Cut-generating functions

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Abstract. In optimization problems such as integer programs or their relaxations, one encounters feasible regions of the form $\{x \in \mathbb{R}^n_+ : Rx \in S\}$ where R is a general real matrix and $S \subset \mathbb{R}^q$ is a specific closed set with $0 \notin S$. For example, in a relaxation of integer programs introduced in [ALWW2007], S is of the form $\mathbb{Z}^q - b$ where $b \notin \mathbb{Z}^q$. One would like to generate valid inequalities that cut off the infeasible solution $x = 0$. Formulas for such inequalities can be obtained through cut-generating functions. This paper presents a formal theory of minimal cut-generating functions and maximal S-free sets which is valid independently of the particular S. This theory relies on tools of convex analysis.

Keywords: Integer programming; Convex analysis; Separation; Generalized gauges; S-free sets

1 Introduction

1.1 The separation problem, examples

This paper deals with sets of the form

$$X = X(R, S) := \left\{x \in \mathbb{R}^n_+ \, : \, Rx \in S\right\}, \tag{1}$$

where

$$\begin{array}{l} R = [r_1, \dots, r_n] \text{ is a real } q \times n \text{ matrix}, \\ S \subset \mathbb{R}^q \text{ is a closed set with } 0 \notin S \,. \end{array} \tag{2}$$

In other words, our set X is the intersection of a closed convex cone (the nonnegative orthant) with a reverse image by a linear mapping. Since $0 \notin S$, it is not difficult to show that 0 does not lie in the closed convex hull of X.

We are interested in *separating* 0 from X: we want to generate cuts, i.e. inequalities valid for X, which we write as

$$c^\top x \geqslant 1 \,, \quad \text{for all } x \in X \,. \tag{3}$$

Geometrically, we want to generate *half-spaces* $H^+ = \left\{x \in \mathbb{R}^n \, : \, c^\top x \geqslant 1\right\}$ (note: $0 \notin H^+$) satisfying $H^+ \supset X$. This paper presents an overview of a formal theory of the functions that generate the coefficients c_j of such cuts.

Let us first give some motivation for our model (1), (2), arising in mixed integer programming. Starting from a polyhedron

$$P = \left\{(x,y) \in \mathbb{R}^n_+ \times \mathbb{R}^m \ : \ Ax + y = b \right\}$$

(nonnegativity of the y-variables can also be imposed), assume that $b \notin \mathbb{Z}^m$. Several situations have been considered in the literature.

Example 1 (An integer linear program). Suppose first that all variables must be integers: the set of interest is $P \cap \{\mathbb{Z}^n \times \mathbb{Z}^m\}$, i.e. the set of points $(x, y = b - Ax)$ such that $x \in \mathbb{Z}^n_+$ and $b - Ax \in \mathbb{Z}^m$. Our problem has the form (1), (2) if we set

$$q = n + m\,, \quad R = \begin{bmatrix} I \\ -A \end{bmatrix}, \quad S = \mathbb{Z}^n \times \mathbb{Z}^m - \begin{bmatrix} 0 \\ b \end{bmatrix}. \tag{4}$$

Since $b \notin \mathbb{Z}^m$, the above S is a closed set not containing the origin; (4) is the model considered by Gomory [G1969]. □

Example 2 (A mixed integer linear program). Consider now $P \cap \{\mathbb{R}^n \times \mathbb{Z}^m\}$: the set of interest is the set of points $(x, y = b - Ax)$ such that $x \in \mathbb{R}^n_+$ and $b - Ax \in \mathbb{Z}^m$. Then (4) is replaced by

$$q = m\,, \quad R = -A\,, \quad S = \mathbb{Z}^m - b\,,$$

which is the model considered by Andersen, Louveaux, Weismantel and Wolsey [ALWW2007]. □

We will retain from the above two examples the asymmetry between S (a very particular and highly structured set) and R (an arbitrary matrix). Keeping this in mind, we will consider that (q, S) is given and fixed, while (n, R) is instance-dependent data: our cutting problem can be viewed as *parametrized* by (n, R). A number of papers have appeared in recent years, dealing with the above problem with various special forms for S, see [ALWW2007], [DW2010], [BCCZ2010] and references therein.

1.2 Cut-generating functions and S-free sets

Let (q, S) be given and fixed. To generate cuts in the present situation, it would be convenient to have a mapping, taking instances of (1), (2) as input, and producing cuts as output. What we need for this is a function

$$\mathbb{R}^q \ni r \mapsto \rho(r) \in \mathbb{R}\,.$$

We will apply the function ρ to the columns r_j of R (an arbitrary matrix, with an arbitrary number of columns) to produce the coefficients $c_j := \rho(r_j)$ of a cut (3). In summary, we require that our ρ satisfies, for any instance X of (1),

$$x \in X \quad \Longrightarrow \quad \sum_{j=1}^n \rho(r_j) x_j \geqslant 1\,. \tag{5}$$

Such a ρ can be called a *cut-generating function* (CGF). So far, a CGF is a rather abstract object; but the (vast!) class of functions from $\mathbb{R}^q$ to $\mathbb{R}$ can be drastically reduced from the following observations.

(i) First consider in (3) a vector c' with $c'_j \leqslant c_j$ for $j = 1, \dots, n$; then $c'^\top x \leqslant c^\top x$ for any $x \geqslant 0$. If c' is a cut, it is tighter than c in the sense that it cuts a bigger portion of $\mathbb{R}^n_+$. We can impose some "minimal" character to a CGF, in order to reach some "tightness" of the resulting cuts.

(ii) Next observe that changing R to tR ($t > 0$) divides X by t; the set of cuts is just multiplied by t. Since we seek a minimal ρ, we can impose without loss of generality $\rho(tr) = t\rho(r)$, for any $r \in \mathbb{R}^q$ and $t > 0$: only *positively homogeneous* CGF's are of interest.

(iii) It can be proved that the closed convex hull of a CGF ρ is again a CGF. Moreover, if ρ is positively homogeneous, then the closed convex hull of ρ is positively homogeneous as well.

A function is *sublinear* if it is convex and positively homogeneous. The above observations show that the class of sublinear functions suffices to generate all relevant cuts; a fairly narrow class indeed, which is fundamental in convex analysis. Sublinear functions are in correspondence with closed convex sets and in our context, such a correspondence is based on the mapping $\rho \mapsto V$ defined by

$$V = V(\rho) := \left\{ r \in \mathbb{R}^q \ : \ \rho(r) \leqslant 1 \right\}. \tag{6}$$

Sublinear functions $\rho : \mathbb{R}^q \mapsto \mathbb{R}$ are convex, continuous and satisfy $\rho(0) = 0$, which implies that $V(\rho)$ in (6) is a closed convex neighborhood of 0 in $\mathbb{R}^q$. The set V turns out to be a cornerstone: via Theorem 1 below, (6) establishes a correspondence between the (sublinear) CGF's and the so-called *S-free sets*.

Definition 1 **(S-free set).** *Given a closed set $S \subset \mathbb{R}^q$ not containing the origin, a closed convex neighborhood V of $0 \in \mathbb{R}^q$ is called S-free if its interior contains no point in S:* $\operatorname{int}(V) \cap S = \emptyset$. □

Theorem 1. *Let ρ be a sublinear function from $\mathbb{R}^q$ to $\mathbb{R}$ and $V(\rho)$ the closed convex neighborhood of $0 \in \mathbb{R}^q$ defined in* (6)*. Then ρ is a* CGF *for* (1), (2) *if and only if $V(\rho)$ is S-free.*

As a result, the cut generation problem for X can alternatively be studied from a geometric point of view, involving sets V instead of functions ρ. This situation, common in convex analysis, is often very fruitful.

Definition 2 **(CGF as representation).** *Let $V \subset \mathbb{R}^q$ be a closed convex neighborhood of the origin. A* representation *of V is a (finite-valued) sublinear function ρ satisfying* (6)*. We will say that ρ* represents *V. A (sublinear) cut-generating function for* (1), (2) *is a representation of an S-free set.* □

A sublinear ρ represents a unique $V = V(\rho)$, well-defined by (6). One easily checks

$$\rho \leqslant \rho' \quad \Longrightarrow \quad V(\rho) \supset V(\rho')\,. \tag{7}$$

Hence, minimality of ρ corresponds to maximality of V. By contrast, the mapping $\rho \mapsto V(\rho)$ in (6) is many-to-one and therefore has no inverse. There is a difficulty here: a given neighborhood V may have several representations, and we are interested in the small ones.

1.3 Goals and outline of the paper

The aim of this paper is to present the main points of a formal theory of minimal cut-generating functions and maximal S-free sets which is valid independently of the particular S. This theory of cut-generating functions gathers, generalizes and synthesizes some existing results (see [BCZ2011], [DW2010], [BCCZ2010] and references therein). The complete theory is presented in an extended version of this paper [CCDLM2013]; in particular, the proofs of the results are omitted here, so the reader is referred to [CCDLM2013] to see precisely how things combine.

The paper is organized as follows. We study the mapping (6) in Section 2. We show that the pre-images of a given V (the representations of V) have a unique maximal element γ_V and a unique minimal element μ_V; in view of (i) above, the latter is *the* relevant inverse of $\rho \mapsto V(\rho)$. Then we study in Section 3 the correspondence $V \leftrightarrow \mu_V$. We show that different concepts of minimality come into play for ρ in (i). Geometrically they correspond to different concepts of maximality for V. We also show that they coincide in a number of cases.

2 Largest and smallest representations

In this section, we study the representation operation introduced in Definition 2 and its geometric counterpart. We first recall some basic definitions of convex analysis; The monograph [HL2001] (especially its Chapter C) is suggested for an elementary introduction, while textbooks [HL1993,R1970] are more complete.

2.1 Basic definitions of convex analysis

The *support function* of a set $G \subset \mathbb{R}^q$ is

$$\sigma_G(r) := \sup_{d \in G} d^\top r \,. \tag{8}$$

It is seen to be sublinear, to grow when G grows, but to remain unchanged if G is replaced by its closed convex hull: $\sigma_G = \sigma_{\overline{\mathrm{conv}}(G)}$. Conversely, any sublinear function ρ is the support function of a closed convex set, unambiguously defined by

$$G = G_\rho := \left\{ d \in \mathbb{R}^n \,:\, d^\top r \leqslant \rho(r) \text{ for all } r \in \mathbb{R}^q \right\};$$

we say that ρ supports G. Note that a sublinear function ρ is finite valued if and only if ρ is the support function of a bounded closed convex set.

Another relevant object for our purpose is the *gauge*

$$\mathbb{R}^q \ni r \mapsto \gamma_V(r) := \inf\{\lambda > 0 \,:\, r \in \lambda V\} \tag{9}$$

of our neighborhood V. In fact, results in convex analysis [HL2001, Theorem C.1.2.5 and Proposition C.3.2.4] show that γ_V

– also appears as a representation of V

– is the support function of the polar set of V defined by

$$V^\circ := \big\{d \,:\, d^\top r \leqslant 1 \text{ for all } r \in V\big\} = \big\{d \,:\, \sigma_V(d) \leqslant 1\big\}\,. \tag{10}$$

2.2 Prepolars and representations

From now on in this section, we are given a subset V of $\mathbb{R}^q$, which is a closed convex neighborhood of the origin. If G is such that $G^\circ = V$, we can say that G is a *prepolar* of V, i.e. that σ_G represents V in the sense of Definition 2. As already mentioned, V may have several representations, and there may be several G's such that $G^\circ = V$, that is, several G's may be prepolars of V. Because $(V^\circ)^\circ = V$, the standard polar V° is itself a prepolar – which is somewhat confusing – and turns out to be the largest one; or equivalently γ_V turns out to be the largest representation of V, as shown by Theorem 2 below. This theorem states furthermore that V has also a smallest prepolar, or equivalently a smallest representation; keeping (i) of Section 1 in mind, this is exactly what we want. This result is actually [BCZ2011, Theorem 1]; we give a different treatment here.

The following geometric objects turn out to be relevant:

$$\begin{cases} \widetilde{V}^\circ := \big\{d \in V^\circ \,:\, d^\top r = \sigma_V(d) = 1 \text{ for some } r \in V\big\}\,, \\ \widehat{V}^\circ := \big\{d \in V^\circ \,:\, \sigma_V(d) = 1\big\}\,. \end{cases} \tag{11}$$

For later use, we illustrate this construction with a simple example.

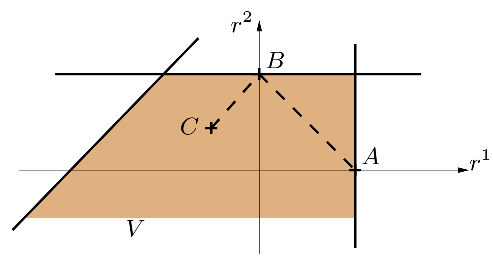

Fig. 1. Constructing $\widetilde{V}^\circ$ or $\widehat{V}^\circ$

Example 3. With $\begin{bmatrix} r^1 \\ r^2 \end{bmatrix} \in \mathbb{R}^2$, take for V the polyhedron given by the following three inequalities (see Figure 1):

$$r^1 \leqslant 1\,, \quad r^2 \leqslant 1\,, \quad r^2 \leqslant 2 + r^1\,.$$

Recalling that extreme points of V° correspond to facets of V, we see that V° has the three extreme points A, B, C defined by the equation $d^\top r = 1$, for r respectively on the three lines making up the boundary of V. We obtain $A = (1, 0)$, $B = (0, 1)$, $C = \frac{1}{2}(-1, 1)$.

In this example, $\widetilde{V}^\circ$ and $\widehat{V}^\circ$ are the same set, namely the union of the two segments $[A, B]$ and $[B, C]$. To obtain V°, convexify them with the fourth point 0; if V had a fourth constraint, say $r^2 \geqslant -1$, then this fourth point would be moved down to $D = (0, -1)$ – and would be part of the sets $\widetilde{V}^\circ$ and $\widehat{V}^\circ$. □

Because $0 \in \operatorname{int} V$, the definition (8) of a support function shows that σ_V is positive whenever it is finite: for some $\varepsilon = \varepsilon(V) > 0$,

$$\varepsilon\|d\| \leqslant \sigma_V(d) \leqslant +\infty \quad \text{for all } d \in \mathbb{R}^q . \tag{12}$$

The two sets in (11) are therefore bounded. Besides, the next proposition shows that they differ very little.

Proposition 1. *We have* $\widetilde{V}^\circ \subset \widehat{V}^\circ \subset \operatorname{cl}\big(\widetilde{V}^\circ\big)$. *It follows that* $\widehat{V}^\circ$ *and* $\widetilde{V}^\circ$ *have the same closed convex hull.*

The closed convex hull revealed by this proposition deserves a notation, as well as its support function: we set

$$V^\bullet := \overline{\operatorname{conv}}\big(\widetilde{V}^\circ\big) = \overline{\operatorname{conv}}\big(\widehat{V}^\circ\big) \quad \text{and} \quad \mu_V := \sigma_{V^\bullet} = \sigma_{\widetilde{V}^\circ} = \sigma_{\widehat{V}^\circ} \tag{13}$$

(in Figure 1, $V^\bullet$ is the triangle $\operatorname{conv}(A, B, C)$). In fact, the next result shows that μ_V is *the* smallest representation we are looking for. From now on, we assume $V \neq \mathbb{R}^q$ (otherwise $V^\bullet = \emptyset$, $\mu_V \equiv -\infty$, a degenerate situation which is trivial).

Proposition 2 (Smallest representation). *Any ρ representing V satisfies $\rho \geqslant \mu_V$. Geometrically, $V^\bullet$ is the smallest closed convex set whose support function represents V.*

Thus, V does have a smallest representation, whose supported set is $V^\bullet$. On the other hand, it is interesting to link it with V°. The intuition suggested by Figure 1 is confirmed by the following result.

Proposition 3. *Appending* 0 *to* $V^\bullet$ *gives the standard polar:*

$$\gamma_V = \max\{\mu_V, 0\} \quad \textit{i.e.} \quad V^\circ = \overline{\operatorname{conv}}\big(V^\bullet \cup \{0\}\big) = [0, 1]V^\bullet .$$

We actually have an equivalence.

Theorem 2 (Representations). *A sublinear function ρ represents V if and only if it satisfies*

$$\mu_V \leqslant \rho \leqslant \gamma_V . \tag{14}$$

Geometrically, the support function of a set G represents V if and only if G is sandwiched between the two extreme prepolars of V:

$$G^\circ = V \quad \Longleftrightarrow \quad V^\bullet \subset \overline{\operatorname{conv}}(G) \subset V^\circ .$$

3 Minimal CGF's and maximal S-free sets

3.1 Minimal CGF's

In our quest for small CGF's, the following definition is natural.

Definition 3 (Minimality). *A* CGF *ρ is called minimal if any* CGF *$\rho' \leqslant \rho$ is ρ itself.* □

A minimal CGF is certainly a smallest representation:

$$\rho \text{ is a minimal CGF} \quad \Longrightarrow \quad \rho = \mu_{V(\rho)} = \sigma_{V(\rho)^\bullet}. \tag{15}$$

(indeed, Theorem 2 states that $\mu_{V(\rho)}$ represents the same set $V(\rho)$ as ρ – and is therefore a CGF if so is ρ).

If ρ is a minimal CGF, $V(\rho)$ must of course be a special S-free set. Take for example $S = \{1\} \subset \mathbb{R}$ and the S-free set $V = [-1, +1]$; $\rho(r) := |r|$ is the smallest (because unique) representation of V but ρ is not minimal: $\rho'(r) := \max\{0, r\}$ is also a CGF, representing $V' =]-\infty, +1]$. From (7), a smaller ρ describes a larger V; so Definition 3 has its geometrical counterpart:

Definition 4 (Maximality). *An S-free set V of Definition* 1 *is called* maximal *if any S-free set $V' \supset V$ is V itself.* □

Actually, this "duality" is deceiving, as the two definitions do not match: the set represented by a minimal CGF need not be maximal. Here is a trivial example.

Example 4. When ρ is linear, the property introduced in Definition 3 holds vacuously: no sublinear function can properly lie below a linear function. Thus, any linear CGF ρ is minimal; yet, a linear ρ represents a neighborhood $V(\rho)$ (a half-space) which is S-free but has not reason to be maximal. See Figure 2: with $n = 1$, the set $V =]-\infty, 1]$ (represented by $\rho(x) = x$) is $\{2\}$-free but is obviously not maximal. □

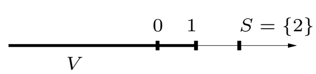

Fig. 2. A linear CGF is always maximal

Note that, if the half-space represented by a linear function is S-free, it actually separates S from 0. A simple assumption such as $0 \in \operatorname{conv} S$ will therefore rule out the above counterexample; but Example 5 below will reveal a more serious deficiency. So a subtlety is necessary, indeed the smallest representation of a maximal V enjoys a stronger property than minimality.

3.2 Strongly minimal CGF's

Let ρ be a CGF, which represents via (6) the set $V = V(\rho)$. The gauge $\gamma_{V(\rho)}$ is then a function of ρ and here comes the correct substitute to Definition 3.

Definition 5 (Strongly minimal CGF). *A CGF ρ is called strongly minimal if any CGF $\rho' \leqslant \gamma_{V(\rho)}$ satisfies $\rho' \geqslant \rho$.*

Needless to say, the class of strong minimality CGF's is a subclass of the class of minimal CGF's. Example 5 below will complement Example 4, showing that the restriction is a real one. At any rate, strong minimality turns out to be *the* appropriate definition in general:

Theorem 3 (Strongly minimal ⇔ maximal). *An S-free set V is maximal if and only if its smallest representation μ_V of (13) is a strongly minimal CGF.*

In fact, the concept of minimality involves two properties from a sublinear function:
– it must be the *smallest* representation of some V (recall (15)),
– the neighborhood V must enjoy some maximality property.
In view of the first property, a CGF can be imposed to be not only sublinear but also to support a set that is a *smallest* prepolar. Then Definition 3 has a geometric counterpart: minimality of $\rho = \mu_V = \sigma_{V^\bullet}$ means

$$\begin{array}{ccccc} G' \subset V^\bullet & \text{and } (G')^\circ \text{ is } S\text{-free} & \Longrightarrow & G' = V^\bullet, \text{ i.e. } (G')^\circ = V\,. \\ [\rho' = \sigma_{G'} \leqslant \mu_V] & [\rho' \text{ is a CGF}] & & [\rho' = \rho] \end{array}$$

Likewise for Definition 5: strong minimality of $\rho = \gamma_V = \sigma_{V^\circ}$ means

$$\begin{array}{ccccc} G' \subset V^\circ & \text{and } (G')^\circ \text{ is } S\text{-free} & \Longrightarrow & G' \supset V^\bullet, \text{ i.e. } (G')^\circ \subset V\,. \\ [\rho' = \sigma_{G'} \leqslant \gamma_V] & [\rho' \text{ is a CGF}] & & [\rho' \geqslant \rho] \end{array}$$

These observations allow some more insight into the $(\cdot)^\bullet$ operation:

Proposition 4. *Let $\rho = \mu_V = \sigma_{V^\bullet}$ be a minimal CGF. If an S-free neighborhood W satisfies $W^\bullet \subset V^\bullet$, then $W = V$.*

Thus, the trouble necessitating strong minimality lies in (7): even though the reverse implication holds when $\rho = \gamma_V$, it does not hold for $\rho = \mu_V$: the mapping $V \mapsto V^\bullet$ is not monotonic; and of course, this phenomenon is linked to the presence of the recession cone V_∞. The following example helps for a better understanding.

Example 5. In Example 3, take for S the union of the three lines given respectively by the three equations

$$r^1 = 1\,, \quad r^2 = 1\,, \quad r^2 = 2 + r^1\,,$$

so that V is clearly maximal S-free.

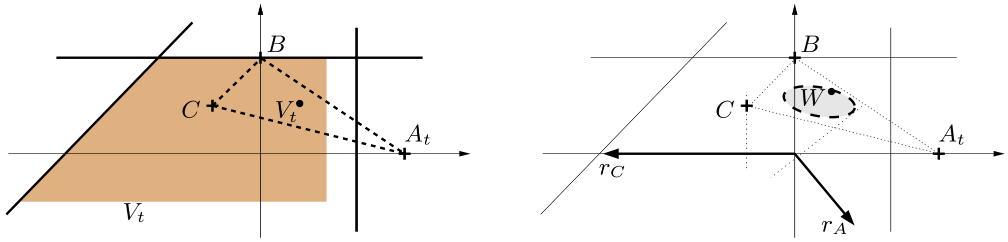

Fig. 3. The mapping $V \mapsto V^\bullet$ is not monotonic

Now shrink V to V_t (left part of Figure 3) by moving its right vertical boundary to $r^1 \leqslant 1-t$. Then A is moved to $A_t = \left(\frac{1}{1-t}, 0\right)$; there is no inclusion between the new $V_t^\bullet = \operatorname{conv}(A_t, B, C)$ and the original $V^\bullet = \operatorname{conv}(A, B, C)$; this is the key to our example.

Let us show that μ_{V_t} is minimal, even though V_t is not maximal. Take for this a CGF $\rho \leqslant \mu_{V_t}$, which represents an S-free set W; by (7), $W \supset V_t$. With the notation (13), we therefore have

$$\sigma_{W^\bullet} = \mu_W \leqslant \rho \leqslant \mu_{V_t} = \sigma_{V_t^\bullet}, \quad \text{i.e.,} \quad W^\bullet \subset V_t^\bullet$$

and we proceed to show that equality does hold, i.e. the three extreme points of $V_t^\bullet$ do lie in $W^\bullet$.

– If $A_t \notin W^\bullet$, the right part of Figure 3 shows that $W^\bullet$ is included in the open upper half-space. Knowing that

$$W = \left(W^\bullet\right)^\circ = \left\{r \,:\, d^\top r \leqslant 1 \text{ for all } d \in W^\bullet\right\}$$

(see the end of Section 2), this implies that the recession cone W_∞ has a vector of the form $r_A = (\varepsilon, -1)$ $(\varepsilon > 0)$; W cannot be S-free.

– If $C \notin W^\bullet$, there is $r_C \in \mathbb{R}^2$ such that $C^\top r_C > \sigma_{W^\bullet}(r_C) = \mu_W(r_C)$ (we denote also by C the 2-vector representing C). For example $r_C = (-2, 0) \in \operatorname{bd}(V)$ (see Figure 3), so that

$$C^\top r_C = 1 > \sigma_{W^\bullet}(-2, 0) = \mu_W(-2, 0)\,.$$

By continuity, $\mu_W(-2-\varepsilon, 0) \leqslant 1$ for $\varepsilon > 0$ small enough. Because μ_W represents W, this implies that $(-2-\varepsilon, 0) \in W$; W (which contains V_t) is not S-free.

– By the same token, we prove that $B \in W^\bullet$ (the separator $r_B = (0, 1) \in \operatorname{bd}(V)$ does the job).

We have therefore proved that $W^\bullet = V_t^\bullet$, i.e $\mu_W = \mu_{V_t}$, i.e. μ_{V_t} is minimal. □

Examples 4 and 5 show that minimality does not imply strong minimality in general. On the other hand, the following theorem provides two favorable cases when this implication holds.

Theorem 4. *Suppose* $0 \in \hat{S} := \overline{\text{conv}} S$ *and that* μ_V *is minimal. Then* μ_V *is strongly minimal under any of the following conditions:*

(i) $V_\infty \cap \hat{S}_\infty = \{0\}$ (*in particular* S *bounded*),

(ii) $V_\infty \cap \hat{S}_\infty = L \cap \hat{S}_\infty$ *where* L *stands for the lineality space of* V, *and* $\hat{S} = G + \hat{S}_\infty$ *where* G *in any nonempty bounded set.*

Theorem 4 generalizes several earlier results. The special case where S is a finite set of points in $\mathbb{Z}^q - b$ was first considered by Johnson [J1981] and more recently by Dey and Wolsey [DW2010]. Theorem 4(ii) was proven by [DW2010] and [BCCZ2010] in the special case where $S = P \cap (\mathbb{Z}^q - b)$ for some rational polyhedron P.

3.3 Asymptotically maximal sets

Finally a natural question arises: how far from being maximal are the S-free sets represented by minimal CGF's? For this, we introduce one more concept, which does not seem to have arisen in the literature on cut-generating functions.

Definition 6. *An* S*-free set* V *of Definition* 1 *is called* asymptotically maximal *if any* S*-free set* $V' \supset V$ *satisfies* $V'_\infty = V_\infty$.

Then we have a partial answer to the question about S-free sets represented by minimal CGF's.

Theorem 5 (Minimal ⇒ asymptotically maximal). *The* S*-free neighborhood represented by a minimal* CGF *is asymptotically maximal.*

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