Cut-generating functions and S-free sets

Michele Conforti
Dip. Mat. Pura e App., Univ. di Padova, Italy
email: conforti@math.unipd.it

Gérard Cornuéjols
Tepper School Business, Carnegie Mellon Univ., Pittsburgh, PA
email: gc0v@andrew.cmu.edu

Aris Daniilidis
Dep. Matemàtiques, Univ. Autònoma de Barcelona, Bellaterra, Spain
email: arisd@mat.uab.cat

Claude Lemaréchal
Inria-Bipop, Montbonnot, France
email: claude.lemarechal@inria.fr

Jérôme Malick
CNRS, Lab. J. Kuntzmann, Grenoble, France
email: jerome.malick@inria.fr

Abstract. We consider the separation problem for sets X that are inverse images of a given set S by a linear mapping. Classical examples occur in integer programming, complementarity problems and other optimization problems. One would like to generate valid inequalities that cut off some point not lying in X, without reference to the linear mapping. Formulas for such inequalities can be obtained through cut-generating functions. This paper presents a formal theory of minimal cut-generating functions and maximal S-free sets. This theory relies on tools of convex analysis.

Key words. Integer programming; Convex analysis; Separation; Generalized gauges; S-free sets.

MSC2000 Subject Classification. Primary: 52A41, 90C11; Secondary: 52A40.

OR/MS Subject Classification. Primary: Combinatorics, Convexity; Secondary: Cutting plane.

1 Introduction

In this paper, we consider sets of the form

$$X = X(R,S) := \{x \in \mathbb{R}^n_+ : Rx \in S\}\,, \tag{1a}$$

$$\text{where } \begin{cases} R = [r_1 \dots r_n] & \text{is a real } q \times n \text{ matrix}\,, \\ S \subset \mathbb{R}^q & \text{is a nonempty closed set with } 0 \notin S\,. \end{cases} \tag{1b}$$

In other words our set X is the intersection of a closed convex cone with a reverse image by a linear mapping. This model is considered in [15], where it is called "multiple right-hand side choice linear program", in the special case where S is a finite set. Because $0 \notin S$, the closed convex hull of X does not contain 0 (see Lemma 2.1 below); we are then interested in *separating* 0 from X: we want to generate *cuts*, i.e. inequalities valid for X, which we write as

$$c^\top x \geqslant 1\,, \quad \text{for all } x \in X\,. \tag{2}$$

1.1 Motivating examples

Starting from a polyhedron

$$P := \{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^m : Ax + y = b\} \tag{3}$$

(nonnegativity of the y-variables can also be imposed), consider first two applications arising in mixed integer programming that have been considered in the literature. Assume that $b \notin \mathbb{Z}^m$.

Example 1.1 (An integer linear program)**.** In a first instance, all variables must be integers: the set of interest is $P \cap (\mathbb{Z}^n \times \mathbb{Z}^m)$, i.e. the set of $(x, y = b - Ax)$ such that $x \in \mathbb{Z}^n_+$ and $b - Ax \in \mathbb{Z}^m$. This model has the form (1) if we set

$$q = n + m\,, \quad R = \begin{bmatrix} I \\ -A \end{bmatrix}, \quad S = \mathbb{Z}^n \times \big(\mathbb{Z}^m - \{b\}\big)\,. \tag{4}$$

Since $b \notin \mathbb{Z}^m$, the above S is a closed set not containing the origin; (4) is the model considered in [11]. The convex hull of $P \cap (\mathbb{Z}^n \times \mathbb{Z}^m)$ is known as the corner polyhedron.

Except for the translation by the basic solution $(x = 0, y = b)$ (a rather simple operation), S is quasi instance-independent. This is actually a crucial feature; it determines the approach developed in this paper, namely cut-generating functions to be developed below. □

Example 1.2 (A mixed integer linear program)**.** In a similar situation, the set of interest is $P \cap (\mathbb{R}^n \times \mathbb{Z}^m)$, i.e. the set of $(x, y = b - Ax)$ such that $x \in \mathbb{R}^n_+$ and $b - Ax \in \mathbb{Z}^m$. Then we are still in the same situation with

$$q = m\,, \quad R = -A\,, \quad S = \mathbb{Z}^m - b\,;$$

this is the model considered in [1] for $m = 2$, and in [7] for general m. □

Model (1) occurs in other areas than integer programming and we give another example.

Example 1.3 (Complementarity problem)**.** Still using P of (3), let

$$E \subset \{1, 2, \ldots, m\} \times \{1, 2, \ldots, m\} \quad \text{and} \quad C := \{y \in \mathbb{R}^m_+ : y^i y^j = 0\,,\ (i, j) \in E\}\,.$$

The set of interest is then $P \cap (\mathbb{R}^n \times C)$. It can be modeled by (1) where

$$q = m\,, \quad R = -A\,, \quad S = C - b\,;$$

Cuts have been used for complementarity problems of this type, for example in [16]. □

We will retain from these examples the dissymetry between S (a very particular and highly structured set) and R (an arbitrary matrix). Keeping this in mind, we will consider that (q, S) is given and fixed, while (n, R) is instance-dependent data: our cutting problem can be viewed as *parametrized* by (n, R).

1.2 Introducing cut-generating functions

To generate cuts in the present situation, it would be convenient to have a mapping, taking instances of (1) as input, and producing cuts as output. What we need for this is a function

$$\mathbb{R}^q \ni r \mapsto \rho(r) \in \mathbb{R}$$

which, applied to the columns r_j of a $q \times n$ matrix R (an arbitrary matrix, with an arbitrary number of columns) will produce the n coefficients $c_j := \rho(r_j)$ of a cut (2). Thus, we require from our ρ to satisfy

$$x \in X \quad \Longrightarrow \quad \sum_{j=1}^{n} \rho(r_j) x_j \geqslant 1\,, \tag{5}$$

for every instance X of (1). Such a ρ can then justifiably be called a *cut-generating function* (CGF). The notation ρ refers to *representation*, which will appear in Definition 2.6 below. One of the most well-known cut-generating functions in integer programming is the so-called Gomory function, used to generate Gomory's mixed-integer cuts through a simple closed-form formula [10]. Because these cuts can be generated quickly, they are a powerful tool in computations; indeed, they drastically speed up integer-programming solvers [6].

So far, a CGF is a rather abstract object, as it lies in the (vast!) set of functions from $\mathbb{R}^q$ to $\mathbb{R}$; but the following observation allows a drastic reduction of this set.

Remark 1.4 (Dominating cuts)**.** Consider in (2) a vector c' with $c'_j \leqslant c_j$ for $j = 1, \ldots, n$; then $c'^\top x \leqslant c^\top x$ whenever $x \geqslant 0$. If c' is a cut, it is tighter than c in the sense that it cuts a bigger portion of $\mathbb{R}^n_+$. We can impose some "minimal" character to a CGF, in order to reach some "tightness" of the resulting cuts. □

With this additional requirement, the decisive Theorem 2.3 below will show that a CGF can be imposed to be *convex positively homogeneous* (and defined on the whole of $\mathbb{R}^q$); positive homogeneity means $\rho(tr) = t\rho(r)$ for all $r \in \mathbb{R}^q$ and $t > 0$. This is a fairly narrow class of functions indeed, which is fundamental in convex analysis. Such functions are in correspondence with closed convex sets and in our context, such a correspondence is based on the mapping $\rho \mapsto V$ defined by

$$V = V(\rho) := \left\{ r \in \mathbb{R}^q : \rho(r) \leqslant 1 \right\}, \tag{6}$$

which turns out to be a cornerstone: via Theorem 2.5 below, (6) establishes a correspondence between the CGF's and the so-called *S-free sets*. As a result, cut-generating functions can alternatively be studied from a geometric point of view, involving sets V instead of functions ρ. This situation, common in convex analysis, is often very fruitful. With regard to Remark 1.4, observe that $V(\rho)$ increases when ρ decreases: small ρ's give large V's. However the converse is not true because the mapping in (6) is many-to-one and therefore has no inverse. A first concern will therefore be to specify appropriate correspondences between (cut-generating) functions and (S-free) sets.

1.3 Scope of the paper

The aim of the paper is to present a formal theory of minimal cut-generating functions and maximal S-free sets, valid independently of the particular S. Such a theory would gather and synthetize a number of papers dealing with the above problem for various special forms for S: [17, 1, 7, 9, 3, 4] and references therein. For this, we use basic tools from convex analysis and geometry. Readers not familiar with this field may use [14] (especially its Chap. C) for an elementary introduction, while [13, 18] are more complete.

The paper is organized as follows.

- Section 2 states more accurately the concepts of CGF's and S-free sets.
- Section 3 studies the mapping (6). We show that the pre-images of a given V (the representations of V) have a unique maximal element γ_V and a unique minimal element μ_V; in view of Remark 1.4, the latter then appears as *the* relevant inverse of $\rho \mapsto V(\rho)$.
- In Section 4, we study the correspondence $V \leftrightarrow \mu_V$. We show that different concepts of minimality come into play for ρ in Remark 1.4. Geometrically they correspond to different concepts of maximality for V.
- We also show in Section 5 that these minimality concepts coincide in a number of cases.
- Finally we have a conclusion section, with some suggestions for future research.

2 Cut-generating functions: definitions and first results

We begin with making sure that our framework is consistent. We will use $\operatorname{conv}(X)$ [resp. $\overline{\operatorname{conv}}(X)$] to denote the convex hull [resp. closed convex hull] of a set X.

Lemma 2.1. *With X given as in* (1)*,* $0 \notin \overline{\operatorname{conv}}(X)$*.*

Proof. Assume $X \neq \emptyset$, otherwise we have nothing to prove. Because 0 does not lie in the closed set S, there is $\varepsilon > 0$ such that $s \in S$ implies $\|s\|_1 \geqslant \varepsilon$; and by continuity of the mapping $x \mapsto Rx$, there is $\eta > 0$

such that $\|x\|_1 \geqslant \eta$ for all $x \in X$. Because $X \subset \mathbb{R}^n_+$, this means

$$\|x\|_1 = \sum_{j=1}^{n} |x_j| = \sum_{j=1}^{n} x_j \geqslant \eta\,, \quad \text{for all } x \in X\ .$$

In other words, the hyperplane $\sum_j x_j \geqslant \eta$ separates 0 from X, hence from $\overline{\text{conv}}(X)$. ☐

Remember that we are interested in functions ρ satisfying (5) for any (n, R) in (1), and that it is desirable to reduce the class of all possible such functions. The following lemma, inspired from Claim 1 in the proof of [3, Lem. 23], is instrumental for this.

Lemma 2.2. *Let ρ be a* CGF*. For all sets of K vectors $r_k \in \mathbb{R}^q$ and nonnegative coefficients α_k, the following relation holds:*

$$\sum_{k=1}^{K} \alpha_k r_k = 0 \quad \Longrightarrow \quad \sum_{k=1}^{K} \alpha_k \rho(r_k) \geqslant 0\,.$$

Proof. Call $e \in \mathbb{R}^q$ the vector of all ones and $\alpha \in \mathbb{R}^K$ the vector of α_k's; take $t \geqslant 0$ and define the vectors in $\mathbb{R}^{K+q}$

$$x := \begin{bmatrix} 0 \\ e \end{bmatrix}, \quad d := \begin{bmatrix} \alpha \\ 0 \end{bmatrix}, \quad \text{so that} \quad x + td = \begin{bmatrix} t\alpha \\ e \end{bmatrix} \in \mathbb{R}^{K+q}_+\,.$$

Then pick $s \in S$; make an instance of (1) with $n = K + q$ and $R := \big[r_1 \dots r_K \mid D(s)\big]$, where the $q \times q$ matrix $D(s)$ is the diagonal built on s. Observing that

$$R(x + td) = t \sum_k \alpha_k r_k + D(s)e = s\,,$$

$x + td$ is feasible in the resulting instance of (1a): (5) writes

$$t \sum_{k=1}^{K} \alpha_k \rho(r_k) \geqslant 1 - z\,,$$

where z is a fixed number gathering the result of applying ρ to the columns of $D(s)$. Letting $t \to +\infty$ proves the claim. ☐

Now we introduce some notation. The *domain* and *epigraph* of a function $\rho\colon \mathbb{R}^q \to \mathbb{R} \cup \{+\infty\}$ are

$$\text{dom}\,\rho := \big\{r \in \mathbb{R}^q : \rho(r) < +\infty\big\} \quad \text{and} \quad \text{epi}\,\rho := \big\{(r, z) \in \mathbb{R}^{q+1} : z \geqslant \rho(r)\big\}\,.$$

If $\text{dom}\,\rho$ is the whole of $\mathbb{R}^q$, we say that ρ is finite-valued; a convex finite-valued function is continuous on $\mathbb{R}^q$. A function is said to be *sublinear* if it is convex and positively homogeneous; or if its epigraph is a convex cone. The *conical hull* $\text{cone}\,(\text{epi}\,\rho)$ of $\text{epi}\,\rho$ is the set of nonnegative combinations of points $(r, z) \in \text{epi}\,\rho$:

$$r = \sum_{k=1}^{K} \alpha^k r^k\,,\ z = \sum_{k=1}^{K} \alpha^k z^k\,, \quad \text{with} \quad z^k \geqslant \rho(r^k)\,,\ \alpha^k \geqslant 0\,,\ k = 1, \dots, K\,,$$

where K is an arbitrary integer. This conical hull is itself the epigraph of a sublinear function: the "sublinear hull" of ρ, whose value at r is the smallest possible of the above z's:

$$\bar{\rho}(r) := \inf\Big\{\sum_{k=1}^{K} \alpha^k \rho(r^k) : \sum_{k=1}^{K} \alpha^k r^k = r,\ \alpha^k \geqslant 0\Big\}\,. \tag{7}$$

Theorem 2.3. *If ρ is a* CGF*, then $\bar{\rho}$ of* (7) *is nowhere $-\infty$ and is again a* CGF.

Proof. Express every $r \in \mathbb{R}^q$ as a nonnegative combination: $\sum_k \alpha^k r^k - r = 0$, hence (Lemma 2.2) $\sum_{k=1}^K \alpha^k \rho(r^k) + \rho(-r) \geqslant 0$ and the sublinear hull is bounded from below: $\bar\rho(r) \geqslant -\rho(-r) > -\infty$.

Then take an instance $R = [r_j]_{j=1}^n$ of (1b). If it produces $X = \emptyset$ in (1a), there is nothing to prove. Otherwise fix $\bar x \in X$.

For each j, take a positive combination $r_j = \sum_k \alpha_j^k r_j^k$. Then

$$\bar s := R\bar x = \sum_{j=1}^n \bar x_j r_j = \sum_{j=1}^n \bar x_j \sum_{k=1}^K \alpha_j^k r_j^k = R_+ x_+ \,,$$

where $x_+ \in \mathbb{R}^{nK}$ denotes the vector with coordinates $\alpha_j^k \bar x_j \geqslant 0$ and R_+ the matrix whose nK columns are r_j^k. Because R_+ is a possible instance of (1b) and $R_+ x_+ = \bar s \in S$, the CGF ρ separates x_+ from 0:

$$1 \leqslant \sum_{k,j} \rho(r_j^k)\big(\alpha_j^k \bar x_j\big) = \sum_{j=1}^n \Big(\sum_{k=1}^K \alpha_j^k \rho(r_j^k)\Big)\bar x_j \,. \tag{8}$$

Apply the definition of an infimum: for each $\varepsilon > 0$ we can choose our decompositions (r_j^k, α_j^k) so that

$$\sum_{k=1}^K \alpha_j^k \rho(r_j^k) \leqslant \bar\rho(r_j) + \varepsilon \,, \quad \text{for } j = 1, \dots, n$$

which yields with (8)

$$1 \leqslant \sum_{j=1}^n \big(\bar\rho(r_j) + \varepsilon\big)\bar x_j = \sum_{j=1}^n \bar\rho(r_j)\bar x_j + \varepsilon \sum_{j=1}^n \bar x_j \,.$$

Because ε is arbitrarily small – while $\bar x$ is fixed – we see that $\bar\rho$ does satisfy (5). ☐

In view of Remark 1.4, Theorem 2.3 allows us to restrict our attention to CGF's that are sublinear; and they are finite-valued by definition. By continuity and because $\rho(0) = 0$, $V(\rho)$ in (6) is a closed convex neighborhood of 0 in $\mathbb{R}^q$. Besides, its interior and boundary are respectively

$$\operatorname{int}(V(\rho)) = \{r \in V : \rho(r) < 1\} \,, \quad \operatorname{bd}(V(\rho)) = \{r \in V : \rho(r) = 1\} \,. \tag{9}$$

This comes from the Slater property $\rho(0) = 0$ (see, e.g., [14, Prop. D.1.3.3]) and can be checked directly:
– by continuity, $\rho(\bar r) < 1$ implies $\rho(r) \leqslant 1$ for r close to $\bar r$;
– by positive homogeneity, $\rho(\bar r) = 1$ implies $\rho(r) = 1 + \varepsilon$ for $r = (1 + \varepsilon)\bar r$.
The relevant such neighborhoods for our purpose are the following:

Definition 2.4 (S-free set)**.** Given a closed set $S \subset \mathbb{R}^q$ not containing the origin, a closed convex neighborhood V of $0 \in \mathbb{R}^q$ is called S-free if its interior contains no point in S: $\operatorname{int}(V) \cap S = \emptyset$. ☐

Let us make clear the importance of this definition.

Theorem 2.5. *Let the sublinear function $\rho\colon \mathbb{R}^q \to \mathbb{R}$ and the closed convex neighborhood V (of $0 \in \mathbb{R}^q$) satisfy* (6)*. Then ρ is a* CGF *for* (1) *if and only if V is S-free.*

Proof. Let V be S-free; in view of (9), $\rho(r) \geqslant 1$ for all $r \in S$. In particular, take a $q \times n$ matrix R, $x \in X$ of (1a) and set $r := Rx \in S$. Then, using sublinearity,

$$1 \leqslant \rho(Rx) = \rho\Big(\sum_{j=1}^n x_j r_j\Big) \leqslant \sum_{j=1}^n x_j \rho(r_j) \,;$$

ρ is a CGF.

Conversely, suppose V is not S-free: again from (9), there is some $r_1 \in S$ such that $\rho(r_1) < 1$. Take in (1b) the instance $(n, R) = (1, [r_1])$. Then $1 \in X$ ($r_1 \in S$), so $c_1 := \rho(r_1) < 1$ cannot be a cut. ☐

This allows a new definition of CGF's, much more handy than the original one:

Definition 2.6 (CGF as representation)**.** Let $V \subset \mathbb{R}^q$ be a closed convex neighborhood of the origin. A *representation* of V is a finite-valued sublinear function ρ such that

$$V = \left\{ r \in \mathbb{R}^q : \rho(r) \leqslant 1 \right\}.$$

We will say that ρ *represents* V.

A sublinear cut-generating function for (1) is a representation of an S-free set. □

A finite-valued sublinear function ρ represents a unique $V = V(\rho)$, well-defined by (6). One easily checks monotonicity of the mapping $V(\cdot)$:

$$\rho \leqslant \rho' \quad \Longrightarrow \quad V(\rho) \supset V(\rho'). \tag{10}$$

By contrast, a given neighborhood V may have several representations, and we are interested in the small ones; this is the subject of the next section.

3 Largest and smallest representations

In this section, we study the representation operation introduced in Definition 2.6 and its geometric counterpart. This indeed puts [5] in perspective and has its own interest in convex analysis. In fact, observing that V in (6) has no reason to be bounded, it somehow extends gauge theory to unbounded sets.

3.1 Some elementary convex analysis

First recall some basic theory (see, e.g., [14, Chap. C]), which will be central in our development. The *support function* of a set $G \subset \mathbb{R}^q$ is

$$\sigma_G(r) := \sup_{d \in G} d^\top r, \tag{11}$$

which may assume the value $+\infty$ if G is unbounded. It is easily seen to be sublinear, to grow when G grows, but to remain unchanged if G is replaced by its closed convex hull: $\sigma_G = \sigma_{\overline{\mathrm{conv}}(G)}$. Besides, σ_G is finite-valued if and only if G is bounded. Conversely, every [finite-valued] sublinear function ρ is the support function of a [bounded] closed convex set, unambiguously defined by

$$G = G_\rho := \left\{ d \in \mathbb{R}^n : d^\top r \leqslant \rho(r) \text{ for all } r \in \mathbb{R}^q \right\};$$

we then say that ρ supports G_ρ, which is the *subdifferential* $\partial\rho(0)$ of ρ at 0. This defines a one-to-one mapping between [finite-valued] sublinear functions and [bounded] closed convex sets.

Besides, the *polar* of G

$$G^\circ := \left\{ r \in \mathbb{R}^q : d^\top r \leqslant 1 \text{ for all } d \in G \right\} = \{ r \in \mathbb{R}^q : \sigma_G(r) \leqslant 1 \} \tag{12}$$

is also a closed convex set (being an intersection of half-spaces, indexed by G). This G° is a neighborhood of the origin when σ_G is finite-valued (i.e. when G is bounded). We see from Definition 2.6 that the support function of G represents its polar G°. Given some $V \subset \mathbb{R}^n$, a set G such that $G^\circ = V$ can be called a *prepolar* of V, i.e. a set G such that σ_G represents V in the sense of Definition 2.6.

From now on in this section, we are given a subset V of $\mathbb{R}^q$, which is a closed convex neighborhood of the origin. Because $0 \in \operatorname{int} V$, the definition (11) of a support function shows that σ_V is positive on $\mathbb{R}^q \backslash \{0\}$; even more: for some $\varepsilon > 0$, V contains the ball $B(\varepsilon)$ centered at 0 of radius ε, hence

$$\varepsilon \|d\| = \sigma_{B(\varepsilon)}(d) \leqslant \sigma_V(d) \quad \text{for all } d \in \mathbb{R}^q. \tag{13}$$

Then V° is bounded since the relation $\sigma_V(d) \leqslant 1$ implies $\|d\| \leqslant 1/\varepsilon$.

A very relevant object for our purpose is the *gauge*

$$\mathbb{R}^q \ni r \mapsto \gamma_V(r) := \inf \{ \lambda > 0 : r \in \lambda V \} \tag{14}$$

of our neighborhood V. In fact, [14, Thm. C.1.2.5 and Prop. C.3.2.4] show that γ_V

– appears as a representation of V

– is the support function of its polar: $V^\circ = \left\{ d : \sigma_V(d) \leqslant 1 \right\} = \partial\gamma_V(0)$.

Now, as already mentioned, V may have several representations, which make up just as many prepolars. Because $(V^\circ)^\circ = V$, the standard polar V° is itself a prepolar – which is somewhat confusing – and turns out to be the largest one; or equivalently γ_V turns out to be the largest representation of V. These facts will be established in Corollary 3.2 below. The main result of this section states that V has also a smallest prepolar, or equivalently a smallest representation; keeping Remark 1.4 in mind, this is exactly what we want. This result is actually [5, Thm. 1]; here we use elementary results of convex analysis, and we insist more on the geometric aspect.

3.2 Largest representation

Introduce the recession cone V_∞ of V. Because $0 \in V$, it can be defined as

$$V_\infty = \{d \in \mathbb{R}^q : td \in V \text{ for all } t > 0\} = \bigcap_{\lambda > 0} \lambda V\,,$$

and the second relation shows that V_∞ is closed. We also see that

$$V_\infty \subset V\,. \tag{15}$$

One then easily sees from (14) that $\gamma_V(r) = 0$ if $r \in V_\infty$. Yet, for any other representation ρ of V, (6) just imposes $\rho(r) \leqslant 0$ at this r and we may a priori have $\rho(r) < 0$: the possible representations of V may differ on V_∞. We make this more precise.

Lemma 3.1 (Representations and recession cone)**.** *For all representations ρ of the closed convex neighborhood V,*

$$\rho(r) \leqslant 0 \iff r \in V_\infty \quad \textit{and} \quad \rho(r) < 0 \implies r \in \operatorname{int}(V_\infty)\,.$$

Besides, all representations coincide on the complement of $\operatorname{int}(V_\infty)$*.*

Proof. By positive homogeneity, $\rho(r) \leqslant 0$ implies $\rho(tr) \leqslant 0 < 1$ (hence $tr \in V$) for all $t > 0$; this implies $r \in V_\infty$. Conversely, $\rho(r) > 0$ implies $\rho(tr) > 1$ for t large enough: using $0 \in V$ again, r cannot lie in V_∞.

To prove the second implication, invoke continuity of ρ: if $\rho(r) < 0$, ρ is still negative in a neighborhood of r, this neighborhood is contained in V_∞.

Besides, take a half-line emanating from 0 but not contained in V_∞; it certainly meets the boundary of V, at a point $\bar r$ which is unique (see, e.g., [14, Rem. A.2.1.7]). By (9), every representation ρ satisfies $\rho(\bar r) = 1$; and by positive homogeneity, the value of this representation is determined all along the half-line. In other words, all possible representations of V coincide on the complement W of V_∞; and by continuity, they coincide also on the closure of W, which is the complement of $\operatorname{int}(V_\infty)$. □

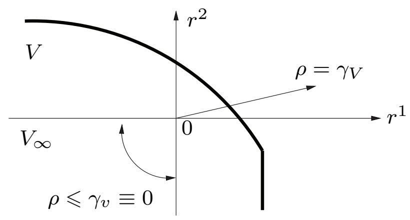

Figure 1: All representations coincide except in $\operatorname{int}(V_\infty)$

Figure 1 illustrates the difference between the recession cone (where the gauge is "maximal") and the rest of the space (where it is *the* representation). Altogether, the gauge appears as the largest representation:

Corollary 3.2 (Maximality of the gauge)**.** *All representations ρ of V satisfy $\rho \leqslant \gamma_V$, with equality on the complement of* $\operatorname{int}(V_\infty)$*.*

Geometrically, all prepolars G are contained in the polar of V:

$$G^\circ = V \quad \implies \quad G \subset V^\circ\,.$$

In particular, V has a unique representation $\rho = \gamma_V$ (and a unique prepolar V°) whenever $\operatorname{int}(V_\infty) = \emptyset$*.*

Proof. Just apply Lemma 3.1, observing from (14) that the gauge is nonnegative.

Geometrically, the inequality between support functions becomes an inclusion: the set G supported by ρ is included in the set V° supported by γ_V (see, e.g., [14, Thm. C.3.3.1]). □

3.3 Smallest representation

The previous subsection dealt with polarity in the usual sense, viewing the gauge as a special representation. However, we are rather interested in *small* representations. Geometrically, we are interested in small prepolars, and the following definitions are indeed relevant:

$$\begin{cases} \widetilde{V}^\circ := \{d \in V^\circ : d^\top r = \sigma_V(d) = 1 \text{ for some } r \in V\}\,, \\ \widehat{V}^\circ := \{d \in V^\circ : \sigma_V(d) = 1\}\,. \end{cases} \tag{16}$$

Because of (9), $\widehat{V}^\circ \neq \emptyset$ if V has a boundary, i.e. if $V \neq \mathbb{R}^q$. Obviously, $\widetilde{V}^\circ \subset \widehat{V}^\circ$. Besides, (13) implies that the two sets are bounded. They turn out to have the same closed convex hull, which is our required smallest prepolar.

Lemma 3.3. *The sets in* (16) *satisfy* $\widetilde{V}^\circ \subset \widehat{V}^\circ \subset \operatorname{cl}\big(\widetilde{V}^\circ\big)$. *It follows that* $\widehat{V}^\circ$ *and* $\widetilde{V}^\circ$ *have the same closed convex hull. In particular,* $\widetilde{V}^\circ \neq \emptyset$ *whenever* $\widehat{V}^\circ \neq \emptyset$.

Proof. The first inclusion is clear. To prove the second inclusion, recall two properties:
– the domain $\operatorname{dom}\partial\sigma_V$ of a subdifferential is dense in the domain $\operatorname{dom}\sigma_V$ of the function itself: see, e.g., [14, Thm. E.1.4.2];
– the subdifferential $\partial\sigma_V(d)$ is the face of V exposed by d: see, e.g., [14, Prop. C.3.1.4].

Thus, $d \notin \widetilde{V}^\circ$ implies $\partial\sigma_V(d) = \emptyset$; in other words, $\widetilde{V}^\circ \supset \operatorname{dom}\partial\sigma_V$. Taking closures,

$$\operatorname{cl}\widetilde{V}^\circ \supset \operatorname{cl}(\operatorname{dom}\partial\sigma_V) \supset \operatorname{dom}\sigma_V\,;$$

the required inclusion follows, since the last set obviously contains $\widehat{V}^\circ$.

It follows from the second inclusion that

$$\overline{\operatorname{conv}}\big(\widehat{V}^\circ\big) \subset \overline{\operatorname{conv}}\Big(\operatorname{cl}\big(\widetilde{V}^\circ\big)\Big)\,.$$

On the other hand, the first inclusion implies that $\overline{\operatorname{conv}}\big(\widehat{V}^\circ\big)$ (a closed set) contains the closure of $\widetilde{V}^\circ$: $\operatorname{cl}\big(\widetilde{V}^\circ\big) \subset \overline{\operatorname{conv}}\big(\widehat{V}^\circ\big)$. This inclusion remains valid by taking the closed convex hulls:

$$\overline{\operatorname{conv}}\Big(\operatorname{cl}\big(\widetilde{V}^\circ\big)\Big) \subset \overline{\operatorname{conv}}\big(\widehat{V}^\circ\big)\,;$$

the two sets coincide. The last statement is clear since the closure of the empty set is the empty set. □

For later use, we illustrate this construction with a simple example.

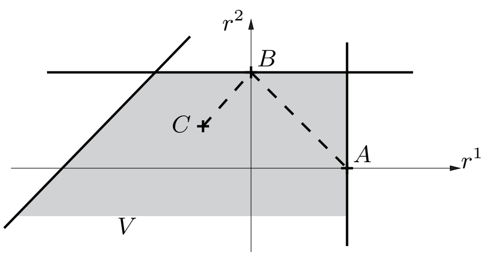

Figure 2: Constructing $\widetilde{V}^\circ$ or $\widehat{V}^\circ$

Example 3.4. We denote by (r^1, r^2) the coordinates of a point in $\mathbb{R}^2$. Take for V the polyhedron of Figure 2, defined by the three inequalities

$$r^1 \leqslant 1\,, \quad r^2 \leqslant 1\,, \quad r^2 \leqslant 2 + r^1\,.$$

Remembering that extreme points of V° correspond to facets of V, we see that V° has the three extreme points A, B, C defined by the equation $d^\top r = 1$, for r respectively on the three lines making up the boundary of V. We obtain $A = (1, 0)$, $B = (0, 1)$, $C = \frac{1}{2}(-1, 1)$.

In a word, $\widetilde{V}^\circ$ and $\widehat{V}^\circ$ are the same set, namely the union of the two segments $[A, B]$ and $[B, C]$. To obtain V°, convexify them with the fourth point 0; if V had a fourth constraint, say $r^2 \geqslant -1$, then this fourth point would be moved down to $D = (0, -1)$ – and enter $\widetilde{V}^\circ$ and $\widehat{V}^\circ$. □

The status of the boundary of the two sets in (16) is ambiguous. The left part of Figure 3 uses the same V as in Figure 1; d_1 lies in $\widehat{V}^\circ$ but not in $\widetilde{V}^\circ$; hence $\widetilde{V}^\circ$ need not be closed. By contrast, d_2 lies in both: on this example, $\widehat{V}^\circ$ is closed; but this is not true in general, the right part of Figure 3 is a classical counter-example. There, $V \subset \mathbb{R}^2$ is the parabolic set defined by the constraint $r^2 \leqslant 1 - \frac{1}{2}(r^1)^2$. A direction $d = (d^1, d^2)$ with $d^2 > 0$ exposes the point $r(d)$ which goes unbounded when $d^2 \downarrow 0$. By direct calculations, $\widehat{V}^\circ$ is defined by the equation

$$\sigma_V(d^1, d^2) = d^2 + \frac{(d^1)^2}{2d^2} = 1\,.$$

This is a curve passing through the origin; yet 0 cannot lie in $\widehat{V}^\circ$, since $\sigma_V(0) = 0 \neq 1$.

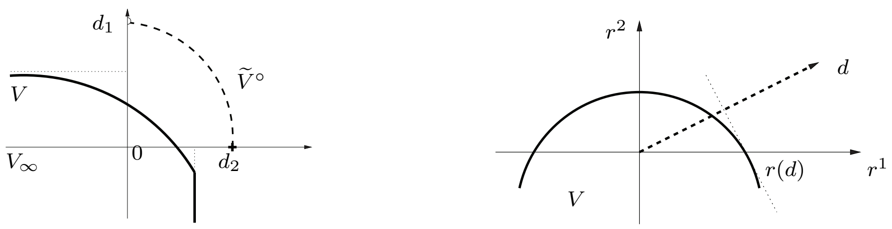

Figure 3: Activity in V°; trouble appears if the unbounded V has no asymptote

The closed convex hull revealed by Lemma 3.3 deserves a notation, as well as its support function: we set

$$V^\bullet := \overline{\operatorname{conv}}\big(\widetilde{V}^\circ\big) = \overline{\operatorname{conv}}\big(\widehat{V}^\circ\big) \quad \text{and} \quad \mu_V := \sigma_{V^\bullet} = \sigma_{\widetilde{V}^\circ} = \sigma_{\widehat{V}^\circ}\,. \tag{17}$$

For example in Figure 2, $V^\bullet$ is the triangle $\operatorname{conv}\{A, B, C\}$. In fact, the next result shows that μ_V is *the* small representation we are looking for. From now on, we assume $V \neq \mathbb{R}^q$, otherwise $V^\bullet = \emptyset$, $\mu_V \equiv -\infty$; a degenerate situation, which lacks interest anyway.

Proposition 3.5 (Smallest representation)**.** *Any ρ representing $V \neq \mathbb{R}^q$ satisfies $\rho \geqslant \mu_V$.*

Geometrically, $V^\bullet$ is the smallest closed convex set whose support function represents V.

Proof. Our assumption implies that neither $\widehat{V}^\circ$ nor $\widetilde{V}^\circ$ is empty (recall Lemma 3.3). Then take an arbitrary d in $\widetilde{V}^\circ$. We have to show that $d^\top r \leqslant \rho(r)$ for all $r \in \mathbb{R}^q$; this inequality will be transmitted to the supremum over d, which is $\mu_V(r)$.

Case 1. First let r be such that $\rho(r) > 0$. Then $\bar{r} := r/\rho(r)$ lies in V, so that $d^\top \bar{r} \leqslant \sigma_V(d) = 1$. In other words, $d^\top \bar{r} = \frac{d^\top r}{\rho(r)} \leqslant 1$, which is the required inequality.

Case 2. Let now r be such that $\rho(r) \leqslant 0$, so that $r \in V_\infty$ by Lemma 3.1. Because $d \in \widetilde{V}^\circ$, we can take $r_d \in V$ such that $d^\top r_d = 1$. Being exposed, r_d lies on the boundary of V: by (9), $\rho(r_d) = 1$.

By definition of the recession cone, $r_d + tr \in V$ for all $t > 0$ and, by continuity of ρ, $\rho(r_d + tr) > 0$ for t small enough. Apply Case 1:

$$d^\top r_d + t d^\top r = d^\top (r_d + tr) \leqslant \rho(r_d + tr) \leqslant \rho(r_d) + t\rho(r)\,,$$

where we have used sublinearity. This proves the required inequality since the first term is $1 + td^\top r$ and the last one is $1 + t\rho(r)$.

The geometric counterpart is proved just as in Corollary 3.2. $\square$

Thus, V does have a smallest representation, which is the support function of $V^\bullet$.

3.4 The set of prepolars

First of all, it is interesting to link the two extreme representations/prepolars introduced so far. The intuition suggested by Figure 2 or 3 is confirmed by the following result.

Proposition 3.6. *Appending* 0 *to* $V^\bullet$ *gives the standard polar:*

$$\gamma_V = \max\{\mu_V, 0\} \quad \textit{i.e.} \quad V^\circ = \overline{\mathrm{conv}}\big(V^\bullet \cup \{0\}\big) = [0,1]V^\bullet\,.$$

Proof. For $r \in V_\infty$, $\gamma_V(r) = 0$, while $\mu_V(r) \leqslant 0$ (Theorem 3.5). For $r \notin V_\infty$, Lemma 3.1 gives $\gamma_V(r) = \mu_V(r) > 0$ because γ_V and μ_V are two particular representations.

Altogether, the first equality holds. Its geometric counterpart is [14, Thm. C.3.3.2]; and because $V^\bullet$ is convex compact, its closed convex hull with 0 is the sets of $\alpha d + (1-\alpha)0$ for $\alpha \in [0,1]$. $\square$

Let us summarize our results:

Theorem 3.7. *The representations of* V *(a closed convex neighborhood of the origin) are the finite-valued sublinear functions* ρ *satisfying*

$$\sigma_{V^\bullet} = \mu_V \leqslant \rho \leqslant \gamma_V = \sigma_{V^\circ}\,. \tag{18}$$

Geometrically, the prepolars of V*, i.e. the sets* G *whose support function represents* V*, are the sets sandwiched between the two extreme prepolars of* V*:*

$$G^\circ = V \quad \Longleftrightarrow \quad V^\bullet \subset \overline{\mathrm{conv}}(G) \subset V^\circ\,.$$

Proof. In view of Corollary 3.2 and Proposition 3.5, we just have to prove that a ρ satisfying (18) does represent V. Indeed, if $r \in V$ then $\rho(r) \leqslant \gamma_V(r) \leqslant 1$; if $r \notin V$, then $1 < \mu_V(r) \leqslant \rho(r)$. The geometric counterpart is again standard calculus with support functions. $\square$

We end this section with a deeper study of prepolars, which will be useful in the sequel. The next result introduces the polar cone $(V_\infty)^\circ$. When G is a cone, the righthand side "1" in (12) can be replaced by "0":

$$(V_\infty)^\circ = \{r \in \mathbb{R}^q : \sigma_{V_\infty}(r) \leqslant 0\}\,. \tag{19}$$

Instead of $(V_\infty)^\circ$, we use the notation V_∞° for simplicity – although $(V_\infty)^\circ$ differs from $(V^\circ)_\infty$, which is $\{0\}$ since V° is bounded.

Lemma 3.8 (Additional properties of prepolars)**.** *Use the notation* (16), (17).

(i) V_∞° *is the closure of* $\mathrm{dom}\,\sigma_V$*;*

(ii) $\mathbb{R}_+\widehat{V}^\circ = \mathbb{R}_+V^\bullet = \mathbb{R}_+V^\circ = \mathrm{dom}\,\sigma_V$.

Proof. First of all, let $d \notin V_\infty^\circ$: there is $r \in V_\infty$ ($\mathbb{R}_+r \in V$) and $d^\top r > 0$; then $d^\top(tr) \to +\infty$ for $t \to +\infty$ and $\sigma_V(d)$ cannot be finite, i.e. $d \notin \mathrm{dom}\,\sigma_V$. Because V_∞° is closed, $\mathrm{dom}\,\sigma_V \subset \mathrm{cl}\,(\mathrm{dom}\,\sigma_V) \subset V_\infty^\circ$.

To prove the converse inclusion, take $r \notin (\mathrm{dom}\,\sigma_V)^\circ$: there is d such that $\sigma_V(d) < +\infty$ and $d^\top r > 0$. Then $d^\top(tr) \to +\infty$ when $t \to +\infty$; if r were in V_∞, then tr would lie in V and $\sigma_V(d)$ would be $+\infty$, a contradiction. Thus we have proved $V_\infty \subset (\mathrm{dom}\,\sigma_V)^\circ$. Taking polars and knowing that $\mathrm{dom}\,\sigma_V$ is a cone, $V_\infty^\circ \supset (\mathrm{dom}\,\sigma_V)^{\circ\circ} = \mathrm{cl}\,(\mathrm{dom}\,\sigma_V)$ (see [14, Prop. A.4.2.6]). This proves (i).

To prove (ii), observe first that $\widehat{V}^\circ \subset V^\bullet \subset V^\circ \subset \mathrm{dom}\,\sigma_V$; and because $\mathrm{dom}\,\sigma_V$ is a cone,

$$\mathbb{R}_+\widehat{V}^\circ \subset \mathbb{R}_+V^\bullet \subset \mathbb{R}_+V^\circ \subset \mathrm{dom}\,\sigma_V\,. \tag{20}$$

On the other hand, take $0 \neq d \in \mathrm{dom}\,\sigma_V$, so that $\sigma_V(d) > 0$ by (13) and $\frac{1}{\sigma_V(d)}d \in \widehat{V}^\circ$: $d \in \mathbb{R}_+\widehat{V}^\circ$. Since 0 also lies in $\mathbb{R}_+\widehat{V}^\circ$, we do have $\mathrm{dom}\,\sigma_V \subset \mathbb{R}_+\widehat{V}^\circ$; (20) is actually a chain of equalities. To complete the proof, observe from Proposition 3.6 that $\mathbb{R}_+V^\circ = \mathbb{R}_+V^\bullet$. $\square$

The situation illustrated by the right part of Figure 3 is really pathological; useful properties come when it does not occur.

Corollary 3.9 (Safe prepolars)**.** *If* $0 \notin V^\bullet$*, then*

$$\mathbb{R}_+\widehat{V}^\circ = \mathbb{R}_+V^\bullet = \mathbb{R}_+V^\circ = \operatorname{dom}\sigma_V = V_\infty^\circ \tag{21}$$

and $\operatorname{int} V_\infty \neq \emptyset$ *(the polar* V_∞° *is a so-called pointed cone).*

Proof. When $0 \notin V^\bullet$, $\mathbb{R}_+V^\bullet$ is closed ([14, Prop. A.1.4.7]). Then apply Lemma 3.8: by (ii) $\operatorname{dom}\sigma_V$ is closed and (21) follows from (i).

Besides, separate 0 from $V^\bullet$: there is some r such that $\sigma_{V^\bullet}(r) < 0$. By continuity of the finite-valued convex function $\sigma_{V^\bullet}$, this inequality is still valid in a neighborhood of r: $\sigma_{V^\bullet} \leqslant 0$ over some nonzero ball B around r. By Lemma 3.8(ii),

$$\sigma_{V_\infty^\circ}(d) = \sigma_{\mathbb{R}_+V^\bullet}(d) = \sup_{t\geqslant 0}\sup_{d\in V^\bullet} td^\top r = \sup_{t\geqslant 0} t\sigma_{V^\bullet}(d)\,,$$

so that $\sigma_{V_\infty^\circ}$ enjoys the same property: by (19), B is contained in $(V_\infty^\circ)^\circ$. Proposition A.4.2.6 of [14] finishes the proof. ☐

Let us put this section in perspective. The traditional gauge theory defines via (14), (12) the polarity correspondence $V \leftrightarrow V^\circ$ for *compact* convex neighborhoods of the origin. We generalize it to *unbounded* neighborhoods, whose standard gauge is replaced via Definition 2.6 by their family of representations. Each representation ρ, which may assume negative values, gives birth to $\partial\rho(0)$ – which we call a prepolar of V. Theorem 3.7 establishes the existence of a largest element (the usual polar V°) and of a smallest element ($V^\bullet$) in the family of (closed convex) prepolars of V.

4 Minimal CGF's, maximal S-free sets

4.1 Minimality

In our quest for small CGF's (remember Remark 1.4), the following definition is natural.

Definition 4.1 (Minimality)**.** A CGF ρ is called minimal if the only possible CGF $\rho' \leqslant \rho$ is ρ itself. ☐

Knowing that a CGF ρ represents $V(\rho)$ and that $\mu_{V(\rho)} \leqslant \rho$ represents the same set, a minimal CGF is certainly a smallest representation:

$$\rho \text{ is a minimal CGF} \quad\Longrightarrow\quad \rho = \mu_{V(\rho)} = \sigma_{V(\rho)^\bullet}\,. \tag{22}$$

In addition, $V(\rho)$ must of course be a special S-free set when ρ is minimal. Take for example $S = \{1\} \subset \mathbb{R}$, $V = [-1, +1]$; $\rho(r) := |r|$ is the smallest (because unique) representation of V but is not minimal: $\rho'(r) := \max\{0, r\}$ is also a CGF, representing $V' =]-\infty, +1]$. From (10), a smaller ρ describes a larger V; so Definition 4.1 has its geometrical counterpart:

Definition 4.2 (Maximality)**.** An S-free set V of Definition 2.4 is called *maximal* if the only possible S-free set $V' \supset V$ is V itself. ☐

We first make sure that maximal S-free sets do exist.

Theorem 4.3. *Every* S*-free set is contained in a maximal* S*-free set.*

Proof. Let V be an S-free set. In the partially ordered family $(\mathcal{F}, \cup)$ of all S-free sets containing V, let $\{W_i\}_{i\in I}$ be a totally ordered subfamily (a chain) and define $W := \cup_{i\in I} W_i$. Clearly, W is a neighborhood of the origin; its convexity is a known property (easily established), let us show that its closure is S-free.

Remember from [14, Thm. C.3.3.2(iii)] that the support function of an union is the (closure of the) supremum of the support functions:

$$\sigma_{\operatorname{int}(W)} = \sigma_W = \operatorname{cl}\left(\sup_{i\in I}\sigma_{W_i}\right) = \operatorname{cl}\left(\sup_{i\in I}\sigma_{\operatorname{int}(W_i)}\right) = \sigma_{\cup_i \operatorname{int}(W_i)}\,.$$

Having the same support function, the two open convex sets int (W) and $\cup_i$ int (W_i) coincide: $r \in$ int (W) means $r \in$ int (W_i) for some i; because W_i is S-free, $r \notin S$ and our claim is proved. Thus, the chain $\{W_i\}$ has an upper bound in $\mathcal{F}$; in view of Zorn's lemma, $\mathcal{F}$ has a maximal element. □

The maximal S-free sets can be explicitly described for some special S's: $\mathbb{Z}^q$ [17], the intersection of $\mathbb{Z}^q$ with an affine subspace [3], or with a rational polyhedron [4].

However, the "duality" between minimal CGF's and maximal S-free sets is deceiving, as the two definitions do not match: the set represented by a minimal CGF need not be maximal. Here is an example.

Example 4.4. When ρ is linear, the property introduced in Definition 4.1 holds vacuously: no sublinear function can properly lie below a linear function. Thus, a linear CGF ρ is always minimal; yet, a linear ρ represents a neighborhood $V(\rho)$ (a half-space) which is S-free but has not reason to be maximal. See Figure 4: with $n = 1$, the set $V =]-\infty, 1]$ (represented by $\rho(x) = x$) is $\{2\}$-free but is obviously not maximal. □

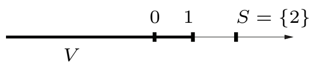

Figure 4: A linear CGF is always maximal

This example is rather naive but Example 4.8 below will reveal a more serious deficiency. So a subtlety is necessary: indeed the smallest representation of a maximal S-free set V enjoys a stronger property than minimality.

4.2 Strong minimality

Let ρ be a CGF, which represents via (6) the set $V = V(\rho)$. The largest representation $\gamma_{V(\rho)}$ somehow depends on ρ and here comes the correct substitute to Definition 4.1.

Definition 4.5 (Strongly minimal CGF)**.** A CGF ρ is called strongly minimal if every CGF $\rho' \leqslant \gamma_{V(\rho)}$ satisfies $\rho' \geqslant \rho$.

Needless to say, strongly minimal CGF's are minimal. Example 4.8 below will complement Example 4.4, showing that the two definitions do differ. At any rate, strong minimality does correspond to maximality.

Theorem 4.6 (Strongly minimal $\Leftrightarrow$ maximal)**.** *An S-free set V is maximal if and only if its smallest representation μ_V of* (17) *is a strongly minimal* CGF.

Proof. Consider first a maximal V. Every CGF ρ' satisfying $\rho' \leqslant \gamma_V$ represents an S-free set V', which contains V – see (6) – so that $V' = V$ by maximality and ρ' represents V as well, hence $\rho' \geqslant \mu_V$ by Theorem 3.5. Thus, μ_V is strongly minimal.

Assume now that ρ is a strongly minimal CGF, so in particular $\rho = \mu_V$ for some S-free V. Let $V' \supset V$ be S-free; there holds $(V')^\circ \subset V^\circ$, i.e.

$$\gamma_{V'} = \sigma_{(V')^\circ} \leqslant \sigma_{V^\circ} = \gamma_V \,.$$

Now $\rho' := \gamma_{V'}$ represents the S-free set V' and is therefore a CGF; in view of Definition 4.5, $\rho' \geqslant \rho$. Altogether, ρ' is a sublinear function satisfying (18): it represents not only V' but also V; hence $V' = V$, i.e. V is maximal. □

In §3 we have systematically developed the geometric counterpart of representations; this exercise can be continued here. In fact, the concept of minimality involves two properties from a sublinear function:

– it must be the *smallest* representation of some neighborhood V – remember (22),

– this neighborhood must enjoy some maximality property.

In view of the first property, a CGF can be imposed to be not only sublinear but also to support a set that is a *smallest* prepolar. Then Definition 4.1 has a geometric counterpart: minimality of $\rho = \mu_V = \sigma_{V^\bullet}$ means

$$\begin{array}{ccccc} G' \subset V^\bullet & \text{and} & (G')^\circ \text{ is } S\text{-free} & \Longrightarrow & G' = V^\bullet, \text{ i.e. } (G')^\circ = V \\ [\rho' = \sigma_{G'} \leqslant \rho] & & [\rho' \text{ is a CGF}] & & [\rho' = \rho] \end{array} .$$

Likewise for Definition 4.5: strong minimality of $\rho = \gamma_V = \sigma_{V^\circ}$ means

$$\begin{array}{ccccc} G' \subset V^\circ & \text{and} & (G')^\circ \text{ is } S\text{-free} & \Longrightarrow & G' \supset V^\bullet, \text{ i.e. } (G')^\circ \subset V \\ [\rho' = \sigma_{G'} \leqslant \gamma_V] & & [\rho' \text{ is a CGF}] & & [\rho' \geqslant \rho] \end{array} .$$

These observations allow some more insight into the $(\cdot)^\bullet$ operation:

Proposition 4.7. *Let $\rho = \mu_V = \sigma_{V^\bullet}$ be a minimal* CGF*. If an S-free neighborhood W satisfies $W^\bullet \subset V^\bullet$, then $W = V$.*

Proof. Because W is S-free, its smallest representation $\rho' := \mu_W = \sigma_{W^\bullet}$ is a CGF; and from monotonicity of the support operation, $\rho' \leqslant \rho$. Then minimality of ρ implies $\rho' = \rho$, i.e. $W^\bullet = V^\bullet$, an equality transmitted to the polars: $W = (W^\bullet)^\circ = (V^\bullet)^\circ = V$. ☐

Thus, the trouble necessitating strong minimality lies in (10). Even though the reverse implication holds when $\rho = \gamma_V$, it does not hold for $\rho = \mu_V$. Geometrically, $V \subset V'$ does imply $V^\circ \supset (V')^\circ$ but does not imply $V^\bullet \supset (V')^\bullet$. The mapping $V \mapsto V^\bullet$ is not monotonic, a phenomenon linked to the presence of V_∞. The following example helps for a better understanding.

Example 4.8. In Example 3.4, take for S the union of the three lines with respective equations

$$r^1 = 1\,, \quad r^2 = 1\,, \quad r^2 = 2 + r^1\,,$$

so that V is clearly maximal S-free.

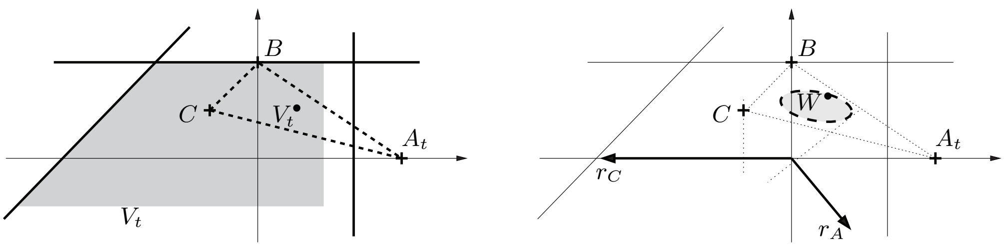

Figure 5: The mapping $V \mapsto V^\bullet$ is not monotonic

Now shrink V to V_t (left part of Figure 5) by moving its right vertical boundary to $r^1 \leqslant 1 - t$. Then A is moved to $A_t = \left(\frac{1}{1-t}, 0\right)$; there is no inclusion between the new $V_t^\bullet = \operatorname{conv}\{A_t, B, C\}$ and the original $V^\bullet = \operatorname{conv}\{A, B, C\}$; this is the key to our example.

Let us show that μ_{V_t} is minimal, even though V_t is not maximal. Take for this a CGF $\rho \leqslant \mu_{V_t}$, which represents an S-free set W; by (10), $W \supset V_t$. We therefore have

$$\sigma_{W^\bullet} = \mu_W \leqslant \rho \leqslant \mu_{V_t} = \sigma_{V_t^\bullet}\,, \quad \text{i.e.,} \quad W^\bullet \subset V_t^\bullet$$

and we proceed to show that equality does hold, i.e. the three extreme points of $V_t^\bullet$ do lie in $W^\bullet$.

– If $A_t \notin W^\bullet$, the right part of Figure 5 shows that $W^\bullet$ is included in the open upper half-space. Knowing that
$$W = \left(W^\bullet\right)^\circ = \left\{r : d^\top r \leqslant 1 \text{ for all } d \in W^\bullet\right\},$$
this implies that W_∞ has a vector of the form $r_A = (\varepsilon, -1)$ $(\varepsilon > 0)$; W cannot be S-free.

– If $C \notin W^\bullet$, there is $r_C \in \mathbb{R}^2$ such that $C^\top r_C > \sigma_{W^\bullet}(r_C) = \mu_W(r_C)$ (we denote also by C the 2-vector representing C). For example $r_C = (-2,0) \in \mathrm{bd}\,(V)$ (see the right part of Figure 5), so that

$$C^\top r_C = 1 > \sigma_{W^\bullet}(-2,0) = \mu_W(-2,0)\,.$$

By continuity, $\mu_W(-2-\varepsilon,0) \leqslant 1$ for $\varepsilon > 0$ small enough. Because μ_W represents W, this implies that $(-2-\varepsilon,0) \in W$; W (which contains V_t) is not S-free.

– By the same token, we prove that $B \in W^\bullet$ (the separator $r_B = (0,1) \in \mathrm{bd}\,(V)$ does the job).

We have therefore proved that $W^\bullet = V_t^\bullet$, i.e. $\mu_W = \mu_{V_t}$, i.e. μ_{V_t} is minimal. □

4.3 Asymptotic maximality

Then comes a natural question: how maximal are the S-free sets represented by minimal CGF's? For this, we introduce one more concept:

Definition 4.9. An S-free neighborhood V is called *asymptotically maximal* if every S-free set $V' \supset V$ satisfies $V'_\infty = V_\infty$.

It allows a partial answer to the question.

Theorem 4.10 (Minimal ⇒ asymptotically maximal)**.** *The S-free neighborhood represented by a minimal* CGF *is asymptotically maximal.*

Proof. Let μ_V be a minimal CGF and take an S-free neighborhood $V' \supset V$. Introduce the set $G := V^\bullet \cap \left(V'_\infty\right)^\circ$. Inclusions translate to inequalities between support functions:

$$\sigma_G \leqslant \sigma_{V^\bullet} = \mu_V \tag{23}$$

and we proceed to prove that this is actually an equality. Let us compute the set $W := G^\circ$ represented by σ_G. The support function of an intersection is obtained via an inf-convolution (formula (3.3.1) in [14, Chap. C)] for example): $\sigma_G(\cdot)$ is the closure of the function

$$r \mapsto \inf\left\{\sigma_{V^\bullet}(r_1) + \sigma_{(V'_\infty)^\circ}(r_2) : r_1 + r_2 = r\right\}.$$

In this formula, $\sigma_{V^\bullet} = \mu_V$ and the support function of the closed convex cone $\left(V'_\infty\right)^\circ$ is the indicator of its polar V'_∞: the above function is

$$r \mapsto \inf\left\{\mu_V(r_1) : r_1 + r_2 = r,\, r_2 \in V'_\infty\right\}.$$

Now use (9): because σ_G represents W, to say that $r \in \mathrm{int}\,(W)$ is to say that the above infimum is strictly smaller than 1, i.e. that there are r_1, r_2 such that

$$r_1 + r_2 = r,\; r_2 \in V'_\infty,\; \mu_V(r_1) < 1 \qquad \text{i.e.} \qquad r_1 + r_2 = r,\; r_2 \in V'_\infty,\; r_1 \in \mathrm{int}\,V\,.$$

In a word:

$$\mathrm{int}\,(W) = V'_\infty + \mathrm{int}\,(V) \supset \mathrm{int}\,(V) \ni 0\,,$$

where we have used the property $0 \in V'_\infty$. Remembering the inclusion $V \subset V'$ and the definition of a recession cone, we also have

$$\mathrm{int}\,(W) = V'_\infty + \mathrm{int}\,(V) \subset V'_\infty + \mathrm{int}\,(V') \subset V'_\infty + V' \subset V'\,.$$

Altogether,

$$0 \in \mathrm{int}\,(W) \subset \mathrm{int}\,(V')\,.$$

As a result, $W (= G^\circ)$ is an S-free closed convex neighborhood of the origin: its representation σ_G is a CGF and minimality of $\mu_V = \sigma_{V^\bullet}$ implies with (23) that $\sigma_G = \sigma_{V^\bullet}$.

Because $V^\bullet$ and $G = V^\bullet \cap \left(V'_\infty\right)^\circ$ are both closed convex, this just means $G = V^\bullet$, i.e. $\left(V'_\infty\right)^\circ \supset V^\bullet$. By polarity, $V'_\infty \subset \left(V^\bullet\right)^\circ = V$ (invoke Theorem 3.7). The cone V'_∞, contained in the neighborhood V, is also contained in its recession cone: $V'_\infty \subset V_\infty$. Since the converse inclusion is clear from $V' \supset V$, we have proved $V'_\infty = V_\infty$: V is asymptotically maximal. □

5 Favourable cases

Despite Example 4.8, a number of papers have established the equivalence between maximal S-free sets and minimal CGF's, for various forms of S. Accordingly, we investigate in this section the question: when does minimality imply strong minimality? So we consider an S-free set V, whose smallest representation $\mu_V = \sigma_{V^\bullet}$ is minimal; making use of Theorem 4.6, we want to exhibit conditions under which V is maximal. Our result is the following:

Theorem 5.1. *Suppose* $0 \in \hat{S} := \overline{\mathrm{conv}}(S)$. *A minimal* μ_V *is strongly minimal whenever one of the following two properties holds:*

(i) $V_\infty \cap \hat{S}_\infty = \{0\}$ *(in particular S bounded),*

(ii) $V_\infty \cap \hat{S}_\infty = L \cap \hat{S}_\infty$ *and* $\hat{S} = G + \hat{S}_\infty$ *with G bounded; here L is the lineality space of V.*

This theorem generalizes several earlier results. The special case where S is a finite set of points in $\mathbb{Z}^q - b$ was first considered by Johnson [15] and more recently by Dey and Wolsey [9]. The case $S = \mathbb{Z}^n$ was considered in [7] and intersection of $\mathbb{Z}^n$ with an affine space in [3]. Case (ii) was proven in [9, 4] in the special case where $S = P \cap (\mathbb{Z}^q - b)$ for some rational polyhedron P; then (ii) holds for all maximal V.

The proof of Theorem 5.1, rather involved, is based on Theorem 4.10. Remembering that the whole issue lies in unboundedness of V, we will construct a sequence of neighborhoods V^k such that $V^k_\infty \supsetneq V_\infty$ (hence V^k is not S-free), and containing an unbounded sequence $\{r^k\} \subset \mathrm{int}\,(V^k) \cap S$. Then we use this sequence to give conditions under which V is maximal.

For a reason that will appear in (30) below, we may assume $0 \notin V^\bullet$. To construct V^k, we take an extreme ray $\mathbb{R}_+ d_V$ of V°_∞. By (21), its intersection with $V^\bullet$ is a nonempty segment $[d_V, t_V d_V]$, with $1 \leqslant t_V < +\infty$. Given a positive integer k, we construct an open neighborhood of $[d_V, t_V d_V]$:

$$N^k := [d_V, t_V d_V] + B\Big(0, \frac{1}{k}\Big) = \bigcup_{1 \leqslant t \leqslant t_V} B\Big(t d_V, \frac{1}{k}\Big)\,, \tag{24}$$

where $B(d, \delta)$ is the open ball of center d and radius δ. We deprive $V^\bullet$ from N^k, thus obtaining a set C, closed hence compact; its convex hull

$$G^k := \mathrm{conv}\, C\,, \quad \text{with} \quad C := V^\bullet \backslash N^k = \Big\{d \in V^\bullet : \|d - t d_V\| \geqslant \frac{1}{k} \text{ for all } t \in [1, t_V]\Big\}$$

is convex compact. Note for future use that the distance from every $d \in [d_V, t_V d_V]$ to C does not exceed $1/k$; and the same holds for $G^k \supset C$. Formally:

$$\forall \bar{d} \in [d_V, t_V d_V]\,,\ \exists d_k \in G^k \ \text{ such that } \ \|d_k - \bar{d}\| \leqslant \frac{1}{k}\,. \tag{25}$$

Viewing G^k as a prepolar, we set

$$V^k := (G^k)^\circ\,.$$

Of course, $V^\bullet \supset G^{k+1} \supset G^k$ and $V \subset V^{k+1} \subset V^k$. The closed convex neighborhood V^k enjoys all of the properties listed in §3, in particular those coming from $0 \notin G^k$. Figure 6 illustrates our construction, motivated by the following result.

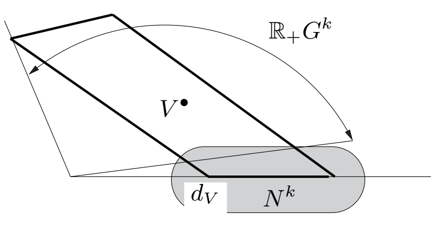

Figure 6: Chopping off $V^\bullet$ near an extreme ray

Lemma 5.2 (Enlarging V_∞). *Assume $0 \notin V^\bullet$; let $\mathbb{R}_+ d_V$ be an extreme ray of V_∞° and assume that V_∞° is not reduced to $\mathbb{R}_+ d_V$. Given an integer $k > 0$, construct N^k, G^k, V^k as above. Then $G^k \neq \emptyset$ for k large enough (say $k \geqslant k_0$) and*

(i) $V_\infty \subsetneq V_\infty^k$ *for $k \geqslant k_0$ (V_∞ is properly contained in V_∞^k),*

(ii) $\cap_{k \geqslant k_0} V^k = V$.

Proof. If G^k were empty for all k, we would have $V^\bullet \subset N^k$ for all k, hence $V^\bullet$ would reduce to $[d_V, t_V d_V]$. In view of (21), this would imply $\mathbb{R}_+ d_V = V_\infty^\circ$, which our assumption rules out.

Every $d \in G^k$ is a convex combination $\sum_i \alpha_i d_i$ with each d_i in $V^\bullet \backslash N^k \subset V_\infty^\circ$. None of these d_i's can lie in $[d_V, t_V d_V] \subset N^k$, and none of their convex combinations either because of extremality of $\mathbb{R}_+ d_V$. We conclude that

$$G^k \cap [d_V, t_V d_V] = \emptyset \,. \tag{26}$$

Now, we see from Theorem 3.7 that

$$\mathbb{R}_+ (V^k)^\bullet \subset \mathbb{R}_+ G^k \subset \mathbb{R}_+ (V^k)^\circ \,;$$

but from Proposition 3.6, this is actually a chain of equalities:

$$\mathbb{R}_+ (V^k)^\bullet = \mathbb{R}_+ G^k \,. \tag{27}$$

Besides, $(V^k)^\bullet \subset G^k \subset V^\bullet$, hence $0 \notin (V^k)^\bullet$ and we can apply (21) to V^k. Then we write

$$\begin{aligned} \left(V_\infty^k\right)^\circ &= \mathbb{R}_+ (V^k)^\bullet && \text{[(21)]} \\ &= \mathbb{R}_+ G^k && \text{[(27)]} \\ &\subsetneq \mathbb{R}_+ V^\bullet && \text{[consequence of (26)]} \\ &= V_\infty^\circ \,. && \text{[(21) again]} \end{aligned}$$

Thus, $\left(V_\infty^k\right)^\circ \subsetneq V_\infty^\circ$, which implies (i) since polarity is an involution between closed convex cones.

To prove (ii), take $\bar r$ in $\cap_k V^k$; we have to prove that $\bar r \in V$ (the other inclusion being obvious). If $\bar r \notin V$ there is a separating hyperplane $\bar d$: $\sigma_V(\bar d) < \bar d^\top \bar r$. Normalizing $\bar d$ via (21), we have altogether

$$\bar r \in \bigcap_k V^k \,, \quad \bar d \in \widehat{V}^\circ \,, \quad \bar d^\top \bar r > 1 \,. \tag{28}$$

Because σ_{G^k} represents V^k, we obtain with (28)

$$\sigma_{G^k}(\bar r) \leqslant 1 < \bar d^\top \bar r \,, \quad \text{hence } \bar d \notin G^k \,.$$

Then $\bar d \in V^\bullet \cap N^k$ for all k (large enough), i.e. $\bar d \in [d_V, t_V d_V]$. Introduce $d_k \in G^k$ from (25):

$$\|d_k - \bar d\| \leqslant \frac{1}{k} \quad \text{and} \quad d_k^\top \bar r \leqslant \sigma_{G^k}(\bar r) \leqslant 1 \,.$$

Passing to the limit, $\bar d^\top \bar r \leqslant 1$; a contradiction to (28). Therefore $\bar r \in V$. □

Now we assume the existence of an S-free set W containing V; it satisfies in particular

$$W^\bullet \subset W^\circ \subset V^\circ = [0,1] V^\bullet \,. \tag{29}$$

If $W^\bullet \subset V^\bullet$, this W is of no use to disprove maximality of V (Proposition 4.7). We are therefore in the situation

$$W^\bullet \not\subset V^\bullet \,, \quad \text{which implies from (29):} \quad 0 \notin V^\bullet \,. \tag{30}$$

Thus, $W^\bullet$ contains some points out of $V^\bullet$. The key argument for our analysis is that one of these points lies on an extreme ray of V_∞°.

Lemma 5.3 (Constructing an appropriate extreme ray). *Let $W \supset V$ satisfy (30). There is an extreme ray $\mathbb{R}_+ d_V$ of V_∞° such that the set N^k defined by (24) satisfies $W^\circ \cap N^k = \emptyset$ for k large enough.*

Proof. Because of (30), we are in the framework of Corollary 3.9; Figure 7 is helpful to follow the proof. If $\widehat{W}^\circ \subset V^\bullet$ then $W^\bullet = \overline{\text{conv}}(\widehat{W}^\circ) \subset V^\bullet$, contradiction. So there is $e \in \widehat{W}^\circ$ (hence $\sigma_W(e) = 1$) which does not lie in $V^\bullet$. Because $V \subset W$, i.e. $\sigma_V \leqslant \sigma_W$, this e satisfies $\sigma_V(e) < 1$ (otherwise $\sigma_V(e) = 1$, hence $e \in \widehat{V}^\circ \subset V^\bullet$).

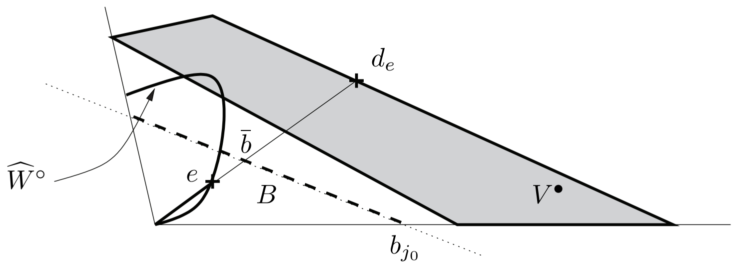

Figure 7: The extreme ray $\mathbb{R}_+ b_{j_0}$ contains some point in $V^\bullet \backslash W^\bullet$

Then construct $d_e := \frac{1}{\sigma_V(e)} e \in \widehat{V}^\circ$ (remember (13): $\sigma_V(e) > 0$). For every $e' \in [0, e]$, the segment $[e', d_e]$ contains e. Being a convex set, $V^\bullet$ cannot contain such an e' (otherwise it would contain e as well). As a result, the compact convex sets $V^\bullet$ and $[0, e]$ can be separated: there is $\ell \in \mathbb{R}^q$ (appropriately scaled) such that

$$\max\{0, e^\top \ell\} < 1 < \min_{d \in V^\bullet} d^\top \ell \,. \tag{31}$$

Observe that

$$1 > e^\top \ell = \sigma_V(e) d_e^\top \ell > 0 \,. \tag{32}$$

Now introduce the closed convex set

$$B := \left\{ b \in V_\infty^\circ : b^\top \ell = 1 \right\} .$$

Clearly, $\mathbb{R}_+ B \subset V_\infty^\circ$. Conversely, apply (21): every nonzero $d \in V_\infty^\circ$ can be scaled to some $td \in V^\bullet$. By (31), $td^\top \ell > 1$, then d can be scaled again to $td/(td^\top \ell)$, which lies in B. We have shown

$$\mathbb{R}_+ B = V_\infty^\circ \,. \tag{33}$$

By (21), every $b \in B$ can be obtained by scaling some $d \in \widehat{V}^\circ$: $b = td$; and $t = \frac{1}{d^\top \ell} \in]0, 1[$ by (31). This means that

$$B \subset]0, 1[\widehat{V}^\circ \subset V^\circ \,; \tag{34}$$

B is therefore bounded (and closed because V_∞° is closed), hence compact.

Using (32), scale e to $\bar{b} := \frac{1}{e^\top \ell} e \in B$ and express $\bar{b} = \sum_j \alpha_j b_j$ as a convex combination of extreme points b_j of B (Minkowski's Theorem). Then

$$\sigma_W(\bar{b}) = \frac{1}{e^\top \ell} \sigma_W(e) = \frac{1}{e^\top \ell} > 1 \,.$$

By convexity of σ_W, there is some j_0 such that $\sigma_W(b_{j_0}) > 1$ (we may have $\sigma_W(b_{j_0}) = +\infty$). Altogether, we have exhibited

$$b_{j_0} \text{ extreme in } B \text{ and satisfying } 1 < \sigma_W(b_{j_0}) \,.$$

Extremality of b_{j_0} in B implies extremality of the ray $\mathbb{R}_+ b_{j_0}$ in $\mathbb{R}_+ B$, i.e. in V_∞° because of (33). The intersection of W° with this extreme ray is some $[0, d_W]$ (d_W may be 0) which, by definition of a polar, does not contain b_{j_0}. Since $b_{j_0}^\top \ell = 1$ (because $b_{j_0} \in B$), $d^\top \ell < 1$ for all $d \in [0, d_W]$. Then, (31) shows that $[0, d_W]$ and $[d_V, t_V d_V]$ are separated.

As a result, the two compact sets W° and $[d_V, t_V d_V]$ are disjoint. If there were $d^k \in W^\circ \cap N^k$ for all k, then the bounded sequence $\{d^k\}$ would have some cluster point d^*; but W° is closed: d^* would lie in $W^\circ \cap [d_V, t_V d_V]$, contradiction. □

The set B constructed in the above proof is a so-called basis of the pointed cone V°_∞. The case $\sigma_W(b_{j_0}) = +\infty$, $d_W = 0$ corresponds to a W as in the right part of Figure 3; it occurs in Figure 7. This latter picture is still helpful to follow the proof of Theorem 5.1, which comes next.

Proof of Theorem 5.1. Take $W \supset V$ such that $W^\bullet \not\subset V^\bullet$ and construct d_V as in Lemma 5.3.

Claim 1: V°_∞ does not reduce to $\mathbb{R}_+ d_V$. Otherwise $\widehat{V}^\circ = V^\bullet = \{d_V\}$ and $V^\circ = [0, d_V]$ (Proposition 3.6): the S-free neighborhood V, represented by σ_{V°, is the half-space $\{r : d_V^\top r \leqslant 1\}$, which separates 0 from $\hat{S}$; this is ruled out by assumption.

We can therefore construct V^k of Lemma 5.2. If μ_V is minimal, V^k cannot be S-free (Lemma 5.2(i) and Theorem 4.10): there exists r^k lying

– in $\operatorname{int} V^k$, hence from (9)

$$1 > \sigma_{G^k}(r^k)\,, \tag{35}$$

– and in S, hence $r^k \notin \operatorname{int} W$: $\sigma_{W^\bullet}(r^k) \geqslant 1$; since $W^\bullet$ is compact,

$$\exists e_k \in W^\bullet \text{ such that } e_k^\top r^k \geqslant 1\,. \tag{36}$$

Claim 2: There is $\delta > 0$ such that

$$t_k e_k \in V^\bullet \cap N^k\,, \quad \text{for some } t_k \geqslant 1+\delta \text{ and all } k \text{ large enough}\,. \tag{37}$$

Using (21), scale e_k (nonzero from its definition) to $t_k e_k \in V^\bullet$; and note from (29) that $t_k \geqslant 1$. Then (36) implies that $t_k e_k \notin G^k$: otherwise

$$1 \leqslant e_k^\top r^k \leqslant t_k e_k^\top r^k \leqslant \sigma_{G^k}(r^k)$$

by definition of a support function; this contradicts (35). It follows that $t_k e_k \in V^\bullet \cap N^k$, which is far from $W^\bullet$ (Lemma 5.3); (37) is proved.

Key Claim: Decompose $r^k = u^k + \ell^k$ with $u^k \in L^\perp$ and $\ell^k \in L$; there is $K \subset \mathbb{N}$ such that

$$\lim_{k\in K} \|r^k\| = +\infty \quad \text{and} \quad \lim_{k \in K} \|u^k\| = +\infty\,.$$

First, let $\bar{d} \in [d_V, t_V d_V]$ be a cluster point of the bounded sequence $\{t_k e_k\}$. Next, use (37), (36), (35) to write for all $d \in G^k$

$$1 + \delta \leqslant t_k \leqslant t_k e_k^\top r^k = (t_k e_k - d)^\top r^k + d^\top r^k < (t_k e_k - d)^\top r^k + 1\,.$$

This holds in particular for $d = d_k$ stated in (25):

$$\delta < (t_k e_k - d_k)^\top r^k\,. \tag{38}$$

Then we obtain with the Cauchy-Schwarz inequality

$$\delta < \|t_k e_k - \bar{d} + \bar{d} - d_k\|\,\|r^k\| \leqslant \Big(\|t_k e_k - \bar{d}\| + \frac{1}{k}\Big)\|r^k\|\,.$$

Furthermore, decompose $r^k = u^k + \ell^k$ in (38) and observe that both $e_k^\top \ell^k$ and $d_k^\top \ell^k$ are 0 ($\ell^k \in L$ while e^k and d^k lie in $V^\circ_\infty \subset L^\perp$). So (38) gives also

$$\delta < (t_k e_k - d_k)^\top u^k \leqslant \Big(\|t_k e_k - \bar{d}\| + \frac{1}{k}\Big)\,\|u^k\|\,.$$

Both statements of our key claim are proved since there is $K \subset \mathbb{N}$ such that $\lim_{k\in K} \|t_k e_k - \bar{d}\| = 0$.

Case (i): Extract a cluster point $\hat{r}$ of the normalized subsequence $\{r^k\}_{k\in K}$: for some $K' \subset K$,

$$\lim_{k\in K'} \frac{r^k}{\|r^k\|} = \hat{r}\,.$$

Then take an arbitrary $M > 0$. We know that $M/\|r^k\| \leqslant 1$ if k is large enough in K' so, because both 0 and r^k lie in $V^k \cap \hat{S}$,

$$\frac{M}{\|r^k\|} r^k \in V^k \cap \hat{S}\,, \quad \text{for large enough } k \in K'\,.$$

By closedness, this implies $M\hat{r} \in \hat{S}$, hence $\hat{r} \in \hat{S}_\infty$ because M is arbitrary. The same argument using Lemma 5.2(ii) gives $\hat{r} \in V_\infty$.

Let us sum up. In case (i), the above $\hat{r}$ cannot exist: no W containing V can have $W^\bullet \not\subset V^\bullet$; from Proposition 4.7, V is maximal, i.e. μ_V is strongly minimal.

Case (ii): Write $u^k = r^k - \ell^k \in V^k - L = V^k + L \subset V^k + V_\infty \subset V^k$. Then proceed as in Case (i): extract a cluster point $\hat{u}$ of $\left\{\frac{u^k}{\|u^k\|}\right\}_K$ and argue that $\frac{M}{\|u^k\|} u^k \in V^k \cap L^\perp$ to exhibit

$$\hat{u} \in V_\infty \cap L^\perp \quad \text{and} \quad \|\hat{u}\| = 1\,. \tag{39}$$

Besides, u^k is the projection onto $L^\perp$ (a linear operator) of $r^k \in S \subset G + \hat{S}_\infty$. Hence

$$u^k \in \operatorname{Proj}_{L^\perp} G + \operatorname{Proj}_{L^\perp} \hat{S}_\infty$$

and, because $\operatorname{Proj}_{L^\perp} G$ is a bounded set, our cluster direction $\hat{u}$ lies in $\operatorname{Proj}_{L^\perp} \hat{S}_\infty$:

$$\hat{u} = \hat{s} - \hat{\ell}\,, \quad \text{for some } \hat{s} \in \hat{S}_\infty \text{ and } \hat{\ell} \in L\,.$$

Use (39):

$$\hat{S}_\infty \ni \hat{s} = \hat{u} + \hat{\ell} \in V_\infty + L = V_\infty\,;$$

then use (ii):

$$\hat{s} \in V_\infty \cap \hat{S}_\infty = L \cap \hat{S}_\infty\,.$$

As a result, $\hat{u} = \hat{s} - \hat{\ell}$ lies in L; use (39) again: $\hat{u} \in L \cap L^\perp$ cannot have norm 1.

Thus, in this case also, no $W \supset V$ exists such that $W^\bullet \not\subset V^\bullet$; μ_V is strongly minimal. □

6 Conclusion and perspectives

In this paper, we have laid down some basic theory toward studying the cutting paradigm for sets of the form (1); thus putting in perspective an abundant literature devoted to S-free sets. We have disclosed the concept of strong minimality, which corresponds to maximal S-freeness; and we have given a result to recover existing theorems [15, 7, 3, 9, 4], dealing with mere minimality. This theory necessitates a generalization of the polarity correspondence to unbounded sets; we have re-proven and revisited the results of [5], revealing the distinction between smallest and largest prepolars.

A number of questions arise from this theoretical work. Some are suggested by Section 3:

Question 1. Given a convex compact set G, can we compute the minimal prepolar of $V = G^\circ$? or at least detect whether $V^\bullet = G$? Similarly, does there exist a simple construction of $V^\bullet$ or μ_V, skipping (16)?

Question 2. Can the polarity theory be further generalized to sets V having the origin on their boundary, or even not containing the origin at all?

These are limited to pure convex analysis; concerning the CGF theory itself, some other questions have a concrete interest.

Question 3. Is it possible to characterize exactly the S-free sets represented by minimal CGF's? a converse form of Theorem 4.10 should be desirable.

Question 4. One might want to consider more general models. For example, it should not be too difficult to replace the "ground set" $\mathbb{R}^n_+$ of (1a) by some other closed convex cone; say the cone of positive semi-definite matrices, which would open the way toward cutting SDP relaxations. Another generalization would be inspired by the approach of [11] of Example 1.1: there, X has the form

$$\left\{x \in \mathbb{Z}^n_+ : -Ax \in \mathbb{Z}^m - b\right\};$$

$S = \mathbb{Z}^m - b$ lies in a smaller space but the ground set $\mathbb{Z}^n_+$ is no longer convex, so sublinear CGF's are now ruled out. Instead, CGF's in this context are subadditive, periodic, and satisfy a certain symmetry condition [12].

Question 5. Perhaps the most crucial question is whether CGF's do generate all possible cuts, i.e., whether (5) is able to produce all possible c's satisfying (2). This turns out to be a tough nut to crack, we conclude the paper with some considerations for future research concerning it.

The following counter-example shows that the answer to Question 5 is no in general.

Example 6.1 (CGF's need not generate all cuts)**.** In $\mathbb{R}^2$, take $S = (0,1) \cup \{(\mathbb{Z}, -1)\}$ (we take row-vectors for typographical convenience). The left part of Figure 8, drawn in the S-space, clearly shows that, if the unit-vector $(1,0)$ lies in the recession cone of an S-free neighborhood V, then it lies on the boundary of this cone.

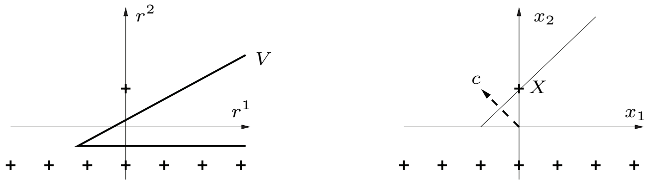

Figure 8: Not all cuts are obtained from a CGF

Now take the identity matrix for R: X reduces to the singleton $(0,1)$ in $\mathbb{R}^2$ (right part of Figure 8). It can be separated from the origin by the cut $x_2 \geqslant x_1 + 1$, obtained with $c = (-1,1)^\top$. Knowing that the first column of R is $r_1 = (1,0)^\top$, a CGF ρ producing this c must therefore have $\rho(r_1) = -1$. In view of Lemma 3.1, $(1,0)$ lies in the interior of V_∞; but we have seen that no V can satisfy this. $\square$

Negative c_j's are therefore troublesome, a general sufficiency theorem is out of reach. To eliminate $c_j < 0$, we can restrict the class of instances:

Proposition 6.2. *If the recession cone of $\overline{\text{conv}}(X)$ is the whole of $\mathbb{R}^n_+$, then every cut c lies in $\mathbb{R}^n_+$.*

Proof. Each basis vector e_j of $\mathbb{R}^n$ lies in $[\overline{\text{conv}}(X)]_\infty$: picking some $x \in X$,

$$c^\top(x + te_j) = c^\top x + tc_j \geqslant 1 \quad \text{for all } t \geqslant 0\,;$$

let $t \to +\infty$ to see that $c_j \geqslant 0$. $\square$

To overcome the above difficulty, a line of attack might follow Question 4 above: restrict the class of CGF's by changing (1a) to

$$\{x \in \mathbb{R}^n_+ \cap B : Rx \in S\}\,.$$

On the other hand, the restriction imposed by Proposition 6.2 does not suffice, as even $c_j = 0$ brings trouble. In fact, make a "more nonlinear" variant of Example 6.1: instead of the horizontal line $r^2 = -1$, take for S the curve $r^2 = -1/|r^1|$ $(r^1 \neq 0)$. This leaves $X = \{(0,1)\}$ unchanged; $c = (0,1)^\top$ is a cut and a CGF ρ generating it has $\rho(r_1) = 0$; this ρ represents a set $V(\rho)$ which has $(\mathbb{R}_+, 0)$ in its recession cone. Being a neighborhood of the origin, $V(\rho)$ contains $A := (0, -\varepsilon)$ for small enough $\varepsilon > 0$; also, $B := (r, 0) \in V(\rho)_\infty \subset V(\rho)$ for all $r > 0$ (see Figure 9); by convexity, the whole segment $[A, B]$ lies in $V(\rho)$, which therefore cannot be S-free.

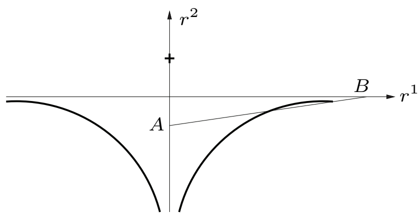

Figure 9: Trouble appears when V_∞ is an asymptote of S

In these two examples, the conical hull of the r_j's does not cover the whole of S. In fact, S contains points that can be reached by no $x \in \mathbb{R}^n_+$; these points have nothing to do with the problem, so forcing

V not to contain them is unduly demanding. In a way, the diffulty is again linked to Question 4. Then one may ask whether CGF's are able to describe all possible cuts, for all possible instances such that $S \subset \text{cone}(r_1, \ldots, r_n)$. This is an open question; here we limit ourselves to a reasonably simple sufficiency result, proved with the help of a "comfortable" assumption.

Theorem 6.3. *Let an instance of* (1) *be as described by Proposition* 6.2 *and assume*

$$\text{cone}(r_1, \ldots, r_n) := \Big\{ \sum_{j=1}^{n} \lambda_j r_j : \lambda_j \geqslant 0,\, j = 1, \ldots, n \Big\} = \mathbb{R}^q .$$

Then every cut can be obtained from a CGF.

Proof. Let $c \in \mathbb{R}^n_+$ and set

$$J_+ := \big\{ j \in \{1, \ldots, n\} : c_j > 0 \big\} , \quad J_0 := \big\{ j \in \{1, \ldots, n\} : c_j = 0 \big\} .$$

Then introduce in $\mathbb{R}^q$ the vectors

$$r'_j := \frac{r_j}{c_j} , \quad \text{for } j \in J_+$$

and the polyhedron

$$V := G + K , \quad \text{with} \begin{cases} G := \text{conv}\{r'_j : j \in J_+\} , \\ K := \text{cone}\{r_j : j \in J_0\} . \end{cases}$$

Claim 1: V is a neighborhood of the origin. In fact, our assumption means that $\mathbb{R}^q = \text{cone}(G) + K$: every $\bar d \in \mathbb{R}^n$ has the form

$$\bar d = \bar t \bar g + \bar k , \quad \text{with} \quad \bar t \geqslant 0 ,\ \bar g \in G ,\ \bar k \in K .$$

Then compute $\sigma_V(\bar d)$ for nonzero $\bar d$.

– Case 1: $\bar t = 0$. Fixing $g \in G$ so that $g + t\bar k \in V$ for all $t \geqslant 0$, we have

$$\sigma_V(\bar d) = \sigma_V(\bar k) \geqslant \bar k^\top (g + t\bar k) = \bar k^\top g + t\|\bar k\|^2 , \quad \text{for all } t > 0 ;$$

let $t \to +\infty$ to see that $\sigma_V(\bar d) = +\infty$.

– Case 2: $\bar t > 0$. Scale $\bar d$ to $\bar t^{-1}\bar d \in G + K = V$ to obtain $\sigma_V(\bar d) \geqslant \bar t^{-1}\|\bar d\|^2 > 0$.

Altogether, we have proved that $\sigma_V(\bar d) > 0$ for all $\bar d \neq 0$, i.e. $0 \in \text{int}(V)$.

Claim 2: V is S-free. Take $\bar r \in \text{int}(V)$. For $\varepsilon > 0$ small enough, $\bar r + \varepsilon \bar r \in V$:

$$(1+\varepsilon)\bar r = \sum_{j \in J_+} \beta_j r'_j + \sum_{j \in J_0} \mu_j r_j , \quad \text{with } \beta_j, \mu_j \geqslant 0, \sum_{j \in J_+} \beta_j = 1 .$$

Divide by $1 + \varepsilon$ and set $\alpha_j = \beta_j/(1+\varepsilon)$, $\lambda_j = \mu_j/(1+\varepsilon)$ to get

$$\bar r = \sum_{j \in J_+} \alpha_j r'_j + \sum_{j \in J_0} \lambda_j r_j, , \quad \text{for } \alpha_j, \lambda_j \geqslant 0 ,\ \sum_{j=1}^{n} \alpha_j < 1 .$$

Introduce the vector $\bar x \in \mathbb{R}^n$ whose coordinates are

$$\bar x_j := \begin{cases} \dfrac{\alpha_j}{c_j} & \text{if } j \in J_+ , \\ \lambda_j & \text{if } j \in J_0 . \end{cases}$$

Observe that $\bar x \geqslant 0$ and that

$$R\bar x = \sum_{j=1}^{n} \bar x_j r_j = \sum_{j \in J_+} \frac{\alpha_j}{c_j} r_j + \sum_{j \in J_0} \lambda_j r_j = \bar r .$$

If $\bar{r} \in S$ then $x \in X$ by definition (1a); but

$$c^\top \bar{x} = \sum_{j \in J_+} c_j \frac{\alpha_j}{c_j} = \sum_{j \in J_+} \alpha_j \leqslant \sum_{j=1}^{n} \alpha_j < 1$$

and x cannot lie in X if c is a cut. We have proved that $\operatorname{int}(V) \cap S = \emptyset$, i.e. that V is S-free.

Conclusion: We have proved that the gauge γ_V is a CGF; besides
– for $j \in J_0$, r_j is a direction of recession of V: $\gamma_V(r_j) = 0 = c_j$;
– for $j \in J_+$, the property $r'_j \in V$ gives

$$1 \geqslant \gamma_V(r'_j) = \frac{1}{c_j}\gamma_V(r_j)\,, \quad \text{hence} \quad \gamma_V(r_j) \leqslant c_j\,.$$

In summary, γ_V is a CGF dominating the cut c. □

To make Question 5 less ambitious, one may ask whether CGF's can reproduce the set of cuts "globally". In fact, the set of c's satisfying (2) is a closed convex set: the opposite of the reverse polar X^-, in the terminology of [2, 8]. Then consider the set $\mathcal{R}_S$ of all representations of a given S-free set. Given (n, R), form the set $\mathcal{C}$ of $c \in \mathbb{R}^n$ whose coordinates are $\rho(r_j)$, where ρ describes $\mathcal{R}_S$. Is it true that $\overline{\operatorname{conv}}(\mathcal{C}) = -X^-$? This question is open. If the answer is yes, one more question occurs: Example 4.8 tells us that $\mathcal{R}_S$ cannot be reduced to the maximal S-free sets; then, what sort of maximality can be imposed while preserving "completeness" of $\mathcal{R}_S$? An answer should need answering Question 3 first.

References

[1] K. Andersen, Q. Louveaux, R. Weismantel, and L. Wolsey. Cutting planes from two rows of a simplex tableau. In M. Fischetti and D. Williamson, editors, *Integer Programming and Combinatorial Optimization*, volume 4513 of *Lecture Notes in Computer Science*, pages 1–15. Springer Verlag, 2007.

[2] E. Balas. Disjunctive programming. *Ann. Disc. Math.*, 5:3–51, 1979.

[3] A. Basu, M. Conforti, G. Cornuéjols, and G. Zambelli. Maximal lattice-free convex sets in linear subspaces. *Math. Oper. Res.*, 35(3):704–720, 2010.

[4] A. Basu, M. Conforti, G. Cornuéjols, and G. Zambelli. Minimal inequalities for an infinite relaxation of integer programs. *SIAM J. on Disc. Math.*, 24(1):158–168, 2010.

[5] A. Basu, G. Cornuéjols, and G. Zambelli. Convex sets and minimal sublinear functions. *J. Convex Anal.*, 18(2):427–432, 2011.

[6] R. Bixby and E. Rothberg. Progress in computational mixed integer programming – a look back from the other side of the tipping point. *Ann. Op. Res.*, 149(1):37–41, 2007.

[7] V. Borozan and G. Cornuéjols. Minimal valid inequalities for integer constraints. *Math. Oper. Res.*, 34(3):538–546, 2009.

[8] G. Cornuéjols and C. Lemaréchal. A convex-analysis perspective on disjunctive cuts. *Math. Program.*, 106(3):567–586, 2005.

[9] S. S. Dey and L. A. Wolsey. Constrained infinite group relaxations of mips. *SIAM J. on Opt.*, 20(6):2890–2912, September 2010.

[10] R. E. Gomory. An algorithm for integer solutions to linear programs. In R. L. Graves and P. Wolfe, editors, *Recent Advances in Mathematical Programming*, pages 269–302. McGraw-Hill, 1963.

[11] R. E. Gomory. Some polyhedra related to combinatorial problems. *Lin. Alg. Appl.*, 2(4):451–558, 1969.

[12] R. E. Gomory and E. L. Johnson. Some continuous functions related to corner polyhedra i. *Math. Program.*, 3:23–85, 1972.

[13] J.-B. Hiriart-Urruty and C. Lemaréchal. *Convex Analysis and Minimization Algorithms.* Springer Verlag, 1993.

[14] J.-B. Hiriart-Urruty and C. Lemaréchal. *Fundamentals of Convex Analysis.* Springer Verlag, 2001.

[15] E. L. Johnson. Characterization of facets for multiple right-hand side choice linear programs. *Math. Program. Study 14*, pages 112–142, 1981.

[16] J. J. Júdice, H. Sherali, I. M. Ribeiro, and A. M. Faustino. A complementarity-based partioning and disjujnctive cut algorithm for mathematical programming problems with equilibrium constraints. *J. Global Opt.*, 136:89–114, 2006.

[17] L. Lovász. Geometry of numbers and integer programming. In M. Iri and K. Tanabe, editors, *Mathematical Programming: Recent Developements and Applications*, pages 177–210. Kluwer, 1989.

[18] R.T. Rockafellar. *Convex Analysis.* Princeton University Press, 1970.