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LINEAR STRUCTURE OF FUNCTIONS WITH MAXIMAL CLARKE SUBDIFFERENTIAL*

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Abstract. In this paper we establish that the set of Lipschitz functions $f : \mathcal{U} \to \mathbb{R}$ ($\mathcal{U}$ a nonempty open subset of ℓ^1_d) with maximal Clarke subdifferential contains a linear subspace of uncountable dimension (in particular, an isometric copy of $\ell^\infty(\mathbb{N})$). This result follows along a similar line to that of a previous result of Borwein and Wang (see [*Proc. Amer. Math. Soc.*, 128 (2000), pp. 3221–3229; *Bull. Aust. Math. Soc.*, 72 (2005), pp. 491–496]). However, while the latter was based on Baire's category theorem, our current approach is constructive and is not linked to uniform convergence. In particular, we establish lineability (and spaceability for the Lipschitz norm) of the above set inside the set of all Lipschitz continuous functions.

Key words. Lipschitz function, maximal Clarke subdifferential, lineability, spaceability

AMS subject classifications. 49J52, 46G05, 46B04

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1. Introduction.

Let X be a separable Banach space and $\mathcal{U}$ a nonempty open subset of X. We denote by $\overline{B}_*$ the closed unit ball of the dual space X^* and by $\|f\|_{\text{Lip}}$ the Lipschitz constant of a Lipschitz function $f : \mathcal{U} \to \mathbb{R}$ (see (2) below). We also denote by $\text{Lip}^{[k]}(\mathcal{U})$ the set of Lipschitz functions f defined on $\mathcal{U}$ of Lipschitz constant $\|f\|_{\text{Lip}} \leq k$. This space, when endowed with the metric of uniform convergence over bounded subsets of $\mathcal{U}$, is complete.

In the above setting, Borwein and Wang showed in [9, 10] that the set of Lipschitz functions with maximal Clarke subdifferential (that is, $\partial^\circ f(x) \equiv \|f\|_{\text{Lip}} \overline{B}_*$ for all $x \in \mathcal{U}$) is generic in $\text{Lip}^{[k]}(\mathcal{U})$. The result was obtained via a standard application of Baire's category theorem. However, this result depends crucially on the chosen metric, the reason being that wild functions with oscillating derivatives can be obtained as uniform limits of well-behaved ones (piecewise linear or quadratic). An explicit construction of such a wild function with maximal Clarke subdifferential is given in [8].

Therefore, in some generic sense, most Lipschitz functions are *Clarke-saturated* (see the forthcoming Definition 1), but this genericity is strongly related to the chosen topology. To further illustrate this fact, let us fix a nonempty compact subset K of $\mathcal{U}$ and let us consider $\text{Lip}^{[k]}(K)$ as a closed subset of the Banach space $(\mathcal{C}(K), \|\cdot\|_\infty)$ (a uniform limit of Lipschitz continuous functions of Lipschitz constant bounded by k is Lipschitz). Then $\|\cdot\|_\infty$-limits of piecewise polynomial functions in $\text{Lip}^{[k]}(K)$ may give rise to Lipschitz functions with maximal Clarke subdifferentials. A completely

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different behavior appears if one instead uses the Lipschitz norm (see (2) below) to describe convergence: in this case $\|\cdot\|_{\mathrm{Lip}}$-limits of polynomials are $\mathcal{C}^1$-functions (and therefore $\partial^\circ f(x) \equiv \{df(x)\}$ for all $x \in K$). The reason is that for smooth functions the Lipschitz norm $\|\cdot\|_{\mathrm{Lip}}$ coincides with the norm of uniform convergence of the derivatives, and under this norm $\mathcal{C}^1(K)$ is a Banach subspace of $\mathrm{Lip}(K)$.

If $X = \mathbb{R}^d$, then important subclasses of Lipschitz functions, such as semialgebraic (more generally, o-minimal) Lipschitz functions, or finite selections of $\mathcal{C}^d$-smooth functions have small Clarke subdifferentials: indeed, the aforementioned classes satisfy a Morse–Sard theorem for their generalized critical values (see [7, Corollary 5(ii)] and [4, Theorem 5], respectively), while every point (and consequently every value) of a Clarke-saturated Lipschitz function is critical.

In this work we complement the results of [8, 9, 10] by establishing a topology-independent result (Theorem 12(i)), namely, that the set of Clarke-saturated Lipschitz functions contains an infinite-dimensional linear space of uncountable dimension; in particular, it is *lineable*, according to the terminology of [20], and consequently algebraically large. Moreover, surprisingly, $(\mathrm{Lip}(K), \|\cdot\|_{\mathrm{Lip}})$ contains a closed nonseparable subspace of Clarke-saturated functions, and hence this set is also *spaceable*. We refer the reader to [2] for related terminology and an exposition on the state of the art of this emerging field, nowadays known as lineability and spaceability, having its origins in Vladimir Gurariy's early work on this field; see [19, 3]. We also refer the reader to [1, 15, 23] and the expository paper [6] for recent results. In some sense, our results were anticipated in [6, Page 114].

2. Preliminaries and notation. For any integer $d \geq 1$ and real $p \in [1, \infty]$, we denote by ℓ_d^p the finite-dimensional vector space $\mathbb{R}^d$ endowed with the classical p-norm. It is a well-known fact that this space is reflexive, with $(\ell_d^p)^* = \ell_d^q$, where $\frac{1}{p} + \frac{1}{q} = 1$. We denote by $\langle \cdot, \cdot \rangle : \ell_d^p \times \ell_d^q \to \mathbb{R}$ the duality mapping. When no confusion occurs, we will simply denote the norm of ℓ_d^p by $\|\cdot\|$ and of ℓ_d^q by $\|\cdot\|_*$ (the dual norm).

We denote by $\mathrm{Lip}(\mathcal{U})$, for $\mathcal{U} \subseteq \ell_d^p$, the vector space of all Lipschitz functions $f : \mathcal{U} \to \mathbb{R}$, that is, those functions for which there exists a constant $L > 0$ such that

$$|f(x) - f(y)| \leq L\|x - y\| \quad \text{for all } x, y \in \mathcal{U}. \tag{1}$$

We denote by $\|f\|_{\mathrm{Lip}}$ the infimum of the above constants, that is,

$$\|f\|_{\mathrm{Lip}} = \inf\{L > 0 \,:\, |f(x) - f(y)| \leq L\|x - y\| \text{ for all } x, y \in \mathcal{U}\}, \tag{2}$$

which in turn is equivalent to

$$\|f\|_{\mathrm{Lip}} = \sup_{x,y \in \mathcal{U},\, x \neq y} \frac{|f(x) - f(y)|}{\|x - y\|}. \tag{3}$$

It is well known that $\|\cdot\|_{\mathrm{Lip}}$ is a seminorm on $\mathrm{Lip}(\mathcal{U})$. Fixing $x_0 \in \mathcal{U}$ and considering the space $\mathrm{Lip}_{x_0}(\mathcal{U})$ of all Lipschitz functions such that $f(x_0) = 0$, the aforementioned seminorm becomes a norm, and $\mathrm{Lip}_{x_0}(\mathcal{U})$ becomes a Banach space under $\|\cdot\|_{\mathrm{Lip}}$.

Recall that every Lipschitz function is differentiable almost everywhere (Rademacher's theorem). If $\mathcal{D}_f$ stands for the set of points where f is differentiable, and $Df(x)$ for the derivative of f at $x \in \mathcal{D}_f$, then the Clarke subdifferential of f at $x \in \mathcal{U}$ is given by [12, Chapter 2]

$$\partial^\circ f(x) = \overline{\mathrm{conv}} \left\{ \lim_{x_n \to x} Df(x_n) \,:\, \{x_n\} \subseteq \mathcal{D}_f \right\}. \tag{4}$$

It follows that $\partial^\circ f(x)$ is a nonempty convex compact subset of ℓ_d^q and for every $x^* \in \partial^\circ f(x)$ it holds that $\|x^*\|_* \leq \|f\|_{\mathrm{Lip}}$. Therefore $\partial^\circ f(x) \subset \|f\|_{\mathrm{Lip}} \overline{B}_*$.

Definition 1 (Clarke-saturated function). *We say that $f \in \mathrm{Lip}(\mathcal{U})$ has a maximal Clarke subdifferential at $x_0 \in \mathcal{U}$ whenever $\partial^\circ f(x_0) = \|f\|_{\mathrm{Lip}} \overline{B}_*$, that is, the Clarke subdifferential equals the closed ball of ℓ_d^q centered at 0 and with radius $\|f\|_{\mathrm{Lip}}$. If this is valid for every $x \in \mathcal{U}$, we say that f is Clarke-saturated.*

The first example of a Clarke-saturated Lipschitz function in one dimension was given (up to an obvious modification) by Lebourg in [22, Proposition 1.9]. The function was given by an explicit formula based on a *splitting* subset A of $\mathbb{R}$ with respect to the family of nontrivial intervals of $\mathbb{R}$, that is, a measurable subset A satisfying

$$0 < \lambda(A \cap I) < \lambda(I) \qquad \text{for every (nontrivial) interval } I \subset \mathbb{R}, \tag{5}$$

where λ denotes the Lebesgue measure. An explicit construction of such a splitting set can be found in [21] in a general setting (an atomless measure space). In the next section we shall extend this construction to the particular case of the real line and come up with a countable family of disjoint splitting sets. This family will be fundamental to the proof of our main result.

Let us recall that $L^\infty(\mathcal{U}; \ell_d^q) = L^1(\mathcal{U}; \ell_d^p)^*$ (see, e.g., [14, Page 98]). We shall need the following recent result about the space $\mathrm{Lip}_{x_0}(\mathcal{U})$, which relates this space to some subspace of $L^\infty(\mathcal{U}; \ell_d^q)$, the space of essentially bounded Lebesgue-measurable functions $g : \mathcal{U} \subseteq \ell_d^p \to \ell_d^q$. This result has been established independently in [16] (see also [17]) and in [13].

Theorem 2 (isometric injection of $\mathrm{Lip}_{x_0}(\mathcal{U})$ into $L^\infty(\mathcal{U}, \ell_d^q)$). *Let $\mathcal{U} \subseteq \ell_d^p$ be a nonempty open convex set and $x_0 \in \mathcal{U}$. Then, the linear operator*

$$\begin{cases} \hat{\mathrm{D}} : \mathrm{Lip}_{x_0}(\mathcal{U}) \to L^\infty(\mathcal{U}; \ell_d^q), \\ \\ \qquad \hat{\mathrm{D}} f = Df \quad a.e., \end{cases}$$

defines an isometry between $\mathrm{Lip}_{x_0}(\mathcal{U})$ and the following subspace of $L^\infty(\mathcal{U}; \ell_d^q)$:

$$\hat{\mathrm{D}}(\mathrm{Lip}_{x_0}(\mathcal{U})) = \{g \in L^\infty(\mathcal{U}; \ell_d^q) \,:\, \partial_i g_j = \partial_j g_i \text{ for every } i, j \in \{1, \ldots, n\}\}. \tag{6}$$

Here, $\partial_i g_j$ stands for the partial derivative of the jth component of g with respect to x_i in the sense of distributions. That is, if $\mathcal{C}_0^\infty(\mathcal{U})$ denotes the space of test functions (compactly supported $\mathcal{C}^\infty$-functions on $\mathcal{U}$) then (6) becomes

$$\int_{\mathcal{U}} g_j(x) \frac{\partial \varphi}{\partial x_i}(x) dx = \int_{\mathcal{U}} g_i(x) \frac{\partial \varphi}{\partial x_j} dx \quad \text{for every } \varphi \in \mathcal{C}_0^\infty(\mathcal{U}).$$

3. Main result. In this section we establish our main result, which consists in exhibiting a linear space of uncountable dimension of Clarke-saturated Lipschitz functions whenever $\mathcal{U} \subseteq \ell_d^1$ is a nonempty open convex set. More precisely, endowing $\mathrm{Lip}_{x_0}(\mathcal{U})$ with the Lipschitz norm $\|\cdot\|_{\mathrm{Lip}}$ we obtain a closed subspace of Clarke-saturated elements, which turns out to be isometrically isomorphic to $\ell^\infty(\mathbb{N})$. Our technique is as follows: we first prove the result for the one-dimensional case and we then extend the construction for the d-dimensional case. In both cases, we first obtain countably many linearly independent Clarke-saturated functions in $\mathrm{Lip}_{x_0}(\mathcal{U})$, and in the last subsection we use these functions to obtain the final result.

3.1. The $d = 1$ case. The construction for the aforementioned family of functions relies on some basic results concerning the Lebesgue measure. We refer the reader to [18] for prerequisites in measure theory. Let us start with a typical example of a subset of $[0, 1]$ that is closed, nowhere dense, and has positive measure.

DEFINITION 3 (Smith–Volterra–Cantor set). *Consider the subsets $F_n \subset [0, 1]$ defined as follows:*

- $F_0 = [0, 1]$ *and* $F_1 = [0, \frac{3}{8}] \cup [\frac{5}{8}, 1]$*;*
- F_{n+1} *is obtained, for $n \geq 1$, by removing the middle open interval of length $\frac{1}{4^{n+1}}$ from each of the 2^n closed intervals whose union is F_n.*

Let $F = \bigcap_{n\geq 0} F_n$. Then F is closed and contains no intervals. Moreover, F is Lebesgue measurable with measure $1/2$.

In what follows we shall use the term *fat Cantor set* for any Cantor-type set (that is, a set built in this way) with positive measure. It is clear that this procedure can be carried out over any (open or closed) interval thanks to the homogeneity and invariance of the Lebesgue measure.

Let us now give the following definition.

DEFINITION 4 (everywhere positive-measured set). *A subset A of $\mathbb{R}$ is called everywhere positive-measured if it intersects any nontrivial interval in a set of positive measure.*

Notice that a set A has the splitting property (5) for the family of intervals of $\mathbb{R}$ if both A and $\mathbb{R} \setminus A$ are everywhere positive-measured. The following lemma asserts the existence of a countable partition of $\mathbb{R}$ into splitting sets. This result goes back to Bruckner [11] (see also [24, Lemma 4.1]). We include a proof for completeness.

LEMMA 5 (countable splitting partition). *There exists a countable partition $\{A_k\}_{k\in\mathbb{N}}$ of $\mathbb{R}$, each of which splits the family of intervals.*

Proof. Let us first notice that it suffices to obtain a partition of $[0, 1)$ with the above property, since we can translate those sets over every interval of the form $[m, m + 1)$, $m \in \mathbb{Z}$. To this end, let $\{I_n\}_{n\in\mathbb{N}}$ be an enumeration of the subintervals of $(0, 1)$ with rational end points, say $I_n = (a_n, b_n)$. We split I_1 into two open contiguous intervals, that is, we take $c \in (a_1, b_1)$ and consider the intervals (a_1, c) and (c, b_1). Then let $T_1^{(1)}$ and $B^{(1)}$ be two fat Cantor sets over (a_1, c) and (c, b_1), respectively. Since $T_1^{(1)} \cup B^{(1)}$ is nowhere dense, there exists $(a'_2, b'_2) \subseteq I_2$ such that

$$(a'_2, b'_2) \bigcap \left(T_1^{(1)} \cup B^{(1)}\right) = \emptyset.$$

We now proceed inductively as follows: given $T_k^{(i)}$, $B^{(i)}$ for $1 \leq k \leq i \leq n - 1$, since their union is a nowhere dense closed subset of $(0, 1)$, there exists a subinterval (a'_n, b'_n) of I_n that is disjoint from this union. We now split the interval (a'_n, b'_n) into $n + 1$ contiguous open intervals and define $T_k^{(n)}$, $B^{(n)}$ (where $k \in \{1, \dots, n\}$) to be fat Cantor sets over each one of these intervals. In this way we inductively obtain disjoint fat Cantor subsets $T_k^{(n)}$, $B^{(n)}$ of $(0, 1)$, where $1 \leq k \leq n$ and $n \in \mathbb{N}$. We then define

$$A_k = \bigcup_{n\geq k} T_k^{(n)}, \qquad A_0 = [0, 1) \setminus \left(\bigcup_{k\geq 1} A_k\right), \qquad B = \bigcup_{n\geq 1} B^{(n)}.$$

We claim that the family $\{A_k\}_{k\geq 0}$ is the partition of $[0, 1)$ we are looking for.

Indeed, the sets $\{A_k\}_{k\geq 1}$ are mutually disjoint: let $1 \leq k < k'$ and, working towards a contradiction, assume that $x \in A_k \cap A_{k'}$. Then, there exists $n \geq k$ and $n' \geq k'$ such that $x \in T_k^{(n)}$ and $x \in T_{k'}^{(n')}$, which is impossible by construction. Notice further that $B \subseteq A_0$ (the argument is the same as before) and that $A_0 \subseteq [0,1) \setminus A_k$ for every $k \geq 1$. Now, let $[a,b] \subseteq [0,1)$ be any interval. For $k \geq 1$, let $n \geq k$ such that $I_n \subseteq [a,b]$. It follows that

$$\lambda(A_k \cap [a,b]) \geq \lambda(A_k \cap I_n) \geq \lambda(T_k^{(n)} \cap I_n) = \lambda(T_k^{(n)}) > 0.$$

On the other hand

$$\begin{aligned}\lambda(A_0 \cap [a,b]) &\geq \lambda(B \cap [a,b]) \geq \lambda(B \cap I_n)\\ &\geq \lambda(B^{(n)} \cap I_n) = \lambda(B^{(n)}) > 0,\end{aligned}$$

yielding the result. □

Now let $\mathcal{U} \subseteq \mathbb{R}$ be a nontrivial open interval and fix $x_0 \in \mathcal{U}$. Define the family of functions

$$g_k(x) = \mathbb{1}_{A_{2k+1}}(x) - \mathbb{1}_{A_{2k}}(x), \quad x \in \mathcal{U}, \tag{7}$$

and set

$$f_k(x) = \int_{x_0}^{x} g_k(t)dt. \tag{8}$$

We list below some properties of the family of functions $\mathcal{F} = \{f_k : k \in \mathbb{N}\}$ defined by (8). In what follows, we denote by c_{00} the space of finitely supported sequences, that is, $\mu = (\mu_n)_{n\in\mathbb{N}}$ if and only if $\mathrm{supp}(\mu) := \{n : \mu_n \neq 0\}$ is finite.

(i) $\mathcal{F} \subset \mathrm{Lip}_{x_0}(\mathcal{U})$. More precisely, for every $k \in \mathbb{N}$, f_k is Lipschitz, with $\|f_k\|_{\mathrm{Lip}} = 1$.

This is straightforward from the fact that the functions $g_k = f_k'$ belong to $L^\infty(\mathcal{U})$, with $\|g\|_\infty = 1$.

(ii) The family $\mathcal{F}$ is linearly independent.

Let $\mu \in c_{00}$. Then

$$\sum_{k\in\mathbb{N}} \mu_k f_k = 0 \iff \int_0^x \left(\sum_{k\in\mathbb{N}} \mu_k g_k(t)\right) dt = 0 \quad \forall x \in \mathcal{U}.$$

In virtue of Rademacher's theorem and Lebesgue's differentiation theorem, the above yields that

$$\sum_{k\in\mathbb{N}} \mu_k g_k(x) = 0 \quad \text{almost everywhere on } \mathcal{U}.$$

Since $\{A_k\}_{k\in\mathbb{N}}$ are disjoint, everywhere positive-measured sets, we can choose $x_k \in A_{2k+1} \cap \mathcal{U}$ for every $k \in \mathbb{N}$. Then $x_k \notin A_{2k}$, and in view of (7) we have $g_k(x_k) = 1$ and $g_k(x_{k'}) = 0$ for $k \neq k'$. From this, we deduce that for every $k \in \mathbb{N}$, $\mu_k = 0$, and therefore $\{f_k\}_{k\in\mathbb{N}}$ is a linearly independent family.

(iii) The functions f_k are Clarke-saturated for every $k \in \mathbb{N}$.

Since $f_k' = g_k$ almost everywhere on $\mathcal{U}$, it follows that f_k' takes each one of the values $\{-1, 0, 1\}$ on an everywhere positive-measured (and a fortiori in a dense) subset of $\mathcal{U}$. It follows by (4) that $\partial f_k^\circ(x) = [-1,1] = \bar{B}_*$ for every $x \in \mathcal{U}$.

Let us now show that linear combinations inherit the property of Clarke-saturation from the family $\mathcal{F}$.

PROPOSITION 6 (lineability in the one-dimensional case). *Every linear combination of the functions* $\{f_k\}_{k\in\mathbb{N}}$ *has maximal Clarke subdifferential.*

Proof. Let $\mu \in c_{00}$ and set $f = \sum_{k\in\mathbb{N}} \mu_k f_k$ (finite combination). Then it holds almost everywhere on $\mathcal{U}$ that

$$f'(x) = \sum_{k\in\mathbb{N}} \mu_k\, f_k'(x) = \sum_{k\in\operatorname{supp}(\mu)} \mu_k\, g_k(x).$$

Notice that for a given $x \in \mathcal{U}$ there exists at most one $k \in \mathbb{N}$ such that $g_k(x) \neq 0$ (namely, $g_k(x) = 1$ or -1), and therefore f' can only take the values $\{\pm\mu_k\}_{k\in\mathbb{N}}$ and 0. Using the same argument as before, we deduce that each of these values is taken on a dense subset of $\mathcal{U}$. Therefore

$$\partial^\circ \left(\sum_{k\in\mathbb{N}} \mu_k f_k\right)(x) = \|\mu\|_\infty[-1,1] = \|\mu\|_\infty \bar{B}_* \quad \text{for every } x \in \mathcal{U}.$$

Moreover,

$$\|f\|_{\text{Lip}} = \left\|\sum_{k\in\mathbb{N}} \mu_k f_k\right\|_{\text{Lip}} = \left\|\sum_{k\in\mathbb{N}} \mu_k g_k\right\|_\infty = \|\mu\|_\infty.$$

We conclude that this linear combination has maximal Clarke subdifferential everywhere, that is, it is Clarke-saturated. ∎

3.2. The $d > 1$ case. In this section we extend the above method from the one-dimensional case to higher dimensions. For technical reasons we equip $\mathbb{R}^d$ with the 1-norm $\|\cdot\|_1$, so that the dual norm is $\|\cdot\|_\infty$. This facilitates establishing Clarke-saturation. We do not know whether or not this result remains true under a different choice of norm. To simplify notation we set $\ell_d^1 := (\mathbb{R}^d, \|\cdot\|_1)$.

Let $\mathcal{U} \subseteq \ell_d^1$ be a nonempty open convex set and let $\hat{\mathrm{D}}$ stand for the isometry in Theorem 2. For $k \in \mathbb{N}$ and $x = (x_1, \ldots, x_d) \in \mathcal{U}$ we define the function

$$\text{(9)} \qquad \begin{cases} G^k : \mathcal{U} \to \ell_d^\infty, \\ G^k(x) := (g_k(x_1), \ldots, g_k(x_d)) \\ \qquad\quad = (\mathbb{1}_{A_{2k+1}}(x_1) - \mathbb{1}_{A_{2k}}(x_1), \ldots, \mathbb{1}_{A_{2k+1}}(x_d) - \mathbb{1}_{A_{2k}}(x_d)). \end{cases}$$

In other words,

$$\langle G^k(x), e_i\rangle = g_k(\langle x, e_i\rangle),$$

where g_k are given by (7) and $\{e_i\}_{i=1,\ldots,d}$ is the canonical basis of $\mathbb{R}^d$.

Let us first show that the functions $\{G^k\}_{k\in\mathbb{N}}$ are "derivatives" of functions of $\text{Lip}_{x_0}(\mathcal{U})$. This part relies on Theorem 2.

PROPOSITION 7 (G^k are derivatives). *For every* $k \in \mathbb{N}$, $G^k \in \hat{\mathrm{D}}(\text{Lip}_{x_0}(\mathcal{U}))$.

Proof. Let $i, j \in \{1, \ldots, d\}$ with $i \neq j$ and $\varphi \in \mathcal{C}_0^\infty(\mathcal{U})$. Then

$$\langle \partial_j G_i^k, \varphi\rangle = -\int_{\mathcal{U}} G_i^k(x) \frac{\partial\varphi}{\partial x_j}(x)dx = -\int_{\mathcal{U}} g_k(x_i) \frac{\partial\varphi}{\partial x_j}(x)dx.$$

As $\varphi \in \mathcal{C}_0^\infty(\mathcal{U})$, thanks to Fubini's theorem we can first integrate the variable x_j and conclude that the above integral is equal to 0. Therefore, $\partial_i G_j^k = 0$ whenever $i \neq j$. In particular, $\partial_i G_j^k = \partial_j G_i^k$ in the sense of distributions, and using Theorem 2 we deduce that $G^k \in \hat{\mathrm{D}}(\mathrm{Lip}_{x_0}(\mathcal{U}))$. □

In view of the above proposition, we can define the family

$$\mathcal{F} = \{f_k\}_{k\geq 0} \subseteq \mathrm{Lip}_{x_0}(\mathcal{U})$$

as the inverse images of the family $\{G^k\}_{k\geq 0}$, that is,

$$f_k := \hat{\mathrm{D}}^{-1}(G^k) \quad \text{for every } k \in \mathbb{N}. \tag{10}$$

We now verify the same properties as in the previous section for the above functions.

(i) $\mathcal{F} \subset \mathrm{Lip}_{x_0}(\mathcal{U})$. Moreover, $\|f_k\|_{\mathrm{Lip}} = 1$.

Notice that the values of G^k are vectors $v \in \mathbb{R}^d$ whose components take the values $\{-1, 0, 1\}$, each of them over everywhere positive-measured sets. Therefore $\|G^k\|_\infty = 1$ and the result follows from the fact that $\hat{\mathrm{D}}$ is an isometry.

(ii) The family $\mathcal{F}$ is linearly independent.

It suffices to prove that the family $\{G^k\}_{k\geq 0}$ is linearly independent, since $\hat{\mathrm{D}}$ is an isometry. Let $\mu \in c_{00}$ (finitely supported sequence) and assume

$$\sum_{k\in\mathbb{N}} \mu_k G^k = 0, \quad \text{that is,} \quad \sum_{k\in\mathbb{N}} \mu_k G^k(x) = 0 \text{ a.e. on } \mathcal{U}.$$

For every $k \geq 0$ let $x^k \in (A_k \times \cdots \times A_k) \cap \mathcal{U}$. Given the definition of the functions G^k, we have that, for $i \in \{1, \ldots, d\}$,

$$\left(\sum_{k\in\mathbb{N}} \mu_k G^k(x^k)\right)_i = \begin{cases} \mu_{2n+1} & \text{if } k = 2n+1, \\ -\mu_{2n} & \text{if } k = 2n. \end{cases}$$

Since $(A_k \times \cdots \times A_k) \cap \mathcal{U}$ has positive measure everywhere, we conclude that $\mu = 0$, and therefore $\{G^k\}_{k\geq 0}$ is linearly independent and the assertion follows.

(iii) The functions f_k are Clarke-saturated.

Notice that every extreme point of the unit ball of ℓ_d^∞ is taken as a value of G^k on a subset of $\mathcal{U}$ that has positive measure everywhere. Since $Df_k = G^k$ almost everywhere on $\mathcal{U}$, we conclude that $\partial^\circ f_k(x) = \bar{B}_*$ for all $x \in \mathcal{U}$.

Similarly to the one-dimensional case, we now establish that Clarke-saturation is preserved under linear combinations of elements of $\mathcal{F}$.

Proposition 8 (lineability). *Every linear combination of the functions $(f_k)_{k\in\mathbb{N}}$ has maximal Clarke subdifferential.*

Proof. Let $\mu \in c_{00}$. Then we have

$$D\left(\sum_{k\in\mathbb{N}} \mu_k f_k\right)(x) = \sum_{k\in\mathbb{N}} \mu_k G^k(x) \quad \text{for a.e. } x \in \mathcal{U}.$$

The values of this last function are exclusively vectors $v \in \mathbb{R}$ with components in the set $\{\pm\mu_k : k \geq 0\}$. Moreover, each component takes each one of the values $\{\pm\mu_k\}_{k\in\mathbb{N}}$ on subsets of $\mathcal{U}$ that have everywhere positive measure. It follows readily from (4) that, for every $x \in \mathcal{U}$,

$$\partial^\circ \left(\sum_{k\in\mathbb{N}} \mu_k f_k \right)(x) = \|\mu\|_\infty \bar{B}_*.$$

In addition, using the isometry $\hat{\mathrm{D}}$ we deduce that

$$\|f\|_{\mathrm{Lip}} = \left\| \sum_{k\in\mathbb{N}} \mu_k f_k \right\|_{\mathrm{Lip}} = \left\| \sum_{k\in\mathbb{N}} \mu_k G^k \right\|_\infty = \|\mu\|_\infty.$$

The proof is complete. ∎

3.3. The space of Clarke-saturated functions. In the previous section we constructed a countable family of linearly independent Clarke-saturated functions f_k belonging to $\mathrm{Lip}_{x_0}(\mathcal{U})$, where $\mathcal{U} \subseteq \ell_d^1$ is a nonempty open convex set and $x_0 \in \mathcal{U}$. We shall now describe in terms of the isometry $\hat{\mathrm{D}}$ (Theorem 2) the closure of the space generated by these functions. In what follows we denote by $\ell^\infty(\mathbb{N})$ the (nonseparable) Banach space of bounded sequences.

Proposition 9. *Let $T : \ell^\infty(\mathbb{N}) \to L^\infty(\mathcal{U}; \ell_d^\infty)$ be given by*

$$T(\mu) = \sum_{k\geq 0} \mu_k G^k \quad \text{for all } \mu = (\mu_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N}).$$

Then T is well defined and establishes a linear isometric injection of $\ell^\infty(\mathbb{N})$ into $L^\infty(\mathcal{U}; \ell_d^\infty)$.

Proof. Let $\{A_k\}_{k\in\mathbb{N}}$ be the countable partition of $\mathbb{R}$ given by Lemma 5. Let $x = (x_1, \ldots, x_d) \in \mathcal{U}$. Since each A_k is everywhere positive-measured, there exists $j_1, \ldots, j_d \geq 0$ such that $x_i \in A_{j_i}$ for $i \in \{1, \ldots, d\}$. This implies that the sum

$$\sum_{k\geq 0} \mu_k G^k(x)$$

is finite for every $x \in \mathcal{U}$, with norm less than or equal to $\|\mu\|_\infty$. Therefore $T(\mu) \in L^\infty(\mathcal{U}; \ell_d^\infty)$, with $\|T\mu\|_\infty \leq \|\mu\|_\infty$. Moreover, if $x \in (A_{2n+1} \times \cdots \times A_{2n+1})$ and $x' \in (A_{2n} \times \cdots \times A_{2n})$, then

$$T(\mu)(x) = -T(\mu)(x') = (\mu_k, \ldots, \mu_k),$$

which leads to $\|T\mu\|_\infty = \|\mu\|_\infty$. Since T is obviously linear, it follows that T defines a linear isometry between $\ell^\infty(\mathbb{N})$ and $T(\ell^\infty(\mathbb{N}))$. ∎

Now, we state the relation between $T(\ell^\infty(\mathbb{N}))$ and $\hat{\mathrm{D}}(\mathrm{Lip}_{x_0}(\mathcal{U}))$. This relation is obtained in a similar way to the case of linear combinations.

Proposition 10. $T(\ell^\infty(\mathbb{N})) \subseteq \hat{\mathrm{D}}(\mathrm{Lip}_{x_0}(\mathcal{U}))$.

Proof. Let $\mu \in \ell^\infty(\mathbb{N})$. We need to prove that $T(\mu)$ is the gradient of some Lipschitz function. Let $i, j \in \{1, \ldots, d\}$ with $i \neq j$. Then

$$(T\mu)_i(x) = \sum_{k\geq 0} \mu_k g_k(x_i).$$

If $\varphi \in \mathcal{C}_0^\infty(\mathcal{U})$, we have that

$$\langle \partial_j (T\mu)_i, \varphi \rangle = -\int_{\mathcal{U}} \left(\sum_{k\geq 0} \mu_k g_k(x_i) \right) \frac{\partial \varphi}{\partial x_j}(x) dx = -\int_{\mathcal{U}} \sum_{k\geq 0} \left(\mu_k g_k(x_i) \frac{\partial \varphi}{\partial x_j}(x) \right) dx.$$

We define, for $n \geq 0$,

$$\psi_n(x) = \sum_{k=0}^{n} \left(\mu_k g_k(x_i) \frac{\partial \varphi}{\partial x_j}(x) \right) \quad \text{and} \quad \psi(x) = \sum_{k\geq 0} \left(\mu_k g_k(x_i) \frac{\partial \varphi}{\partial x_j}(x) \right).$$

Notice that for $x = (x_1, \ldots, x_d) \in \mathcal{U}$ and $i \in \{1, \ldots, d\}$ we have

$$g_k(x_i) \neq 0 \iff x_i \in A_{2k+1} \cup A_{2k},$$

and in this case $g_{k'}(x_i) = 0$ for all $k' \neq k$. Therefore, there exists some $N \geq 0$ large enough such that

$$\psi_n(x) = \sum_{k=0}^{n} \mu_k g_k(x_i) \frac{\partial \varphi}{\partial x_j}(x) = \begin{cases} 0, & n < N, \\ \mu_N \; g_N(x_i) \; \frac{\partial \varphi}{\partial x_j}(x), & n \geq N, \end{cases}$$

yielding

$$\psi_n \to \psi \ \ \text{(pointwise)} \quad \text{and} \quad |\psi_n| \leq \|\mu\|_\infty \left| \frac{\partial \varphi}{\partial x_j} \right| \in L^1(\mathcal{U}).$$

By virtue of Lebesgue's dominated convergence theorem, we have that

$$\langle \partial_j (T\mu)_i, \varphi \rangle = -\sum_{k\geq 0} \left(\int_{\mathcal{U}} \mu_k g_k(x_i) \frac{\partial \varphi}{\partial x_j}(x) dx \right).$$

But thanks to Fubini's theorem, we can first integrate with respect to the x_j variable, and since φ has compact support, we conclude that all the integrals are equal to 0. Then $\partial_j (T\mu)_i = 0$ whenever $i \neq j$, which leads to $T(\mu) \in \hat{\mathrm{D}}(\mathrm{Lip}_{x_0}(\mathcal{U}))$. ∎

PROPOSITION 11. *Let $f \in \mathrm{Lip}_{x_0}(\mathcal{U})$ be such that $\hat{\mathrm{D}} f = T(\mu)$. Then f is Clarke-saturated.*

Proof. It suffices to notice that

$$\|f\|_{\mathrm{Lip}} = \|\hat{\mathrm{D}} f\|_\infty = \|T(\mu)\|_\infty = \|\mu\|_\infty$$

and that for every extreme point v of the dual ball $\overline{B}_*$ and $k \geq 0$ there exists an everywhere positive-measured set $A \subseteq \mathcal{U}$ such that

$$\hat{\mathrm{D}} f(x) = T\mu(x) = \mu_k v \quad \text{for every } x \in A.$$

Since f is differentiable almost everywhere, we conclude that

$$\partial^\circ f(x) = \|\mu\|_\infty \overline{B}_* = \|f\|_{\mathrm{Lip}} \overline{B}_*,$$

which finishes the proof. ∎

We are ready to state our main result.

THEOREM 12 (spaceability of Clarke-saturated functions). *Let $d \geq 1$ and $\mathcal{U} \subseteq \ell_d^1$ be a nonempty open convex set. Then we have the following.*

(i) (Lineability.) *The space* $\mathrm{Lip}(\mathcal{U})$ *of Lipschitz functions contains a linear subspace of Clarke-saturated functions of uncountable dimension.*

(ii) (Spaceability.) *For any $x_0 \in \mathcal{U}$, the Banach space* $(\mathrm{Lip}_{x_0}(\mathcal{U}), \|\cdot\|_{\mathrm{Lip}})$ *contains a (proper) linear subspace of Clarke-saturated functions isometric to $\ell^\infty(\mathbb{N})$.*

In particular, if $\mathcal{F} = \{f_k : k \in \mathbb{N}\}$ is the family defined in (10), then $\mathrm{span}\{f_k\}$ is isometric to c_{00} while $\overline{\mathrm{span}}\{f_k\}$ is isometric to $c_0(\mathbb{N})$ (the Banach space of null sequences).

Proof. Thanks to Propositions 9 and 10, we deduce that $\ell^\infty(\mathbb{N})$ is isometric to the subspace

$$Z = \hat{\mathrm{D}}^{-1}(T(\ell^\infty(\mathbb{N})))$$

of $\mathrm{Lip}_{x_0}(\mathcal{U})$. This subspace is proper (any strictly differentiable function $h \in \mathrm{Lip}_{x_0}(\mathcal{U}) \setminus \{0\}$ does not belong to Z). This proves (ii), and yields directly that Clarke-saturated functions contain a linear subspace of uncountable dimension. Therefore (i) holds, since $\mathrm{Lip}_{x_0}(\mathcal{U})$ is a linear subspace of $\mathrm{Lip}(\mathcal{U})$. Finally, an easy computation shows that if $\mu \in c_{00}$, then $\hat{\mathrm{D}}^{-1}(T(\mu)) \in \mathrm{span}\{f_k\}$, whence c_{00} is isometric to $\mathrm{span}\{f_k\}$. It follows readily by continuity that $c_0(\mathbb{N})$ is isometric to $\overline{\mathrm{span}}\{f_k\}$. ∎

We conclude this work with the following straightforward consequence of Theorem 12.

COROLLARY 13. *Let $p \in \mathbb{R}^d$ and $r > 0$. Then, there exists $f \in \mathrm{Lip}(\mathcal{U})$ such that $\partial^\circ f(x) = p + r\overline{B}_*$ for every $x \in \mathcal{U}$.*

Proof. Let $\mu \in \ell^\infty$ be such that $\|\mu\|_\infty = r$. Set $h_1 = D^{-1}T\mu$ and $h_2 = \langle p, \cdot \rangle$. Then $\partial^\circ h_1(x) = p + r\overline{B}_*$ and $\partial^\circ h_2(x) = \{p\}$ for every $x \in \mathcal{U}$, where we used that h_2 is strictly differentiable. Again thanks to that, if $f = h_1 + h_2$ then, for every $x \in \mathcal{U}$,

$$\partial^\circ f(x) = \partial^\circ(h_1 + h_2)(x) = \partial^\circ h_1(x) + \partial^\circ h_2(x) = p + r\overline{B}_*.$$

The proof is complete. ∎

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