $C$, since the point $x_0 = 0$ is extreme, we infer the following corollary.

**Corollary 2.** *Let $C$ be a closed convex pointed cone of $X$. The following statements are equivalent:*
*(i) $0$ is a denting point of $C$*
*(ii) $0$ is a point of continuity of $C$*

The above corollary together with Proposition 6 shows in particular that Petschke's result (Theorem 3) and Gong's result (Theorem 4(ii)) are equivalent. Consequently, it remains widely open whether we can efficiently relax (or omit) the assumption of a bounded cone in Theorem 3, without giving up the norm approximation result theorem.

In the following theorem we survey the statements of Proposition 4, of Proposition 6 and of the previous corollary in the following theorem, see also [8]. The equivalence of (ii) and (iv) has also been observed in [16].

**Theorem 5.** *Let $C$ be a closed convex pointed cone of $Y$. The following statements are equivalent:*
*(i) $0$ is a strongly exposed point of $C$*
*(ii) $0$ is a denting point of $C$*
*(iii) $0$ is a point of continuity of $C$*
*(iv) $\exists\varepsilon > 0, \quad 0 \notin \overline{co}(C\backslash B(0,\varepsilon))$*
*(v) $C$ has a bounded base*
*(vi) $intC^* \neq \emptyset$*

The following proposition is a local density result which extends in particular Theorem 4(i). The essence of this result comes actually from Corollary 1. We recall that a norm is said to have the *Kadec-Klee property* ([11] eg.*)*, if the relative norm and the relative weak topologies on the unit ball $B_X$ coincide at any point of the unit sphere $S_X := \{x \in X : \|x\| = 1\}$. We also recall that every reflexive Banach space admits a Kadec-Klee renorming.

**Proposition 8.** *Let $A$ be a $w$-compact convex subset of $X$ and $x_0 \in Max(A, C)$. Consider the following conditions:*
*(i) $x_0$ is a point of continuity of the set $A$.*
*(ii) $0$ is a point of continuity of the cone $C$*
*(iii) There exists $y \in X$, such that for some $n_0 \geq 2$, $x_0$ is the farthest point of $y$ for the set $A_{n_0} := (x_0 + C_{n_0}) \cap A$, (i.e. $\|y - x_0\| \geq \|y - x\|$, for all $x \in A_{n_0}$), with respect to an equivalent norm $\|\cdot\|$ of $X$ having the Kadec-Klee property.*
*(iv) For some $n_0 \geq 2$, $x_0$ is a point of continuity of the set $A_{n_0}$.*
*If any of the conditions (i)-(iv) holds, we have*

$$x_0 \in \overline{Pos(A, C)}^{\|\cdot\|}$$

*Proof.* In virtue of the Corollary 1, it suffices to show that each of the conditions (i)-(iii) implies condition (iv).
Since $A_{n_0} \subseteq A$, it follows directly that condition (i) implies (iv).
Let us now assume that (ii) holds. Then from Theorem 5 it follows that $C$ has a bounded base $V$. Following the construction of the proof of Theorem 2, we observe that the cones $C_n$ also have a bounded base, hence applying again Theorem 5 we conclude that (iv) holds.
Let us finally assume that (iii) holds. Then $x_0$ is a boundary point of the closed ball $B_r(y)$ centered at $y$ with radius $r = \|y - x_0\|$. Since the norm $\|\cdot\|$ has the Kadec-Klee property, it follows that $x_0$ is a point of continuity of the set $B_r(y)$. Since $A_{n_0} \subseteq B_r(y)$ it follows that (iv) holds.

□

**Remark:** Since condition (ii) is equivalent to the existence of a bounded base (see Theorem 5), the above proposition gives in particular an alternative (and simpler) way to prove Theorem 3 of Petschke.

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**Analyse convexe et quasi-convexe ; Applications en optimisation**

**Résumé :** Ce document de synthèse s'articule autour de l'analyse convexe, de l'analyse quasi-convexe et des applications en optimisation. Dans le premier domaine on aborde les thèmes de la continuité, de la différentiabilité et des critères de coïncidence pour les fonctions convexes, puis la convexification des fonctions semi-continues inférieurement. Pour l'étude des fonctions quasi-convexes deux approches sont adoptées : une approche analytique, via un sous-différentiel généralisé, et une approche géométrique, basée sur les normales aux tranches. La dernière partie est consacrée à des applications à l'intégration d'opérateurs multivoques, aux inéquations variationnelles et à des problèmes d'optimisation multi-critères en dimension finie et infinie. Parmi les nouveautés de ce travail, on trouve la notion de monotonie fortement cyclique, qui caractérise le sous-différentiel d'une fonction convexe dont la restriction à son domaine est continue, la quasi-monotonie cyclique, qui est une propriété intrinsèque du sous-différentiel d'une fonction quasi-convexe avec des applications importantes en économie mathématique, et la notion de quasi-monotonie propre, qui caractérise les opérateurs pour lesquels l'inéquation variationnelle associée a toujours des solutions sur toute sous-partie convexe et faiblement compacte de leur domaine. Notons encore une nouvelle caractérisation de la propriété de Radon-Nikodym, et une extension à la dimension infinie d'un résultat de Janin concernant l'intégration d'un opérateur maximal cycliquement sous-monotone, résultat qui généralise le théorème classique de Rockafellar pour les opérateurs maximaux cycliquement monotones.

**Mots clés :** Convexité, convexité généralisée, analyse non-lisse, sous-différentiel, intégration des opérateurs multivoques, inéquation variationnelle, optimisation multi-critère, économie mathématique.

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**Convex and Quasiconvex Analysis ; Applications in Optimization**

**Abstract:** This document is a research contribution on Convex Analysis, on Generalized Convexity and on their applications in Optimization Theory. The first part deals with several fundamental questions concerning continuity, differentiability and criteria of coincidence for the class of convex functions. Convexification processes for lower semicontinuous functions are also studied. For the class of quasiconvex functions two approaches are used: an analytic approach, in the spirit of non-smooth analysis, and a geometric one, based on the notion of normal cones to sublevel sets. The second part is devoted to applications to the integration of multivalued operators, to Variational Inequality Problems and to finite and infinite dimensional multicriteria optimization problems. Among the concepts that are introduced for the first time in this work are: the notion of strong cyclic monotonicity, which characterizes the subdifferential of a convex function with a continuous restriction on its domain; the notion of cyclic quasimonotonicity, an intrinsic property of the subdifferentials of quasiconvex functions with important applications in Mathematical Economics; and the notion of proper quasimonotonicity, which characterizes the class of operators for which the associated Minty Variational Inequality problem has at least one solution on every nonempty convex and weakly compact subset of their domains. Let us finally mention a new characterization of the Radon-Nikodym property, and an extension to infinite dimensions of a result of Janin concerning the integration of the class of maximal cyclically submonotone operators, which generalizes a classical result of Rockafellar for maximal cyclically monotone operators.

**Key words:** Convexity, generalized convexity, non-smooth analysis, subdifferential, integration of multivalued operators, variational inequalities, multicriteria optimization, mathematical economics.