An appropriate subdifferential for quasiconvex functions

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Abstract In this paper we introduce and study a subdifferential, that is related to the quasiconvex functions in a similar way as the Fenchel-Moreau subdifferential is related to the convex ones. It is defined for any lower semicontinuous function, through an appropriate combination of an abstract subdifferential with the normal cone to sublevel sets. We show that this "quasiconvex" subdifferential is always a cyclically quasimonotone operator that coincides with the Fenchel-Moreau subdifferential whenever the function is convex, and under mild assumptions, the density of its domain in the domain of the function is equivalent to the quasiconvexity of the function. We also show that the "quasiconvex" subdifferential of a lower semicontinuous function contains the derivatives of its differentiable quasi-affine supports. As a consequence, it contains the subdifferential introduced by Martínez-Legaz and Sach in a recent paper. Several other properties and calculus rules are also established.

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1 Introduction

In the last thirty years, several notions of subdifferential for quasiconvex functions have been proposed. The oldest ones are the Greenberg-Pierskalla

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subdifferential [6] and the tangential introduced by Crouzeix [4]. These two subdifferentials have in common that they are convex cones, and are therefore too large to give enough information on the function. The lower subdifferential of Plastria [13] is smaller but still unbounded, as are the related $\alpha-$lower subdifferentials [10]. All these subdifferentials arise in the context of some quasiconvex conjugation scheme. Of a different nature is the weak lower subdifferential [9], which is more in the spirit of nonsmooth analysis in that its support function partially coincides with the directional derivative; however this set is not quite satisfactory either, as it is even bigger than the lower subdifferential of Plastria. Trying to remedy this drawback, Martínez-Legaz and Sach [11] recently introduced the Q-subdifferential. Given that it is a subset of the Greenberg-Pierskalla subdifferential, it shares with all other quasiconvex subdifferentials the property that its nonemptiness on the domain of a lower semicontinuous function implies quasiconvexity of the function, which justifies the claim that it is a quasiconvex subdifferential; on the other hand, unlike all other subdifferentials previously introduced in quasiconvex analysis, it can be regarded as a small set, as it is contained in the Fréchet subdifferential. But this advantage is, at the same time, the main drawback of this subdifferential, as one has to impose rather strong assumptions on a quasiconvex function to ensure the nonemptiness of its Q-subdifferential on a dense subset of the domain.

In view of all these considerations, one can reasonably say that the problem of defining a sufficiently good subdifferential for quasiconvex functions is still open. To solve it, one has first to set the objectives that such a concept should meet. In this sense, we can formulate the general principle that a quasiconvex subdifferential should be related to quasiconvex functions in a similar way as the classical Fenchel-Moreau subdifferential relates to convex functions. Let us be more precise. The Fenchel-Moreau subdifferential is well defined for an arbitrary function, while, under mild conditions, its nonemptiness on a dense subset of the domain of a lower semicontinuous function is equivalent to convexity of the function. Concurrently, a quasiconvex subdifferential should be defined for arbitrary functions, but its nonemptiness on the domain of a lower semicontinuous function should be equivalent (under mild assumptions) to quasiconvexity of the function. Another desirable property of any (quasiconvex) subdifferential is that it should reduce to the Fenchel-Moreau subdifferential in the case of convex functions. As we shall prove below, the quasiconvex subdifferential introduced in this paper satisfies all these requirements. Moreover, it is smaller than all previously defined quasiconvex subdifferentials (except the Q subdifferential), as it is contained in the upper Dini subdifferential.

The new subdifferential is defined through an appropriate combination of an abstract subdifferential (in the sense of the axiomatic scheme of Aussel-Corvellec-Lassonde [2]) with geometrical considerations based on the notion of the normal cone to sublevel sets, in a way that it retains important properties from both. For instance, for the class of quasiconvex functions our subdifferential is identical (under mild conditions) to the abstract subdifferential, so that it inherits the same calculus rules; on the other hand, for any continuous function f, the existence of a non-zero element of the subdifferential at x_0 implies that f is "quasiconvex with respect to x_0", in the sense that if $x_0 = \lambda x + (1-\lambda)y$, with $0 \leq \lambda \leq 1$, then $f(x_0) \leq \max\{f(x), f(y)\}$.

The rest of the paper is organized as follows. Section 2 establishes the notation and some preliminaries related to abstract subdifferentials on which our quasiconvex subdifferential is built upon. The central part of the paper is Section 3, where the quasiconvex subdifferential is introduced and compared with other subdifferentials, and its main properties are discussed.

2 Notation and preliminaries

In the sequel, $X \neq \{0\}$ will denote a Banach space and X^* its dual. For any $x \in X$ and $x^* \in X^*$ we denote by $\langle x^*, x\rangle$ the value of x^* at x. For $x \in X$ and $\varepsilon > 0$ we denote by $B_\varepsilon(x)$ the closed ball centered at x with radius $\varepsilon > 0$, while for $x, y \in X$ we denote by $[x, y]$ the closed segment $\{tx + (1-t)y : t \in [0,1]\}$. The segments $]x, y]$, $[x, y[$ and $]x, y[$ are defined analogously.

Throughout the article we shall deal with proper functions $f : X \to \mathbb{R} \cup \{+\infty\}$ (i.e. functions for which $dom(f) := \{x \in X : f(x) < +\infty\}$ is nonempty). For any $a \in \mathbb{R}$ the sublevel (resp. strict sublevel) set of f corresponding to a is the set $S_a(f) = \{x \in X : f(x) \leq a\}$ (resp. $S_a^<(f) = \{x \in X : f(x) < a\}$). We shall use S_a and $S_a^<$ if there is no risk of confusion.

The Fenchel-Moreau subdifferential $\partial^{FM} f(x)$ of f at any $x \in \mathrm{dom}(f)$, is defined by the formula

$$\partial^{FM} f(x) := \{x^* \in X^* : f(y) \geq f(x) + \langle x^*, y - x\rangle, \forall y \in X\}. \tag{1}$$

(If $x \notin \mathrm{dom}(f)$, then we set $\partial^{FM} f(x) = \emptyset$).

Another useful subdifferential is the Greenberg-Pierskalla subdifferential $\partial^{GP} f$, given by

$$\partial^{GP} f(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \geq 0 \Rightarrow f(y) \geq f(x)\}. \tag{2}$$

Given a set $C \subseteq X$ and $x \in X$, the normal cone to C at x is by definition the cone

$$N_C(x) = \{x^* \in X^* : \forall y \in C, \langle x^*, y - x\rangle \leq 0\}.$$

Let $N_f(x) := N_{S_{f(x)}}(x)$ (resp. $N_f^<(x) := N_{S_{f(x)}^<}(x)$) be the normal cone to the sublevel (resp. strict sublevel) set corresponding to the value $f(x)$. The following equivalences are straightforward.

$$x^* \in N_f(x) \iff (\forall y \in X, \langle x^*, y - x\rangle > 0 \Rightarrow f(y) > f(x)) \tag{3}$$

$$x^* \in N_f^<(x) \iff (\forall y \in X, \langle x^*, y - x\rangle > 0 \Rightarrow f(y) \geq f(x)) \tag{4}$$

Combining the above relations it follows

$$\partial^{GP} f(x) \subseteq N_f^<(x) \text{ and } N_f(x) \subseteq N_f^<(x).$$

Besides ∂^{FM} and ∂^{GP} one can define other subdifferentials which, unlike the former ones, depend only on the local properties of the function f. Such is the Fréchet subdifferential $\partial^F f(x)$, defined by

$$\partial^F f(x) := \{x^* \in X^* : f(y) \geq f(x) + \langle x^*, y - x\rangle + o(y - x), \forall y \in X\}$$

where $o : X \to R$ is some real valued function satisfying

$$\lim_{x \to 0} \frac{o(x)}{\|x\|} = 0.$$

Another "local" subdifferential is the upper Dini subdifferential $\partial^{D^+} f$ defined as follows:

$$\partial^{D^+} f(x) = \begin{cases} \{x^* \in X^* : \langle x^*, d\rangle \leq f^{D+}(x, d), \forall d \in X\}, & \text{if } x \in dom(f) \\ \emptyset, & \text{if } x \notin dom(f) \end{cases}$$

where

$$f^{D^+}(x, d) = \limsup_{t \searrow 0^+} \frac{1}{t} (f(x + td) - f(x)). \tag{5}$$

Both the upper Dini and the Fréchet subdifferential belong to a larger class of subdifferentials defined axiomatically. We recall from [2, Definition 2.1] the relevant definition.

Definition 1 *A subdifferential ∂ is an operator that associates to any lower semicontinuous (lsc) function $f : X \to \mathbb{R} \cup \{+\infty\}$ and any $x \in X$ a subset $\partial f(x)$ of X^* so that the following properties are satisfied:*

$$\partial f(x) = \partial^{FM} f(x), \textit{ whenever } f \textit{ is convex;} \tag{P1}$$

$$0 \in \partial f(x), \text{ whenever } f \text{ has a local minimum at } x \tag{P2}$$

$$\partial(f+g)(x) \subseteq \partial f(x) + \partial g(x) \tag{P3}$$

for all convex continuous functions g for which both $\partial g(x)$ and $\partial(-g)(x)$ are nonempty. (Such functions are called ∂-differentiable at x.)

Other subdifferentials satisfying the above properties are the Gâteaux, Hadamard and Clarke-Rockafellar subdifferentials [2].

Remark 2 *Let us observe, in relation to Property (P1), that*

$$\partial^{FM} f \subseteq \partial f \tag{6}$$

for any lsc function f. Indeed, take any $x_0 \in X$ and any $x^* \in \partial^{FM} f(x_0)$. Then relation (1) guarantees that the function

$$g(x) = f(x) - \langle x^*, x - x_0 \rangle$$

has a minimum at x_0, which yields in view of (P2) that $0 \in \partial g(x_0)$. Using now Properties (P3) and (P1) we conclude

$$0 \in \partial f(x_0) + \partial(\langle -x^*, \cdot - x_0 \rangle) = \partial f(x_0) - x^*$$

i.e. $x^* \in \partial f(x_0)$.

For the purposes of the present paper we shall always use a subdifferential ∂ such that $\partial \subseteq \partial^{D^+}$.

We further recall from [2, Definition 2.2] the following definition.

Definition 3 *A norm $\|.\|$ on X is said to be ∂-smooth if the functions of the form $x \mapsto \sum_n \mu_n \|x - v_n\|^2$ are ∂-differentiable, where the sequence (v_n) converges in X, $\mu_n \geq 0$ and the series $\sum_n \mu_n$ is convergent.*

We shall always assume that the space X admits a ∂-smooth renorming. (Note that this condition is automatically satisfied if ∂ is the Clarke-Rockafellar subdifferential; also, all reflexive Banach spaces admit a ∂^F-smooth renorming). In such a case, the following Mean-Value theorem holds [2, Theorem 4.1]:

Theorem 4 *Let f be lsc and ∂ be a subdifferential. If $x, y \in X$ and $f(y) > f(x)$, then there exists $z \in [x, y[$ and sequences $(x_n) \subseteq dom f$, $(x_n^*) \subseteq X^*$, such that $x_n \to z$, $x_n^* \in \partial f(x_n)$ and*

$$\langle x_n^*, z + t(y - x) - x_n \rangle > 0, \text{ for all } t > 0.$$

In particular, $dom(\partial f)$ is dense in $dom(f)$.

Subdifferentials can be used to characterize lsc quasiconvex functions. We recall that a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is called *quasiconvex* if its sublevel sets S_α are convex subsets of X for all $\alpha \in \mathbb{R}$. In [1] it has been shown that a function f is quasiconvex if, and only if, the following property is true:

$$\text{If } x^* \in \partial f(x) \text{ and } \langle x^*, y - x\rangle > 0 \text{ then } f(z) \leq f(y) \text{ for all } z \in [x, y]. \tag{7}$$

An easy consequence of (7) is the following property of lsc quasiconvex functions (for $\partial f \subseteq \partial^{D^+} f$):

$$\text{If } x^* \in \partial f(x) \text{ and } \langle x^*, y - x\rangle > 0 \text{ then } f(y) > f(x). \tag{8}$$

Indeed, $x^* \in \partial f(x)$ and $\langle x^*, y - x\rangle > 0$ yield $f^{D^+}(x, y - x) > 0$; hence for some $t > 0$ (suitably small) we have $f(x) < f(x + t(y - x))$. From (7) it follows that $f(x + t(y - x)) \leq f(y)$ hence the result.

Let now $T : X \rightrightarrows X^*$ be a multivalued operator. Following [5] we say that T is *cyclically quasimonotone*, if for any $n \geq 1$ and any $x_1, x_2, ..., x_n \in X$, there exists $i \in \{1, 2, ..., n\}$ such that

$$\langle x_i^*, x_{i+1} - x_i\rangle \leq 0, \forall x_i^* \in T(x_i) \tag{9}$$

(where $x_{n+1} := x_1$). If we restrict n in (9) to $n = 2$, then T is called *quasimonotone*.

3 The "quasiconvex" subdifferential ∂^q

In this section we introduce the "quasiconvex" subdifferential ∂^q whose definition depends on both local and global properties of the function. We show that this subdifferential seems completely adapted in quasiconvex analysis (as far as one considers that the Fenchel-Moreau subdifferential ∂^{FM} is apt in convex analysis). In Subsection 3.1 we compare the subdifferential ∂^q with the one defined recently in [11], while in Subsection 3.2 we present some interesting properties of ∂^q.

Given an abstract subdifferential ∂ (according to Definition 1) contained in ∂^{D^+}, we introduce below the "quasiconvex" subdifferential ∂^q:

Definition 5 *The quasiconvex subdifferential $\partial^q f : X \rightrightarrows X^*$ of f is defined for all $x \in dom(f)$ as follows:*

$$\partial^q f(x) = \begin{cases} \partial f(x) \cap N_f(x), & \text{if } N_f^<(x) \neq \{0\} \\ \emptyset, & \text{if } N_f^<(x) = \{0\} \end{cases}$$

If $x \notin dom f$, then we set $\partial^q f(x) = \emptyset$.

Here are some fundamental properties of ∂^q.

Proposition 6 *For every proper function f, the operator $\partial^q f$ is cyclically quasimonotone.*

Proof It is sufficient to show that the operator N_f (relation (3)) is cyclically quasimonotone. The proof follows exactly the same pattern as the proof of quasimonotonicity of N_f in [12]. If $x_i \in X$, $i = 1, 2, \ldots n$ and $x_i^* \in N_f(x_i)$ are such that $\langle x_i^*, x_{i+1} - x_i \rangle > 0$ for all i (where $x_{n+1} \equiv x_1$) then (8) implies that $f(x_{i+1}) > f(x_i)$ for all i. By transitivity we conclude $f(x_1) > f(x_1)$, hence a contradiction. $\square$

Proposition 7 *Let f be a radially continuous function (that is the restriction of f on line segments is continuous). Then*

(i) For all $x \in dom(f)$ we have

$$\partial^q f(x) = \begin{cases} \partial f(x) \cap N_f(x), & \text{if } \partial^{GP} f(x) \neq \emptyset \\ \emptyset, & \text{if } \partial^{GP} f(x) = \emptyset \end{cases}$$

In particular for any $x \in X$, if $\partial^q f(x) \neq \emptyset$ then $\partial^{GP} f(x) \neq \emptyset$.

(ii) $\quad \partial^q f(x) \setminus \{0\} \subseteq \partial^{GP} f(x)$.

Proof (i) If $0 \in \partial^{GP} f(x)$ then $\partial^{GP} f(x) = X^*$. Hence, if $\partial^{GP} f(x) \neq \emptyset$ then $N_f^<(x) \neq \{0\}$. So we have only to prove that if $\partial^{GP} f(x) = \emptyset$ then $N_f^<(x) = \{0\}$. Note that from (4) we always have $0 \in N_f^<(x)$. Let us show that $N_f^<(x) \setminus \{0\} \subseteq \partial^{GP} f(x)$. To this end, let $x^* \in N_f^<(x) \setminus \{0\}$ and suppose that $\langle x^*, y - x \rangle \geq 0$. Choose $d \in X$ such that $\langle x^*, d \rangle > 0$. For any $t > 0$ one has $\langle x^*, y + td - x \rangle > 0$ hence $f(y + td) \geq f(x)$. Letting $t \to 0$ and using radial continuity we get $f(y) \geq f(x)$, that is $x^* \in \partial^{GP} f(x)$.

(ii) The second assertion follows from the following inclusions

$$\partial^q f(x) \setminus \{0\} \subseteq N_f(x) \setminus \{0\} \subseteq N_f^<(x) \setminus \{0\} \subseteq \partial^{GP} f(x).$$

The proof is complete. $\square$

Proposition 8 *Suppose that f is lsc and that satisfies one of the following conditions:*

(i) f is convex;

(ii) f is quasiconvex and for all $a > \inf f$ the sublevel sets $S_a(f)$ have

nonempty interior.
Then

$$\partial f = \partial^q f.$$

Proof It follows directly from Definition 5 that $\partial^q f \subseteq \partial f$. To show that equality holds consider any $x^* \in \partial f(x)$.
Suppose first that $x^* \neq 0$. Then (8) and (3) entail that $x^* \in N_f(x)$, hence $x^* \in \partial^q(x)$.
If now $x^* = 0$, then obviously $x^* \in \partial f(x) \cap N_f(x)$. According to Definition 5 it suffices to ensure that $N_f^<(x) \neq \{0\}$. Indeed, if x is a global minimum then $N_f^<(x) = X^*$. If x is not a global minimum, then f cannot be convex, hence assumption (ii) holds. It follows that the convex set $S_{f(x)}^<$ has nonempty interior. Thus by the Hahn-Banach Theorem there exists $y^* \in X^* \setminus \{0\}$ such that $\langle y^*, x\rangle \geq \langle y^*, x'\rangle$ for all $x' \in S_{f(x)}^<$. We now conclude that $y^* \in N_f^<(x)$, i.e. $N_f^<(x) \neq \{0\}$. □

Remark. The same proof shows that Proposition 8 (ii) holds without any assumption on the sublevel sets, in case that X is finite-dimensional.

Note that if f is lsc quasiconvex and radially continuous, then S_a has a nonempty interior for all $a > \inf f$. This is a direct consequence of the following proposition.

Proposition 9 *If f is quasiconvex, lsc and radially continuous then it is continuous.*

Proof Since f is lsc, it suffices to show that $S_a^<$ is open. For any $x \in S_a^<$, let b be such that $f(x) < b < a$. Since f is radially continuous, for any $y \in X$ we can find $\varepsilon > 0$ such that $]x - \varepsilon y, x + \varepsilon y[\subseteq S_b$. Hence $x \in alg \operatorname{int} S_b$. For closed convex sets in Banach spaces the algebraic and the topological interior coincide ([7, pg 139] e.g.). It follows that $x \in \operatorname{int} S_b \subseteq \operatorname{int} S_a^<$. Hence $S_a^<$ is open. □

In the same spirit is the following lemma.

Lemma 10 *Let $K \subseteq X$ be closed. If* $\operatorname{algint} K \neq \emptyset$*, then $intK \neq \emptyset$.*

Proof Let $x \in \operatorname{algint} K$. Then obviously

$$\bigcup_{n \in \mathbb{N}} n(K - x) = X.$$

By Baire's lemma, there exists $n_0 \in \mathbb{N}$ such that $\operatorname{int}(n_0(K-x)) \neq \emptyset$. We conclude that $intK \neq \emptyset$. $\square$

We are now ready to state the following result.

Proposition 11 *Let f be lsc and suppose that either f is radially continuous, or $dom(f)$ is convex and S_a has nonempty interior for all $a > \inf f$.*
(i) If the set $\{x \in X : N_f^{<}(x) \neq \{0\}\}$ is dense in $dom(f)$, then f is quasiconvex.
(ii) f is quasiconvex if, and only if, the domain of $\partial^q f$ is dense in $dom(f)$.

Proof (i) To show that f is quasiconvex, it suffices to show that S_a is convex for all a with $\inf f < a < +\infty$. For this it is sufficient to show that any $x \in X \backslash S_a$ can be strictly separated from S_a by means of a closed hyperplane. By Lemma 10, both assumptions imply that $\operatorname{int} S_a \neq \emptyset$. Choose any $y \in \operatorname{int} S_a$.
Case 1: Suppose that f is radially continuous. Then the restriction of f on the line segment $[x, y]$ takes all the values between $f(x)$ and $f(y)$. Hence there exists $z \in]x, y[$ such that $a < f(z) < +\infty$. In particular $z \in dom(f)$, so (by assumption) we can find $c^* \in N_f^{<}(c) \setminus \{0\}$ where c is as close to z as we wish. Since f is lsc we may assume that $f(c) > a$ and $c \in]x, y'[$ for some $y' \in \operatorname{int} S_a$. Using now (4) we obtain

$$\langle c^*, d \rangle > 0 \Rightarrow f(c+d) \geq f(c).$$

For all $w \in S_a$ we have $\langle c^*, w - c \rangle \leq 0$ (otherwise we would have $f(w) \geq f(c) > a$). In particular, $\langle c^*, w - c \rangle \leq 0$ for all $w \in y' + B_\varepsilon(y')$ for a suitable $\varepsilon > 0$. It follows easily that $\langle c^*, y' - c \rangle < 0$, hence $\langle c^*, x - c \rangle > 0$. Summarizing,

$$\langle c^*, w \rangle \leq \langle c^*, c \rangle < \langle c^*, x \rangle, \forall w \in S_a.$$

Consequently, c^* separates strictly S_a and x.

Case 2: Suppose that $dom(f)$ is convex. If $x \notin \overline{dom}(f)$ then we can strictly separate x and $\overline{dom}(f)$ by means of a closed hyperplane. In particular, the same hyperplane strictly separates x and S_a.

If $x \in \overline{dom}(f)$ then $[y, x[\subseteq int\, dom(f)$. Since S_a is closed and $x \notin S_a$, there exists $z \in]y, x[$ such that $a < f(z) < +\infty$. As in Case 1, it now follows that x and S_a can be strictly separated.

(ii) If f is quasiconvex, then by Proposition 8 we conclude $\partial^q f = \partial f$. Hence (by Theorem 4) $dom(\partial^q f)$ is dense in $dom(f)$. Conversely, if $dom(\partial^q f)$

is dense in $dom(f)$, then the set $\left\{z \in dom(f) : N_f^<(z) \neq \{0\}\right\}$ is dense in $dom(f)$, hence by (i) the function f is quasiconvex. $\square$

Combining Proposition 8, Proposition 11 and Theorem 4, we obtain the following corollary.

Corollary 12 *Let f be a lsc radially continuous function (respectively, f is a lsc function with convex domain and its sublevel sets have nonempty interior). Then the following are equivalent:*
(i) f is quasiconvex.
(ii) $\partial^q f = \partial f$
(iii) $\partial^q f$ satisfies the conclusion of Theorem 4 (Mean Value theorem)
(iv) $dom(\partial^q f)$ is dense in $dom(f)$.

3.1 Comparison of ∂^q with other subdifferentials

We start with the following result.

Proposition 13 *For any lsc function f*

$$\partial^{FM} f \subseteq \partial^q f \subseteq \partial f. \tag{10}$$

Proof The second inclusion follows directly from Definition 5. To prove the first inclusion, consider any $x^* \in \partial^{FM} f(x)$. It is straightforward from (3) that $x^* \in N_f(x) \subseteq N_f^<(x)$. Note also that $N_f^<(x) \neq \{0\}$ (if $x^* = 0$ then (1) yields that $N_f^<(x) = X^*$). Hence (10) follows in view of Remark 2. $\square$

Remark 14 *In view of Proposition 8, the inclusion $\partial^q f \subseteq \partial f$ becomes equality if the function f is quasiconvex and continuous, while both inclusions in (10) become equalities if the function f is convex.*

We shall further compare ∂^q with the subdifferential ∂^Q introduced recently in [11, Def. 2.1]. Before recalling the definition of the latter, we provide a result concerning the representation of lsc quasiconvex functions by means of *quasiaffine* functions. We recall that a function f is called quasiaffine if it is both quasiconvex and quasiconcave. In contrast to the rest of the paper, in the next proposition we allow the functions to take the value $-\infty$.

Proposition 15 *A function* $f : X \to \mathbb{R} \cup \{+\infty, -\infty\}$ *is lsc quasiconvex if, and only if, it satisfies*

$$f(x) = \sup_{q \in Q} q(x)$$

where Q *is the set of continuous quasiaffine minorants* $q : X \to \mathbb{R} \cup \{+\infty, -\infty\}$ *that are differentiable on* $q^{-1}(\mathbb{R})$.

Proof The "if" part of the statement is obvious, since all continuous quasiaffine functions are lsc quasiconvex, and this class is closed under pointwise suprema. To prove the "only if" part, let $f : X \to \mathbb{R} \cup \{+\infty, -\infty\}$ be lsc quasiconvex and define $g : X \to \mathbb{R} \cup \{+\infty\}$ by $g(x) = e^{f(x)}$ (using the conventions $e^{+\infty} = +\infty$ and $e^{-\infty} = 0$). It follows that g is quasiconvex and nonnegative. Combining [8, Theorem 5.15] with implication (ii)⇒(i) in [8, Theorem 5.1], we conclude that g is the pointwise supremum of the collection of its real valued, differentiable quasiaffine minorants with bounded derivatives. It follows that g is also the supremum of a collection of continuous nonnegative quasiaffine functions, which are differentiable at all points where their value is positive. Let us observe that $f(x) = \ln g(x)$ (with the conventions $\ln 0 = -\infty$ and $\ln +\infty = +\infty$), and that the logarithmic function

$$\ln : [0, +\infty] \to \mathbb{R} \cup \{+\infty, -\infty\}$$

is continuous, differentiable on $]0, +\infty[$ and increasing. The proposition follows from the observation that the composition $q = \ln \circ r$ of ln with a continuous quasiaffine function r which is differentiable at all points x such that $r(x) \in]0, +\infty[$ yields a continuous quasiaffine function q differentiable on $q^{-1}(\mathbb{R})$. □

Given a lsc function $f : X \to \mathbb{R} \cup \{+\infty\}$ let us recall the definition of the subdifferential $\partial^Q f$ given in [11].
The subdifferential $\partial^Q f(x)$ of f at $x \in dom(f)$ is the set of all $x^* \in X^*$ such that for some non-decreasing differentiable function $\varphi : \mathbb{R} \to \mathbb{R}$ (depending on x^*) with $\varphi(0) = 0$ and $\varphi'(0) = 1$ the following relation holds:

$$f(y) \geq f(x) + \varphi(\langle x^*, y - x \rangle), \text{ for all } y \in X. \tag{11}$$

Let us observe that the right part of the above inequality defines a differentiable quasiaffine support function of f at x (i.e. a differentiable quasiaffine function g satisfying $f \geq g$ and $f(x) = g(x)$). Therefore $\partial^Q f(x)$ is contained in the set of the derivatives at x of the differentiable quasiaffine supports of f at x.

Proposition 16 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be lsc and suppose that $\partial^F f \subseteq \partial f$.*
(i) If x^* is the derivative of a continuous quasiaffine support of f at x differentiable at x, then $x^* \in \partial^q f(x)$.
(ii) $\partial^Q f(x) \subseteq \partial^q f(x)$.

Proof (i) From Theorem 2.31 of [8] it follows that a continuous function $h : X \to \mathbb{R}$ is quasiaffine if, and only if, there exist $y^* \in X^*$ and a nondecreasing continuous function $\psi : \mathbb{R} \to \mathbb{R}$ such that $h = \psi \circ y^*$. Thus, if h is a quasiaffine support of f at x and x^* is the derivative of h at x, then $x^* = \psi'(\langle y^*, x\rangle)y^*$. Since h is a support of f at x, we have obviously $x^* \in \partial^F f(x)$ thus $x^* \in \partial f(x)$.
Let us first assume that $x^* \neq 0$. Let $y \in X$ be such that $\langle x^*, y - x\rangle > 0$. Since $x^* \in \partial f(x)$ and h is quasiconvex, using (8) we conclude $f(y) \geq h(y) > h(x) = f(x)$. Thus $y \notin S^<_{f(x)}(f)$, which proves that $x^* \in N_f(x) \subseteq N^<_f(x)$. Hence $x^* \in \partial f(x) \cap N_f(x) = \partial^q f(x)$.
Suppose now that $x^* = 0$. Then obviously $x^* \in \partial f(x) \cap N_f(x)$, hence it suffices to show that $N^<_f(x) \neq \{0\}$. This certainly holds if x is a global minimum of f. If this is not the case, then $y^* \neq 0$. Let us prove that, in this case, $y^* \in N^<_f(x)$. Indeed, for $y \in S^<_{f(x)}(f)$ one has $\psi(\langle y^*, y\rangle) \leq f(y) < f(x) = \psi(\langle y^*, x\rangle)$, whence, as ψ is nondecreasing, $\langle y^*, y\rangle < \langle y^*, x\rangle$.
(ii) It follows directly from (i) and (11). □

3.2 Other properties of the subdifferential ∂^q

In this section we establish calculus rules for the quasiconvex subdifferential ∂^q. Let us first remark that inside the class of lsc quasiconvex functions whose sublevel sets have nonempty interior, the quasiconvex subdifferential ∂^q inherits calculus rules from the abstract subdifferential ∂, see Corollary 12. On the other hand, for any lsc function f, Definition 5 yields the following necessary condition for global optimality:

$$f \text{ has a global minimum at } x_0 \implies 0 \in \partial^q f(x_0). \tag{12}$$

Remark: Thanks to Proposition 8, relation (12) holds true also for local minima, whenever f is lsc quasiconvex and for all $a > \inf f$ the sublevel sets $S_a(f)$ have nonempty interior.

Let us further show a calculus rule based on the “supremum”, operation important in Quasiconvex analysis.

Proposition 17 *Suppose that ∂ is either the upper Dini subdifferential ∂^{D^+} or the Fréchet subdifferential ∂^F. Let $\{f_i\}_{i\in I}$ be a family of lsc functions on X and set $f = \sup_{i\in I} f_i$. Then for every $x_0 \in X$*

$$\overline{\mathrm{co}}^{w^*}\left(\bigcup_{i\in I(x_0)} \partial^q f_i\,(x_0)\right) \subseteq \partial^q f\,(x_0)\,; \tag{13}$$

where $I(x_0) := \{i \in I : f_i(x_0) = f(x_0)\}$ and $\overline{\mathrm{co}}^{w^*}(K)$ denotes the w^*-closed convex hull of K.

Proof Let $x_0 \in X$. If $x_0 \notin \mathrm{dom}(f)$, then for all $i \in I(x_0)$, $f_i(x_0) = f(x_0) = +\infty$ and $\partial^q f\,(x_0) = \partial^q f_i\,(x_0) = \emptyset$. Hence we may suppose that $x_0 \in \mathrm{dom}(f)$. Let us observe that $\partial^q f\,(x_0)$ is a w^*-closed and convex subset of X^*. Thus it suffices to show that if $x^* \in \bigcup_{i\in I(x_0)} \partial^q f_i\,(x_0)$ then $x^* \in \partial^q f\,(x_0)$. To do so, let $i \in I(x_0)$ and $x^* \in \partial^q f_i\,(x_0)$. Since $\partial^q f_i\,(x_0) \neq \emptyset$, we deduce that $N^<_{f_i}(x_0) \neq \{0\}$. Using the fact that $f(x_0) = f_i(x_0)$ and $f(x) \geq f_i(x)$ for all $x \in X$, we obtain $N^<_f(x_0) \neq \{0\}$. Thus it remains to show (see Definition 5) that $x^* \in \partial^{D^+} f\,(x_0) \cap N_f\,(x_0)$ (respectively, $x^* \in \partial^F f\,(x_0) \cap N_f\,(x_0)$). But this follows easily from the fact that $N_{f_i}(x_0) \subset N_f\,(x_0)$ and $\partial^{D^+} f_i\,(x_0) \subset \partial^{D^+} f\,(x_0)$ (respectively, $\partial^F f_i\,(x_0) \subset \partial^F f\,(x_0)$). $\square$

Remark

1. Relation (13) holds true whenever ∂ is an abstract subdifferential satisfying $\partial f\,(x_0) \subset \partial g\,(x_0)$, whenever $f(x_0) = g\,(x_0)$ and $f \leq g$.

2. Equality in (13) is generally not true, even if f is the supremum of two continuous quasiconvex functions. Indeed, let

$$f_1(x) = \begin{cases} \sqrt{-x} & \text{if } x \leq 0 \\ -\sqrt{x} & \text{if } x > 0 \end{cases}$$

and $f_2 = -f_1$. Then $f(x) = \max\{f_1(x), f_2(x)\} = \sqrt{|x|}$ and $\partial^q f(0) = \mathbb{R}$, while $\partial^q f_1(0) = \partial^q f_2(0) = \emptyset$.

Let us give a special case where (13) holds with equality. Suppose that $\{f_1, f_2, ..., f_k\}$ be a finite family of locally Lipschitz quasiconvex functions on X that are regular (respectively, strongly regular) at x_0, that is $\partial^{D^+} f_i(x_0) = \partial^o f_i(x_0)$ (respectively, $\partial^F f_i(x_0) = \partial^o f_i(x_0)$), where $\partial^o f_i(x_0)$ stands for the Clarke subdifferential of f_i at x_0 ([3]). If $f = \max f_i$ and $x^* \in \partial^q f\,(x_0)$, then obviously $x^* \in \partial^o f\,(x_0)$, hence by [3, Proposition 2.3.12]

$x^* \in \operatorname{co}\left(\bigcup_{i\in I(x_0)} \partial^o f_i(x_0)\right)$. Thanks to Corollary 12 (ii) and the regularity (respectively, strong regularity) of each f_i, we infer that $\partial^o f_i(x_0) = \partial^q f_i(x_0)$, so equality in (13) follows.

A more general result is given in the following proposition.

Proposition 18 *Let $f = \max_{i\in I} f_i$, where $\{f_i\}_{i\in I}$ is a finite set of lsc quasiconvex functions such that for all $a > \inf f_i$ the sublevel sets $S_a(f_i)$ have nonempty interior, and let $x_0 \in X$. Further, let ∂ be the upper Dini subdifferential, and assume that for all $i \in I$ and $d \in X$*

$$f_i^{D^+}(x_0, d) = \sup\{\langle x^*, d\rangle : x^* \in \partial f_i(x_0)\}. \tag{14}$$

(This condition is in particular satisfied whenever f is regular, or (Pshenichnyi) quasidifferentiable at x_0 with nonempty subdifferential.) Then

$$\overline{co}^{w^*}\left(\bigcup_{i\in I(x_0)} \partial^q f_i(x_0)\right) = \partial^q f(x_0), \tag{15}$$

where $I(x_0) := \{i \in I : f_i(x_0) = f(x_0)\}$.

Proof: Thanks to Proposition 17, we have only show the right hand side inclusion "$\supseteq$". Let us suppose, towards a contradiction, that there exists $x^* \in \partial^q f(x_0) \backslash \overline{co}^{w^*}\left(\bigcup_{i\in I(x_0)} \partial^q f_i(x_0)\right)$. Then by the Hahn-Banach theorem, there exists $d \in X$ and $\varepsilon > 0$ such that for all $z^* \in \overline{co}^{w^*}\left(\bigcup_{i\in I(x_0)} \partial^q f_i(x_0)\right)$ we have $\langle x^*, d\rangle > \langle z^*, d\rangle + \varepsilon$. Since I is finite, it can be easily shown that there exists $i \in I$ such that $f^{D^+}(x_0, d) \le f_i^{D^+}(x_0, d)$. Our assumptions imply (see Proposition 8 (ii)) that $\partial f_i(x_0) = \partial^q f_i(x_0)$. Since $\partial^q f(x_0) \subseteq \partial f(x_0)$ we get $x^* \in \partial f(x_0)$, that is

$$f_i^{D^+}(x_0, d) \ge f^{D^+}(x_0, d) \ge \langle x^*, d\rangle > \langle z^*, d\rangle + \varepsilon, \text{ for all } z^* \in \partial f_i(x_0).$$

This clearly contradicts (14). $\square$

Note that whenever X is finite-dimensional, the assumption on the sublevel sets is superfluous (see the remark after Proposition 8). The following example shows that the assumption that the family is finite cannot be overcome, even if all f_i are convex and the supremum is actually a maximum at each point:

Example Let $f : \mathbb{R} \to \mathbb{R}$ be the convex function

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x + x^2 & \text{if } 0 < x \end{cases} .$$

For each $n \in \mathbb{N}$, let $g_n(x)$ be the equation of the straight line which is tangent to the graph of f at $(1/n, f(1/n))$ and $x_n \in]0, 1/n[$ be the intersection of this tangent with the x-axis. Let us define

$$f_n(x) = \begin{cases} 0 & \text{if } x \leq x_n \\ g_n(x) & \text{if } x_n < x \leq \frac{1}{n} \\ f(x) & \frac{1}{n} < x \end{cases} .$$

Then f_n is convex, $f(x) = \max\limits_{n \geq 1} f(x)$ for each $x \in \mathbb{R}$, $\partial^q f_n(0) = \{0\}$ while $\partial^q f(0) = [0, 1]$. Hence, (15) does not hold.

In the sequel we shall show that ∂^q obeys a chain rule. We start with the corresponding rule for classical subdifferentials.

Proposition 19 *Suppose that ∂ is either ∂^{D^+} or ∂^F, let $f : X \to \mathbb{R} \cup \{+\infty\}$ and suppose that $g : \mathbb{R} \cup \{+\infty\} \to \mathbb{R} \cup \{+\infty\}$ is non-decreasing.*
(i) If g is differentiable at $f(x_0)$ for some $x_0 \in dom(f)$, then

$$g'(f(x_0))\, \partial f(x_0) \subseteq \partial (g \circ f)(x_0); \tag{16}$$

(ii) If, moreover, f is convex and $g'(f(x_0)) > 0$, then (16) holds with equality.

Proof (i) Assume first that $\partial = \partial^{D^+}$. Let $a < f^{D^+}(x_0, d)$. It follows from (5) that for any $\delta > 0$ there exists $0 < t < \delta$ satisfying

$$\frac{f(x_0 + td) - f(x_0)}{t} > a.$$

Hence $f(x_0 + td) > f(x_0) + at$ and $g(f(x_0 + td)) \geq g(f(x_0) + at)$. Since g is differentiable at $f(x_0)$ it follows

$$g(f(x_0) + at) = g(f(x_0)) + g'(f(x_0))\, at + o(at)$$

where $\lim_{t\to 0}\frac{o(t)}{t}=0$. Hence

$$\frac{g\left(f\left(x_0+td\right)\right)-g\left(f\left(x_0\right)\right)}{t}\geq ag'\left(f\left(x_0\right)\right)+\frac{o\left(at\right)}{t}$$

which yields $\left(g\circ f\right)^{D^+}\left(x_0,d\right)\geq ag'\left(f\left(x_0\right)\right)$. Consequently,

$$g'\left(f\left(x_0\right)\right)f^{D^+}\left(x_0,d\right)\leq\left(g\circ f\right)^{D^+}\left(x_0,d\right)$$

hence (16) holds.

Assume now that $\partial=\partial^F$ and take any $x^*\in\partial^F f\left(x_0\right)$. Then

$$\liminf_{\|u\|\searrow 0}\frac{f\left(x_0+u\right)-f\left(x_0\right)-\langle x^*,u\rangle}{\|u\|}\geq 0.$$

Let $a<0$. Then there exists $\delta>0$ such that for all $u\in X$ with $\|u\|<\delta$

$$\frac{f\left(x_0+u\right)-f\left(x_0\right)-\langle x^*,u\rangle}{\|u\|}>a.$$

Since g is non-decreasing, the previous inequality implies

$$g\left(f\left(x_0+u\right)\right)\geq g\left(f\left(x_0\right)+\langle x^*,u\rangle+a\left\|u\right\|\right)$$

and since g is differentiable at $f\left(x_0\right)$,

$$g\left(f\left(x_0+u\right)\right)\geq g\left(f\left(x_0\right)\right)+g'\left(f\left(x_0\right)\right)\left(\langle x^*,u\rangle+a\left\|u\right\|\right)+o\left(\langle x^*,u\rangle+a\left\|u\right\|\right)$$

where $\lim_{t\to 0}\frac{o(t)}{t}=0$. Since $\left(\|x^*\|-a\right)\|u\|\geq\left|\langle x^*,u\rangle+a\left\|u\right\|\right|$ it follows that

$$\liminf_{\|u\|\searrow 0}\frac{\left(g\circ f\right)\left(x_0+u\right)-\left(g\circ f\right)\left(x_0\right)-g'\left(f\left(x_0\right)\right)\langle x^*,u\rangle}{\|u\|}\geq ag'\left(f\left(x_0\right)\right). \tag{17}$$

Since the above relation is true for all $a<0$, the left-hand side is nonnegative. This implies that $g'\left(f\left(x_0\right)\right)\ x^*\in\partial^F\left(g\circ f\right)\left(x_0\right)$, hence (16) holds.

(ii) Suppose now that f is convex. Then the function $t\to f\left(x_0+td\right)$ is right differentiable, hence the same holds also for the function $t\to\left(g\circ f\right)\left(x_0+td\right)$. It follows from the usual chain rule for differentiable functions that

$$g'\left(f\left(x_0\right)\right)f^{D^+}\left(x_0,d\right)=\left(g\circ f\right)^{D^+}\left(x_0,d\right). \tag{18}$$

Hence if $\partial=\partial^{D^+}$, (16) holds with equality.

Suppose now that $\partial = \partial^F$. It is sufficient to show that if $x^* \notin \partial^F f(x_0)$ then $g'(f(x_0))\, x^* \notin \partial^F (g \circ f)(x_0)$. Since f is convex we have $\partial^F f = \partial^{FM} f$, hence from (1) there exists $u \in X$ such that $f(x_0 + u) - f(x_0) < \langle x^*, u \rangle$. Choose $a < 0$ such that

$$f(x_0 + u) - f(x_0) < \langle x^*, u \rangle + a \|u\|. \tag{19}$$

Convexity of f guarantees that the function $t \to \frac{f(x_0+tu)-f(x_0)}{t}$ is nondecreasing for all $t \geq 0$. Thus, for any $0 < t < 1$ we infer from (19) that

$$f(x_0 + tu) - f(x_0) < (\langle x^*, u \rangle + a \|u\|)\, t.$$

Since g is nondecreasing we obtain

$$g(f(x_0 + tu)) \leq g(f(x_0) + t \langle x^*, u \rangle + ta \|u\|)$$

and since g is differentiable at $f(x_0)$

$$g(f(x_0 + tu)) \leq g(f(x_0)) + tg'(f(x_0))(\langle x^*, u \rangle + a\|u\|) + o(t \langle x^*, u \rangle + ta \|u\|)$$

where $\lim_{t \to 0} \frac{o(t)}{t} = 0$. Dividing by $t\|u\|$ and letting $t \to 0$ we deduce

$$\liminf_{t \searrow 0} \frac{(g \circ f)(x_0 + tu) - (g \circ f)(x_0) - g'(f(x_0)) \langle x^*, tu \rangle}{\|tu\|} \leq ag'(f(x_0)).$$

Since $a < 0$ and $g'(f(x_0)) > 0$ it follows that the left-hand side of (17) is negative. Hence, $g'(f(x_0))\, x^* \notin \partial^F (g \circ f)(x_0)$. $\square$

Proposition 20 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be lsc and $g : \mathbb{R} \cup \{+\infty\} \to \mathbb{R} \cup \{+\infty\}$ be non-decreasing. Assume that the subdifferential ∂ satisfies assertions (i) and (ii) of Proposition 19 (for instance $\partial = \partial^F$ or ∂^{D^+}). If g is differentiable at $f(x_0)$ with $g'(f(x_0)) > 0$ for some $x_0 \in dom(f)$, then*

$$g'(f(x_0))\, \partial^q f(x_0) \subseteq \partial^q (g \circ f)(x_0); \tag{20}$$

the above inclusion becomes equality whenever f is convex.

Proof Since g is nondecreasing and $g'(f(x_0)) > 0$ we can easily deduce that

$$N_f^<(x_0) = N_{g \circ f}^<(x_0) \tag{21}$$

and

$$N_f(x_0) = N_{g \circ f}(x_0). \tag{22}$$

Thus, if $x^* \in \partial^q f(x_0)$, then (21) yields $N_{g\circ f}^{<}(x_0) \neq \emptyset$. Since $\partial^q f \subseteq \partial f$, we infer from (16) that

$$g'(f(x_0))\, x^* \in \partial (g \circ f)(x_0).$$

Besides, since $x^* \in N_f(x_0)$ and $N_{g\circ f}(x_0)$ is a cone, (22) implies

$$g'(f(x_0))\, x^* \in N_{g\circ f}(x_0).$$

Hence (20) holds.

If now f is convex, then by Proposition 8, $\partial^q f = \partial^{FM} f = \partial f$. Hence, in order to show the equality in (20), we have to show that $\partial^q (g \circ f)(x_0) = \partial (g \circ f)(x_0)$. It suffices to show that if $x^* \in \partial (g \circ f)(x_0)$ then $x^* \in \partial^q (g \circ f)(x_0)$. Since (16) holds with equality, we have

$$\frac{x^*}{g'(f(x_0))} \in \partial f(x_0) = \partial^q f(x_0).$$

Hence, $N_{g\circ f}^{<}(x_0) = N_f^{<}(x_0) \neq \{0\}$ and (since $N_f(x_0)$ is a cone), $x^* \in N_f(x_0) = N_{g\circ f}(x_0)$. It follows that $x^* \in \partial^q (g \circ f)(x_0)$. □

Let $C \subseteq X$ and let us define the (upper Dini tangent) cone $T_{D^+}(C, x_0)$ of C at $x_0 \in C$ as follows:

$$T_{D^+}(C, x_0) = \{u \in X : \exists \delta > 0\text{: } \forall t \in]0, \delta[,\ x_0 + tu \in C\}.$$

We have the following proposition.

Proposition 21 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ and $x_0 \in f^{-1}(\mathbb{R})$. Then*

$$\begin{aligned}&\{x^* \in X^* : (x^*, -1) \in N_{\mathrm{epi}\, f}(x_0, f(x_0))\} \subseteq \partial^q f(x_0) \\ &\subseteq \{x^* \in X^* : (x^*, -1) \in (T_{D^+}(\mathrm{epi}\, f, (x_0, f(x_0))))^o\}\end{aligned}$$

Proof The first inclusion follows from (10) and the observation that

$$\partial^{FM} f(x_0) = \{x^* \in X^* : (x^*, -1) \in N_{\mathrm{epi}\, f}(x_0, f(x_0))\}.$$

To prove the second inclusion, since $\partial^q \subseteq \partial \subseteq \partial^{D^+}$ it suffices to show that

$$\partial^{D^+} f(x_0) = \{x^* \in X^* : (x^*, -1) \in (T_{D^+}(\mathrm{epi}\, f, (x_0, f(x_0))))^o\}.$$

To this end, let $x^* \in \partial^{D^+} f(x_0)$. For any $(u, v) \in T_{D^+}(\mathrm{epi}\, f, (x_0, f(x_0)))$ there exists $\delta > 0$ such that

$$f(x_0 + tu) \leq f(x_0) + tv$$

for all $t \in]0, \delta[$. It follows that

$$\langle x^*, u\rangle \leq \limsup_{t \searrow 0} \frac{f(x_0 + tu) - f(x_0)}{t} \leq v$$

i.e. $(x^*, -1) \in (T_{D^+}(\operatorname{epi} f, (x_0, f(x_0))))^o$.
Conversely let $x^* \in X^*$ be such that $(x^*, -1) \in (T_{D^+}(\operatorname{epi} f, (x_0, f(x_0))))^o$. For each $u \in X$ set $v = f^{D^+}(x_0, u)$. Then for any $\lambda \in]v, +\infty[$ we can find $\delta > 0$ such that for all $t \in]0, \delta[$

$$\frac{f(x_0 + tu) - f(x_0)}{t} \leq \lambda.$$

It follows that $(u, \lambda) \in T_{D^+}(\operatorname{epi} f, (x_0, f(x_0)))$ hence $\langle x^*, u\rangle \leq \lambda$. Since this is true for all $\lambda \in]v, +\infty[$ we deduce that $\langle x^*, u\rangle \leq v$, hence $x^* \in \partial^{D^+} f(x_0)$.
$\square$

Let us finally state the following corollary.

Corollary 22 *Let $A \subseteq X$ and denote by $\delta_A : X \to \mathbb{R} \cup \{+\infty\}$ the indicator function of A defined by*

$$\delta_A(x) = \begin{cases} 0, & \text{if } x \in A \\ +\infty, & \text{if } x \notin A. \end{cases}$$

For all $x_0 \in A$ we have

$$\partial^q \delta_A(x_0) = N_A(x_0).$$

Proof We have the following equivalences:

$$\begin{aligned} x^* \in \partial^{FM} \delta_A(x_0) &\Leftrightarrow \forall x \in X,\ \langle x^*, x - x_0\rangle \leq \delta_A(x) - \delta_A(x_0) \\ &\Leftrightarrow \forall x \in A,\ \langle x^*, x - x_0\rangle \leq 0 \Leftrightarrow x^* \in N_A(x_0). \end{aligned}$$

Hence (10) implies that $N_A(x_0) \subseteq \partial^q \delta_A(x_0)$. Conversely, if $x^* \in \partial^q \delta_A(x_0)$ then $x^* \in N_{\delta_A}(x_0)$. It is very easy to see that $N_{\delta_A}(x_0) = N_A(x_0)$ and the corollary follows. $\square$

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