Trace convexity and Choquet theory.

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Abstract. We study the notion of trace-convexity for functions and respectively, for subsets of a compact topological space. This notion generalizes both classical convexity of vector spaces, as well as Choquet convexity for compact metric spaces and provides an alternative description for the convexification for sets and functions. We show that the class of upper semicontinuous convex-trace functions attaining their maximum at exactly one Choquet-boundary point is residual and we obtain several enhanced versions of the maximum principle, including a multi-maximum principle for families of convex-trace functions, which generalize both the classical Bauer's theorem as well as its abstract version in the Choquet theory. We illustrate our notions and results with concrete examples of three different types.

Keywords and phrases: Trace-convexity, Choquet convexity, Krein-Milman theorem, Bauer Maximum Principle, Variational Principle, Exposed point, Extreme point.

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Contents

1 Introduction.

The classical Krein-Milman theorem ([15]) states that every nonempty convex compact subset K of a locally convex vector space E can be represented as the closed convex hull of its extreme points, that is, $K = \overline{\text{co}}\,(\text{Ext}(K))$. The result is based on the Hahn-Banach separation theorem together with Zorn's lemma. It asserts in particular the existence of extreme points in every convex compact set. An alternative way to obtain the same conclusion is based on the Bauer maximum principle ([4]), which states that for every nonempty convex compact set, every upper semicontinuous convex function attains its maximum at some extreme point of that set.

A more general version of the Krein-Milman theorem can be obtained via the Choquet representation theory, in terms of Radon measures on K whose support is contained in the closure of the extreme points of K. This theory gives rise to an abstract definition of convexity, further beyond the framework of vector spaces, the so-called Φ-Choquet convexity, where K is a compact metric space and Φ is a closed subspace of the Banach space $\mathcal{C}(K)$ of real-valued continuous functions on K. Then, a function $f \in \mathcal{C}(K)$ is called Φ-Choquet convex (see forthcoming Definition 5) if for every $x \in K$ and probability measure μ on K the following implication holds:

$$\forall \phi \in \Phi,\ \phi(x) = \int_K \phi\, d\mu \quad \Longrightarrow \quad f(x) \leq \int_K f d\mu.$$

In particular, a Φ-Choquet convex function f is, by definition, continuous. Moreover, an abstract version of Bauer's maximum principle holds true for these functions defined on a compact metric space, which evokes the so-called Choquet boundary of K.

In this work, we adopt a geometrical approach to define convexity on a compact (not necessarily metric) space K. Considering the canonical injection $\delta^{\Phi} : K \to \Phi^*$, given by $\delta^{\Phi}(x)(\phi) = \phi(x)$, for all $\phi \in \Phi$ we may identify K with its homeomorphic image $\delta^{\Phi}(K)$, which lies in particular into a convex w^*-compact subset $K(\Phi)$ of Φ^*. (The exact definition of the set $K(\Phi)$ is given in (6).) Under this identification, we call a set (respectively, a function) *convex-trace*, if it is the trace on K of a convex subset of $K(\Phi)$ (respectively, the restriction on K of a real-valued convex function on $K(\Phi)$), see Definition 17 (respectively, Definition 8). Notice that a convex-trace function does not have to be continuous. Nonetheless, for every continuous function f we may associate a continuous convex-trace function $\widehat{f}$, which corresponds to a trace-convexification for f. Moreover, we show that continuous convex-trace functions coincide with Φ-Choquet convex functions. Therefore trace-convexity can be seen as an alternative geometric definition for Choquet convexity.

Our approach pinpoints a natural extension for Choquet convexity. Indeed, we can consider the class of convex-trace functions that are merely upper semicontinuous, which in addition, is the optimal framework to all results related to the maximum principle. In this work, we establish enhanced versions of the maximum principe for such functions defined on a compact (not necessarily metric) space, extending both the classical Bauer's maximum principle for convex functions on locally convex spaces and its abstract version in the Choquet theory on compact metric spaces. Moreover, in the particular case that the compact set K is metrizable, we obtain a genericity result, whose proof does not require Zorn's lemma and is based on a new variational principle established recently in [2, Lemma 3]. This new variational principle is in the spirit of that of Deville-Godefroy-Zizler in [8] and Deville-Revalski in [9], and at the same time, complementary to them: it does not require the existence of a bump function (ie. a function with a nonempty bounded support) but in turn, it requires the set to be compact and metrizable. This assumption applies particularly well when Φ is the space of affine continuous functions or the space of harmonic functions, spaces which do not dispose bump functions.

Let us mention, for completeness, that motivated by Choquet convexity, there exists another abstract notion of convexity, called Φ-convexity, studied in works by Ky Fan [10], M. W. Grossman [12] and B. D. Khanh [14]. This notion is defined algebraically, based on an abstract definition of segments, and aims to extend from convex sets to compact spaces both the classes of convex and of strict quasi-convex functions. Since we are interested in notions extending convexity of functions in a fully compatible way (in the sense that the definition, when applied to a convex subset of a locally convex space, should yield exactly the class of convex functions and not more than this), we shall not deal with this notion in this work. On the other hand, Φ-convex sets (in the theory of Ky Fan) will turn out to be exactly the convex-trace sets (see Remark 26). The same conclusion can also be derived from [16, Proposition 8.22], where the authors used an ostensibly different definition of Φ-convexity (thereby named $\mathcal{H}$-convexity), showed that it eventually corresponds to the property of trace-convexity and established the abstract version of Krein-Milman theorem that we reproduce here, see [16, Chapter 8]. This latter result is based on a well-adapted definition of Φ-extreme points, that goes with the spirit of trace-convexity. These points are in general much less than the Φ-extreme points in the theory of Ky Fan.

Throughout this work, all topological spaces will be assumed Hausdorff. We shall systematically denote by K a nonempty compact space (which might be metrizable or not) and by Φ a closed subspace of $\mathcal{C}(K)$ which separates points and contains the constant functions. The whole theory can also be developed in a more general setting, starting from a completely regular space X (and defining $K := \beta X$ to be the Stone-Čech compactification of X), or considering an open dense subset X of a given compact space K. Notwithstanding, we shall only adopt this more general setting when we deal with convex-trace sets, in order to discuss properly some examples at the end of the manuscript.

The manuscript is organized as follows:

In Section 2, we review concepts related to Choquet convexity in a topological setting (K is a compact space) and fix terminology and notation. In particular we recall the notions of representing measure, of Choquet boundary and of Choquet convex function. This part is quite standard and can be found (under a slightly different notation) in *e.g.* [5] or [16].

In Section 3 we introduce the central notion of this work, that is, the notion of trace-convexity, both for functions (Definition 8) and for sets (Definition 17). As consequence of our first main result (Theorem 11) we show that Choquet convexity can be equivalently restated in terms of trace convexity for continuous functions (Corollary 13). This restatement allows a natural extension by considering traces of upper semicontinuous convex functions, see Definition 5. We also introduce convex-trace sets and a notion of trace convexification for sets and reproduce an abstract Krein-Milman theorem (Theorem 24), which has been previously established via a different approach in [16, Corollary 8.19]. As pointed out by one of the referees, the trace-convexification of a set, under a different terminology, can also be deduced from the results presented in [16]. In view of this, several of the results of Section 3 should rather be viewed as an alternative approach to the theory exposed in [16, Chapter 8].

In Section 4 we show that this new setting fits perfectly to the framework of Bauer's maximum principle (Theorem 28). Moreover, in the specific case that the compact set is metrizable, an enhanced version of the maximum principle (Theorem 35) and a generic maximum principle (Theorem 37) hold true.

Finally, in Section 5 we provide three typical examples of different nature to illustrate these notions and the results of this work.

2 Preliminaries and notation.

Let K be a compact topological space and let $\mathcal{C}(K)$ be the Banach space of continuous real-valued functions on K equipped with the sup-norm: $\|f\|_\infty := \sup_{x\in K}|f(x)|$, for $f \in \mathcal{C}(K)$. Throughout this work, Φ will denote a closed subspace of $\mathcal{C}(K)$ satisfying the following two properties:

(i) Φ separates points in K ;
(that is, for every $x, y \in K$ with $x \neq y$, there exists $\phi \in \Phi$ such that $\phi(x) \neq \phi(y)$)

(ii) Φ contains the constant functions.
(equivalently, the function $\mathbf{1}(x) = 1$, for all $x \in K$ belongs to Φ.)

It is well-known that K admits a canonical injection to $\mathcal{C}(K)^*$ by means of the following Dirac mapping

$$\begin{cases} \delta : K \longrightarrow \mathcal{C}(K)^* \\ \delta(x) = \delta_x \quad \text{with } \delta_x(f) := f(x), \text{ for all } f \in \mathcal{C}(K) \end{cases} \tag{1}$$

If we equip $\mathcal{C}(K)^*$ with the $\sigma(\mathcal{C}(K)^*, \mathcal{C}(K))$–topology (that we simply call w^*-topology), then the above injection is homeomorphic and K is topologically identified to $\delta(K) := \{\delta_x : x \in K\}$ as subset of $(\mathcal{C}(K)^*, w^*)$. We also recall (see [19] eg.) that the dual space $\mathcal{C}(K)^*$ is naturally identified with the Radon measures on K via the duality map

$$\langle \mu, f\rangle = \int_K f d\mu, \quad \text{for all } \mu \in \mathcal{C}(K)^* \text{ and } f \in \mathcal{C}(K). \tag{2}$$

In particular, δ_x is the Dirac measure of x and (1) becomes:

$$\delta_x(f) := \langle \delta_x, f\rangle = f(x).$$

Furthermore, the dual norm $||\mu||_*$ coincides with the total variation of the measure μ.

We denote by $\mathcal{M}^1(K)$ the set of all Borel probability measures on K. This set is a w^*-compact convex subset of $\mathcal{C}(K)^*$ and coincides with the weak* closed convex hull of the set $\delta(K)$, that is,

$$\mathcal{M}^1(K) = \{\mu \in \mathcal{C}(K)^* : \|\mu\|_* = \langle \mu, \mathbf{1}\rangle = 1\} = \overline{\text{conv}}^{w^*}(\delta(K)) \subset \mathcal{C}(K)^*, \tag{3}$$

where $\mathbf{1}(x) = 1$, for all $x \in K$.

Definition 1 (Φ-representing measures)**.** Let $x \in K$. We say that $\mu \in \mathcal{M}^1(K)$ is a Φ-representing (probability) measure for x if

$$\phi(x) = \int_K \phi \, d\mu, \quad \text{for all } \phi \in \Phi.$$

The set of all Φ-representing measures of x is denoted by $\mathcal{M}_x(\Phi)$. Notice that $\delta_x \in \mathcal{M}_x(\Phi)$, for every $x \in K$, therefore $\mathcal{M}_x(\Phi)$ is nonempty. Every $Q \in \Phi^*$ (linear continuous functional on Φ) can be extended, via the Hahn-Banach theorem, to an element of $\mathcal{C}(K)^*$ (linear continuous functional on $\mathcal{C}(K)$) of the same norm. However this extension is neither unique nor canonical. Let us consider the following equivalence relation on $\mathcal{C}(K)^*$:

$$\mu \sim \mu' \iff \langle \mu, \phi \rangle = \langle \mu', \phi \rangle, \text{ for all } \phi \in \Phi. \tag{4}$$

We denote by $[\mu]$ the class of equivalence of $\mu \in \mathcal{C}(K)^*$ under the above binary relation. Since this relation is compatible with the linear structure of $\mathcal{C}(K)^*$, setting $\widehat{\pi}(\mu) = [\mu]$, for all $\mu \in \mathcal{C}(K)^*$, we obtain a linear bounded surjective map $\widehat{\pi} : (\mathcal{C}(K)^*, ||\cdot||_\infty) \longrightarrow (\mathcal{C}(K)^*/\!\sim, ||\cdot||_\sim)$, where $||\cdot||_\sim$ is the quotient norm on $\mathcal{C}(K)^*/\!\sim$, defined as follows:

$$||[\mu]||_\sim := \inf \left\{ ||\mu'||_* : \mu' \sim \mu \right\}.$$

If $\mathcal{C}(K)^*$ is equipped with its w^*-topology, then we denote by τ the final (quotient) topology on $\mathcal{C}(K)^*/\!\sim$ under $\widehat{\pi}$, that is, the finest topology for which the mapping

$$\widehat{\pi} : (\mathcal{C}(K)^*, w^*) \longrightarrow (\mathcal{C}(K)^*/\!\sim, \tau)$$

is continuous. Therefore, $O \in \tau$ if and only if $\widehat{\pi}^{-1}(O)$ is w^*-open in $\mathcal{C}(K)^*$. The following result shows that the space $(\mathcal{C}(K)^*/\!\sim, \tau)$ is in fact linearly homeomorphic to Φ^*, if the latter is considered with its $\sigma(\Phi^*, \Phi)$-topology (which will be also denoted by w^* if no confusion arises). Before we proceed, we observe that the linear surjective map

$$\begin{cases} i^* : \mathcal{C}(K)^* \longrightarrow \Phi^* \\ i^*(\mu) = \mu|_\Phi \end{cases}$$

is (w^*-w^*)-continuous, being the adjoint of the identity map $i : (\Phi, \|\cdot\|_\infty) \to (\mathcal{C}(K), \|\cdot\|_\infty)$ (which is a linear isometric injection).

Lemma 2 (Identification of $C(K)^*/\!\sim$ with (Φ^*, w^*))**.** *The bijective map*

$$\begin{cases} \mathcal{J} : (\mathcal{C}(K)^*/\!\sim, \tau) \longrightarrow (\Phi^*, w^*) \\ \mathcal{J}([\mu]) = \mu|_\Phi \end{cases}$$

is a linear homeomorphism between $(\mathcal{C}(K)^*/\!\sim, \tau)$ and (Φ^*, w^*). Moreover, we have the identity:

$$\widehat{\pi}(\mu) = \mathcal{J}^{-1} \circ i^*(\mu), \quad \textit{for every } \mu \in \mathcal{C}(K)^*. \tag{5}$$

Proof. It is straightforward to see that the mapping $\mathcal{J}$ is a linear bijection and (5) holds. Since $i^* : (\mathcal{C}(K)^*, w^*) \longrightarrow (\Phi^*, w^*)$ is continuous, and $\mathcal{J} \circ \widehat{\pi} = i^*$ it follows from the definition of the final topology τ that $\mathcal{J}$ is (τ,w^*)-continuous. It remains to prove that $\mathcal{J}$ maps τ-closed sets to w^*-closed sets. To this end, let F be τ-closed in $\mathcal{C}(K)^*/\!\sim$. In view of Banach-Dieudonné theorem, it is sufficient to prove that for every $R > 0$ the set $\mathcal{J}(F) \cap \bar{\mathbf{B}}_R$ is w^*-closed in (Φ^*, w^*), where $\bar{\mathbf{B}}_R = i^*(\bar{B}(0, R))$ and $\bar{B}(0, R)$ is the closed ball of $\mathcal{C}(K)^*$ centered at 0 with radius $R > 0$. By the Banach-Alaoglou theorem and the continuity of $\widehat{\pi}$ we deduce that $\widehat{\pi}(\bar{B}(0, R))$ is τ-compact. It follows that the restriction of $\mathcal{J}$ on the (τ-compact) set $\widehat{\pi}(\bar{B}(0, R))$ is an homeomorphism between $\widehat{\pi}(\bar{B}(0, R))$ and the closed ball $\bar{\mathbf{B}}_R = i^*(\bar{B}(0, R))$ of Φ^*. This yields that the set

$$\mathcal{J}(F) \cap \bar{\mathbf{B}}_R = \mathcal{J}(F \cap \bar{B}(0, R))$$

is w^*-closed in Φ^* as asserted. $\square$

Restricting the projection mappings $\widehat{\pi} : \mathcal{C}(K)^* \longrightarrow \mathcal{C}(K)^*/\sim$ and $i^* : \mathcal{C}(K)^* \longrightarrow \Phi^*$ on the set of probability measures $\mathcal{M}^1(K)$ we obtain an affine homeomorphic bijection between (the τ-compact set) $\widehat{\pi}(\mathcal{M}^1(K)) := \mathcal{M}^1(K)/\sim$ and (the w^*-compact set)

$$i^*(\mathcal{M}^1(K)) := K(\Phi) = \{Q \in \Phi^* : \|Q\| = \langle Q, \mathbf{1}\rangle = 1\} \ \subset \ \Phi^*, \tag{6}$$

where we continue to denote by $\langle \cdot, \cdot \rangle$ the duality mapping between Φ^* and Φ. In particular:

Corollary 3 (Identification of $(\mathcal{M}^1(K)/\sim, \tau)$ with $(K(\Phi), w^*)$)**.** *The bijective mapping*

$$\mathcal{J} : (\mathcal{M}^1(K)/\sim, \tau) \to (K(\Phi), w^*)$$

defined by $\mathcal{J}([\mu]) := \mu_{|\Phi}$ *is an affine homeomorphism and*

$$\mathcal{J} \circ \widehat{\pi}(\mu) = i^*(\mu) = \mu|_\Phi, \quad \textit{for all } \mu \in \mathcal{M}^1(K).$$

We also recall the following universal property of the quotient map

$$\widehat{\pi} : (\mathcal{M}^1(K), w^*) \to (\mathcal{M}^1(K)/\sim, \tau)$$

- (*factorization lemma*) If $G : (\mathcal{M}^1(K), w^*) \to Z$ is a continuous map such that $\mu \sim \nu$ implies $G(\mu) = G(\nu)$ for all $\mu, \nu \in \mathcal{M}^1(K)$, then there exists a unique continuous map $H : (\mathcal{M}^1(K)/\sim, \tau) \to Z$ such that $G = H \circ \widehat{\pi}$ (where Z is any topological space).

Combining (1) with (5) we obtain a canonical injection of K into Φ^* as follows:

$$\begin{cases} \delta^\Phi : K \longrightarrow (K(\Phi), w^*) \\ \delta^\Phi = i^* \circ \delta \quad \text{with } \delta^\Phi_x := \delta^\Phi(x) = i^*(\delta_x). \end{cases} \tag{7}$$

Therefore δ^Φ defines a homeomorphism between K and $\delta^\Phi(K)$. In fact, it is often convenient to identify these spaces: $K \equiv \delta^\Phi(K)$. Under the above notation, $\delta^\Phi_x(\phi) = \phi(x)$, for all $\phi \in \Phi$. Similarly to (3), the w^*-compact convex set $K(\Phi)$ in Φ^* coincides with the w^*-closed convex hull of the set $\{\delta^\Phi_x : x \in K\}$.

Definition 4 (Choquet boundary)**.** The Φ-Choquet boundary $\partial_\Phi(K)$ of K (or simply Choquet boundary, if no confusion arises) is defined as follows:

$$\partial_\Phi(K) := \{x \in K : \mathcal{M}_x(\Phi) = \{\delta_x\}\}.$$

Denoting by $\mathcal{C}(K)^*_+$ the cone of positive Borel measures on K, it follows easily that

$$x \in \partial_\Phi(K) \quad \text{ if and only if } \quad [\delta_x] \cap \mathcal{M}^1(K) = \{\delta_x\} = [\delta_x] \cap \mathcal{C}(K)^*_+\,,$$

where $[\delta_x]$ denotes the equivalent class of the Dirac measure δ_x, that is,

$$\mu \in [\delta_x] \quad \text{if and only if} \quad \langle \mu, \phi \rangle = \phi(x), \text{ for all } \phi \in \Phi. \tag{8}$$

In addition, it is well-known (see [17], [18], eg.) that a point $x \in K$ belongs to the Φ-Choquet boundary of K if and only if the canonical injection δ^Φ, given in (7), maps this point to an extreme point of the w^*-compact convex set $K(\Phi)$, that is,

$$\partial_\Phi(K) = \left\{x \in K : \ \delta^\Phi_x \in \mathrm{Ext}\, K(\Phi)\right\}. \tag{9}$$

2.1 Choquet convexity.

Let us now recall the following definition (eg. [7], [17], [18, Prop. 3.6], [16, Definition 3.8]).

Definition 5 (Choquet convex function)**.** A continuous function $f \in \mathcal{C}(K)$ is said to be Φ-Choquet-convex (or simply, Choquet-convex, if no ambiguity arises), if for every $x \in K$ it holds:

$$f(x) \leq \int_K f d\mu, \quad \text{for all } \mu \in \mathcal{M}_x(\Phi).$$

The set of all Choquet-convex functions will be denoted by

$$\Gamma_\Phi(K) := \{f : K \to \mathbb{R} \text{ Choquet-convex function}\} \subset \mathcal{C}(K).$$

Notation. For any convex subset S of a locally convex space E, we set

$$\Gamma(S) := \{f : S \to \mathbb{R} \text{ convex continuous function}\}.$$

We also denote by $\Gamma^{>}(S)$ the set of upper semicontinuous (in short, usc) convex functions, and by $\Gamma^{<}(S)$ the set of lower semicontinuous (in short, lsc) convex functions on S.

Remark 6 (Compatibility of Choquet convexity)**.** If K is a convex subset of a locally convex space E and $\Phi = \mathrm{Aff}(K)$ is the set of affine continuous functions on K, then δ^Φ is an affine homeomorphic bijection between K and $K(\Phi)$ and we have:

$f \in \Gamma_\Phi(K)$ (Choquet convex) if and only if $f \in \Gamma(K)$ (convex continuous).

Before we proceed, let us recall from [17, Key Lemma] the following result.

Lemma 7 (Key Lemma)**.** *For every continuous function on K it holds:*

$$\left\{\int_K f d\mu : \ \mu \in \mathcal{M}_x(\Phi)\right\} = \left[\sup_{\phi\in\Phi,\, \phi\leq f} \phi(x), \ \inf_{\phi\in\Phi,\, \phi\geq f} \phi(x)\right].$$

Therefore, we deduce:

$$\inf_{\mu\in\mathcal{M}_x(\Phi)} \int_K f d\mu \ = \sup_{\phi\in\Phi,\, \phi\leq f} \phi(x) \ \leq \ f(x).$$

It follows directly from Definition 5 and the above Key Lemma that

$$f \in \Gamma_\Phi(K) \quad \Longleftrightarrow \quad f(x) = \inf_{\mu\in\mathcal{M}_x(\Phi)} \int_K f d\mu = \sup_{\phi\in\Phi,\, \phi\leq f} \phi(x), \quad \text{for all } x \in K.$$

3 Extending Choquet convexity on topological spaces.

Using the notation of the previous section, we fix a compact space K and a closed subspace Φ of $\mathcal{C}(K)$ satisfying conditions (i) and (ii). We also consider the w^*-compact convex subset $K(\Phi)$ of (Φ^*, w^*) defined in (6).

3.1 Convex-trace functions.

In the spirit of the definition of Choquet boundary, using the identification of the topological space K with $\delta^{\Phi}(K)$ and the classical convexity of functions defined on the affine variety $K(\Phi)$ we introduce a new notion of convexity for real-valued functions defined on K. The main idea is that since $K(\Phi)$ is an affine variety, we can define therein convex functions (in the classical sense) and consider their traces on $\delta^{\Phi}(K)$. We therefore obtain the class of *convex-trace* functions, which can either be continuous —case in which we recover the class $\Gamma_{\Phi}(K)$ of Choquet convex functions (Corollary 13)— or more generally upper (or lower) semicontinuous, yielding a natural extension of Choquet convexity, that can be used to extend results related to the generalized Bauer's maximum principle. More precisely, we give the following definition.

Definition 8 (Convex-trace functions)**.** Let K be a compact space and Φ a closed subspace of $\mathcal{C}(K)$ that separates points in K and contains the constant functions. Let further $\delta^{\Phi} : K \to (K(\Phi), w^*)$ be the canonical injection given in (7). A function $f : K \to \mathbb{R}$ is called:

(i) continuous convex-trace with respect to Φ (or simply continuous Φ-convex-trace), denoted $f \in T\mathcal{C}(K,\Phi)$, if there exists a convex continuous function $F : (K(\Phi), w^*) \to \mathbb{R}$, that is, $F \in \Gamma(K(\Phi))$ such that
$$f = F \circ \delta^{\Phi}. \tag{10}$$

(ii) usc (respectively, lsc) convex-trace with respect to Φ, or simply, usc (respectively lsc) Φ-convex-trace, denoted $f \in T\mathcal{C}^{>}(K,\Phi)$ (respectively, $f \in T\mathcal{C}^{<}(K,\Phi)$), if there exists $F \in \Gamma^{>}(K(\Phi))$ (respectively, $\Gamma^{<}(K(\Phi))$) such that (10) holds.

In other words, identifying K with its canonical image $\delta^{\Phi}(K)$ in Φ^*, a function f is Φ-convex-trace on K whenever f is the trace of a (usual) convex function on (the affine variety) $K(\Phi)$. Since every $\phi \in \Phi \subset \mathcal{C}(K)$ is obviously a linear w^*-continuous functional on Φ^*, it follows directly that
$$\Phi \subset T\mathcal{C}(K,\Phi).$$
Notice that both Choquet convexity (Definition 5) and trace-convexity (Definition 8) depend on the choice of the closed subspace Φ of $\mathcal{C}(K)$. This being said, whenever no confusion arises, we shall drop Φ and simply talk about usc, lsc or continuous convex-trace functions on K, denoting their class by $T\mathcal{C}^{>}(K)$, $T\mathcal{C}^{<}(K)$ and $T\mathcal{C}(K)$ respectively.

Remark 9 (Compatibility of trace-convexity)**.** Similarly to Remark 6, if K is a convex subset of a locally convex space E, then taking $\Phi = \mathrm{Aff}(K)$, the convex sets $K(\Phi)$ and K can be identified via the affine homeomorphism δ^{Φ}, and the notion of trace-convexity coincides with the classical convexity on K, that is:
$$T\mathcal{C}(K) = \Gamma(K), \quad T\mathcal{C}^{>}(K) = \Gamma^{>}(K) \quad \text{and} \quad T\mathcal{C}^{<}(K) = \Gamma^{<}(K).$$

Remark 10 (Φ-stability)**.** A set of functions $\mathcal{B} \subset \mathbb{R}^K$ is called Φ-stable, if $\Phi + \mathcal{B} \subset \mathcal{B}$. It is straightforward to see that Φ, $\mathcal{C}(K)$ and $\mathbb{R}^K$ are Φ-stable. It follows easily by Definition 5 that $\Gamma_{\Phi}(K)$ is Φ-stable. We leave the reader to verify from Definition 8 that the sets of (usc, lsc, continuous) convex-trace functions $T\mathcal{C}^{>}(K)$, $T\mathcal{C}^{>}(K)$ and respectively, $T\mathcal{C}(K)$ are Φ-stable.

3.2 Trace-convexification $f \mapsto \widehat{f}$.

Let us fix an arbitrary continuous function $f \in \mathcal{C}(K)$. For each $\mu \in \mathcal{M}^1(K)$, we define the w^*-open set

$$\mathcal{W}_{\mu,f} := \{\nu \in \mathcal{C}(K)^* : |\langle \nu - \mu, f\rangle| < 1\} \tag{11}$$

and we set

$$\mathcal{O}_f := \mathrm{co}\left(\bigcup_{\mu \in \mathcal{M}^1(K)} \mathcal{W}_{\mu,f}\right) \supseteq \mathcal{M}^1(K), \tag{12}$$

where $\mathrm{co}(\cup_{\mu\in\mathcal{M}^1(K)}\mathcal{W}_{\mu,f})$ denotes the convex hull of the w^*-open set $\cup_{\mu\in\mathcal{M}^1(K)}\mathcal{W}_{\mu,f}$. We have that $\mathcal{O}_f$ is w^*-open and convex subset of $\mathcal{C}(K)^*$. We define $\widehat{f}$ as follows:

$$\begin{cases} \widehat{f} : K \to \mathbb{R} \\ \widehat{f}(x) = \inf\limits_{\mu \in [\delta_x]\cap\mathcal{O}_f} \int_K f\, d\mu. \end{cases} \tag{13}$$

It is straightforward to see that

$$f(x) = \langle \delta_x, f\rangle \geq \widehat{f}(x), \quad \text{for all } x \in K.$$

Moreover, since $\int_K \phi\, d\mu = \phi(x)$ for every $\phi \in \Phi$ and every $\mu \in [\delta_x]$, it follows that

$$\widehat{\phi} = \phi \quad \text{for every } \phi \in \Phi\,.$$

We shall now show that $\widehat{f}$ is a convex-trace function and consequently, $\widehat{f}$ can be seen as a *trace-convexification* of f on K.

Theorem 11 ($\widehat{f}$ is convex-trace)**.** *For every $f \in \mathcal{C}(K)$ there exists a convex w^*-continuous function $F_f : (K(\Phi), w^*) \to \mathbb{R}$ such that:*

(i). $\widehat{f} = F_f \circ \delta^\Phi$ *(therefore, $\widehat{f} \in T\mathcal{C}(K,\Phi)$) ;*

(ii). $-(\|f\|_\infty + 1) \leq F_f(Q) \leq \|f\|_\infty$, *for all $Q \in K(\Phi)$.*

Proof. Let $f \in \mathcal{C}(K)$, and let $\mathcal{O}_f$ be the w^*-open convex set defined in (12). We define:

$$\begin{aligned} G_f : (\mathcal{O}_f, w^*) &\to \mathbb{R} \\ \mu &\mapsto \inf_{\nu\in[\mu]\cap\mathcal{O}_f} \int_K f\, d\nu, \end{aligned} \tag{14}$$

where $[\mu]$ is the class of equivalence of the measure $\mu \in \mathcal{O}_f \subset \mathcal{C}(K)^*$, according to (4). It is easy to see that:

$$-\|f\|_\infty\, \|\mu\|_* - 1 \,\leq\, G_f(\mu) \,\leq\, \int_K f\, d\mu \,\leq\, \|f\|_\infty \|\mu\|_*, \quad \text{for all } \mu \in \mathcal{O}_f, \tag{15}$$

which guarantees that G_f is well-defined (it takes finite values) on $\mathcal{O}_f$. Notice also that

$$(G_f \circ \delta)(x) := G_f(\delta_x) = \widehat{f}(x) \leq f(x), \quad \text{for all } x \in K.$$

Step 1. The function G_f is convex on $\mathcal{O}_f$.
To this end, let $\mu_1, \mu_2 \in \mathcal{C}(K)^*$ and $t \in (0,1)$ and fix any $\varepsilon > 0$. Then there exist $\nu_i \in [\mu_i] \cap \mathcal{O}_f$, for $i \in \{1,2\}$, such that

$$\langle \nu_i, f \rangle := \int_K f \, d\nu \leq G_f(\mu_i) + \varepsilon.$$

Since $t\nu_1 + (1-t)\nu_2 \in [t\mu_1 + (1-t)\mu_2] \cap \mathcal{O}_f$ (recall that $\mathcal{O}_f$ is a convex set) we obtain

$$G_f(t\mu_1 + (1-t)\mu_2) \leq \langle t\nu_1 + (1-t)\nu_2, f \rangle = t\langle \nu_1, f \rangle + (1-t)\langle \nu_2, f \rangle \leq G_f(\mu_1) + (1-t)G_f(\mu_2) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, convexity of G_f follows.

Step 2. G_f is locally bounded from above on the w^*-open set $\mathcal{O}_f$.
Let $\mu_0 \in \mathcal{O}_f$ and define the w^*-open set $\mathcal{V}_{\mu_0,f} := \{\mu \in \mathcal{O}_f : |\langle \mu - \mu_0, f \rangle| < 1\}$. Then for every $\mu \in \mathcal{V}_{\mu_0}$ we have

$$G_f(\mu) \leq \int_K f \, d\mu < ||f||_\infty \, ||\mu_0||_* + 1,$$

which yields that G_f is bounded from above on $\mathcal{V}_{\mu_0}$.

Step 3. G_f is convex w^*-continuous.
Since G_f is convex, takes finite values on $\mathcal{O}_f$ and it w^*-locally bounded from above, the assertion follows directly from [1, Theorem 5.42] applied to the w^*-open subset $\mathcal{O}_f$ of $\mathcal{C}(K)^*$ (where $\mathcal{C}(K)^*$ equipped with its w^*-topology is considered as a locally convex space).

Step 4. $\widehat{f}$ is continuous convex-trace.
Notice that by its very definition, $G_f(\mu) = G_f(\mu')$, whenever $\mu \sim \mu'$ (with respect to (4)). Since G_f is w^*-continuous, then its restriction

$$G_{f|\mathcal{M}^1(K)} : (\mathcal{M}^1(K), w^*) \to \mathbb{R}$$

is also w^*-continuous. Therefore, by the factorization lemma and the topological quotient, there exists a unique τ-continuous function $H_f : (\mathcal{M}^1(K)/\sim, \tau) \to \mathbb{R}$ such that $G_{f|\mathcal{M}^1(K)} = H_f \circ \widehat{\pi}$. It is straightforward to see that H_f is convex, since G_f is convex and $\widehat{\pi}$ is affine. Using Corollary 3, we deduce that the function $F_f : (K(\Phi), w^*) \to \mathbb{R}$ defined by $F_f = H_f \circ \mathcal{J}^{-1}$ is convex w^*-continuous. Thus, we have that $F_f \circ i^* = G_{f|\mathcal{M}^1(K)}$ on $\mathcal{M}^1(K)$. Moreover, we deduce from the definition of $\widehat{f}$ in (13) that:

$$F_f(\delta_x^\Phi) = (H_f \circ \mathcal{J}^{-1})(\delta_x^\Phi) = H_f([\delta_x]) = G_f(\delta_x) = \widehat{f}(x), \quad \text{for all } x \in K.$$

This shows that $\widehat{f}$ is the trace on $K \equiv \delta^\Phi(K)$ of the convex w^*-continuous function F_f as asserted. The inequality in (ii) follows from the formula (15). $\square$

3.3 Trace-convexity vs Choquet convexity.

In this subsection we establish (see forthcoming Corollary 13 (i)$\Leftrightarrow$(iv)) that the class of Choquet convex functions coincides with the class of continuous convex-trace functions. This latter class admits a natural extension to upper semicontinuous functions, which consists of the most natural framework to state a generalized maximum principle (see Subsection 4.1).

At this stage let us recall from [3] the Φ-conjugate $f^{\times}:\Phi\to\mathbb{R}$ of a proper bounded from below function $f:K\to\mathbb{R}\cup\{+\infty\}$, which is defined as follows:

$$f^{\times}(\phi):=\sup_{x\in K}\{\phi(x)-f(x)\},\quad \phi\in\Phi. \tag{16}$$

The function $f^{\times}$, being defined in a Banach space Φ, admits a usual Fenchel defined by

$$\begin{cases} (f^{\times})^{*}:\Phi^{*}\to\mathbb{R}\cup\{+\infty\}\\ (f^{\times})^{*}(Q):=\sup\limits_{\phi\in\Phi}\{\langle Q,\phi\rangle-f^{\times}(\phi)\},\quad \text{for all } Q\in\Phi^{*}.\end{cases}$$

Restricting the above onto $\delta^{\Phi}(K)$ yields a function $f^{\times\times}:K\to\mathbb{R}\cup\{+\infty\}$ defined as follows:

$$f^{\times\times}(x):=\left((f^{\times})^{*}\circ\delta^{\Phi}\right)(x)=\sup_{\phi\in\Phi}\left\{\langle\delta_x^{\Phi},\phi\rangle-f^{\times}(\phi)\right\}=\sup_{\phi\in\Phi}\left\{\phi(x)-f^{\times}(\phi)\right\},\quad \text{for all } x\in K.$$

It is easily seen that $f^{\times\times}\leq f$. Moreover, we have $\tilde{\phi}(x):=\phi(x)-f^{\times}(\phi)\leq f(x)$, for all $x\in K$. Since Φ contains the constant functions, we have $\tilde{\phi}\in\Phi$ and readily deduce

$$f^{\times\times}(x)=\sup_{\phi\in\Phi}\underbrace{\left\{\phi(x)-f^{\times}(\phi)\right\}}_{:=\tilde{\phi}(x)\,\left(\leq f(x)\right)}\leq\sup_{\phi\in\Phi,\,\phi\leq f}\phi(x).$$

On the other hand, $f^{\times}(\phi)\leq 0$ whenever $\phi\leq f$. Consequently

$$f^{\times\times}(x)\;=\;\sup_{\phi\in\Phi}\left\{\phi(x)-f^{\times}(\phi)\right\}\geq\sup_{\phi\in\Phi,\,\phi\leq f}\left\{\phi(x)-f^{\times}(\phi)\right\}\;\geq\;\sup_{\phi\in\Phi,\,\phi\leq f}\phi(x).$$

Combining the above equations, and using Lemma 7 (Key Lemma) we obtain

$$f^{\times\times}(x)\;=\;\sup_{\phi\in\Phi,\,\phi\leq f}\phi(x)=\inf_{\mu\in\mathcal{M}_x(\Phi)}\int_K f\,d\mu. \tag{17}$$

In particular we obtain the following result.

Proposition 12 (Relation between $\widehat{f}$ and $f^{\times\times}$)**.** *For every $f\in\mathcal{C}(K)$, we have that*

$$\widehat{f}(x)\leq f^{\times\times}(x)=\sup_{\phi\in\Phi,\,\phi\leq f}\phi(x)=\inf_{\mu\in\mathcal{M}_x(\Phi)}\int_K f\,d\mu\leq f(x),\quad \textit{for all } x\in K.$$

Consequently, f is Choquet convex if and only if $\widehat{f}=f$ and in this case, we have that

$$\widehat{f}(x)=f^{\times\times}(x)=\sup_{\phi\in\Phi,\,\phi\leq f}\phi(x)=\inf_{\mu\in\mathcal{M}_x(\Phi)}\int_K f\,d\mu=f(x),\quad \textit{for all } x\in K.$$

Proof. Since $\mathcal{M}^1(K)\subset\mathcal{O}_f$ we deduce that $\mathcal{M}_x(\Phi)=[\delta_x]\cap\mathcal{M}^1(K)\subset[\delta_x]\cap\mathcal{O}_f$ for all $x\in K$. Therefore, it follows readily that:

$$\widehat{f}(x):=\inf_{\mu\in[\delta_x]\cap\mathcal{O}_f}\int_K f\,d\mu\leq\inf_{\mu\in\mathcal{M}_x(\Phi)}\int_K f\,d\mu\leq\langle\delta_x,f\rangle=f(x),\quad \text{for all } x\in K.$$

The assertion follows from (17).

For the second part of the proposition: it follows readily from the above inequalities and Definition 5 that if $f = \widehat{f}$, then f is Choquet convex. For the converse, suppose that f is Choquet convex. Then by Lemma 7 (Key Lemma) and formula (17) we deduce that for every $x \in K$

$$f(x) \leq \inf_{\mu \in \mathcal{M}_x(\Phi)} \int_K f d\mu = \sup_{\phi \in \Phi,\, \phi \leq f} \phi(x) = f^{\times\times}(x) \leq f(x),$$

yielding

$$f(x) = \sup_{\phi \in \Phi,\, \phi \leq f} \phi(x) = f^{\times\times}(x).$$

Therefore we have:

$$\begin{aligned}
\widehat{f}(x) \quad &:= \quad \inf_{\mu \in [\delta_x] \cap \mathcal{O}_f} \int_K f d\mu = \inf_{\mu \in [\delta_x] \cap \mathcal{O}_f} \int_K \left(\sup_{\phi \in \Phi,\, \phi \leq f} \phi \right) d\mu \\
&\geq \quad \inf_{\mu \in [\delta_x] \cap \mathcal{O}_f} \sup_{\phi \in \Phi,\, \phi \leq f} \int_K \phi d\mu \geq \sup_{\phi \in \Phi, \phi \leq f} \inf_{\mu \in [\delta_x] \cap \mathcal{O}_f} \int_K \phi d\mu \\
&= \quad \sup_{\phi \in \Phi,\, \phi \leq f} \phi(x) = f^{\times\times}(x).
\end{aligned}$$

Thus, $\widehat{f} \geq f^{\times\times}$ and so the equalities hold. $\square$

The following corollary resumes the above results and provides a characterization of Choquet convex functions. In particular the class of Choquet convex functions coincides with the class of (continuous) convex-trace functions. The implication (iv) $\Rightarrow$ (i) can also be found in [16, Lemma 4.27].

Corollary 13 ($\Gamma_\Phi(K) = T\mathcal{C}(K)$)**.** *Let $f \in \mathcal{C}(K)$.*
The following are equivalent:

(i). *f is Choquet convex (ie. $f \in \Gamma_\Phi(K)$).*

(ii). *$f(x) = \sup_{\phi \in \Phi,\, \phi \leq f} \phi(x) = f^{\times\times}(x)$, for all $x \in K$.*

(iii). *$f(x) = \widehat{f}(x)$, for all $x \in K$.*

(iv). *f is continuous convex-trace (i.e $f \in T\mathcal{C}(K)$).*

Proof. The equivalence of (i), (ii) and (iii) follows from Proposition 12. Implication $(iii) \Rightarrow (iv)$ follows from Theorem 11. It remains to prove that $(iv) \Rightarrow (iii)$. Let $f \in T\mathcal{C}(K)$. Then there exists $F \in \Gamma(K(\Phi))$ such that $f = F \circ \delta^\Phi$. By assigning the value $+\infty$ outside $K(\Phi)$ we extend F to a w^*-lsc convex function $\tilde{F} \in \Gamma^<(\Phi^*)$. We set

$$\left\{ \begin{array}{l} \tilde{F}^* : \Phi \longrightarrow \mathbb{R} \cup \{+\infty\} \\ \tilde{F}^*(\phi) = \sup_{Q \in \Phi^*} \left\{ \langle Q, \phi \rangle - \tilde{F}(Q) \right\} \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \tilde{F}^{**} : \Phi^* \longrightarrow \mathbb{R} \cup \{+\infty\} \\ \tilde{F}^{**}(Q) = \sup_{\phi \in \Phi} \left\{ \langle Q, \phi \rangle - \tilde{F}^*(\phi) \right\}. \end{array} \right.$$

Then the classical Fenchel-duality yields that $\tilde{F} = \tilde{F}^{**}$. On the other hand, since $f = F \circ \delta^\Phi$, we deduce readily from (16) that

$$f^\times(\phi) := \sup_{x \in K} \{\phi(x) - f(x)\} = \sup_{x \in K} \left\{ \langle \delta_x^\Phi, \phi \rangle - \tilde{F}(\delta_x^\Phi) \right\} \leq \sup_{Q \in \Phi^*} \left\{ \langle Q, \phi \rangle - \tilde{F}(Q) \right\} = \tilde{F}^*(\phi),$$

for all $\phi \in \Phi$, and consequently

$$f(x) \geq f^{\times\times}(x) := (f^{\times})^*(\delta_x^{\Phi}) \geq \tilde{F}^{**}(\delta_x^{\Phi}) = \tilde{F}(\delta_x^{\Phi}) = f(x), \quad \text{for all } x \in K,$$

yielding that equality holds. Therefore, $f = f^{\times\times} = \widehat{f}$. □

Remark 14. (i). Since the function $\widehat{f}$ is convex-trace for every $f \in \mathcal{C}(K)$ (cf. Theorem 11), Corollary 13 yields that $\widehat{\widehat{f}} = \widehat{f}$. Consequently, $\widehat{f}$ is Choquet convex for every $f \in \mathcal{C}(K)$.

(ii). A careful reader might observe that the definition of $\widehat{f}$ depends on the way the neighborhood $\mathcal{O}_f$ is defined. Indeed, taking any $\alpha > 0$ and defining

$$\mathcal{W}_{\mu,f,\alpha} := \{\nu \in \mathcal{C}(K)^* : |\langle \nu - \mu, f\rangle| < \alpha\} \quad \text{and} \quad \mathcal{O}_{f,\alpha} := \mathrm{co}\left(\bigcup_{\mu \in \mathcal{M}^1(K)} \mathcal{W}_{\mu,f,\alpha}\right)$$

we obtain

$$\widehat{f}^{\alpha}(x) = \inf_{\mu \in [\delta_x] \cap \mathcal{O}_{f,\alpha}} \int_K f\, d\mu\,, \quad \text{for } x \in K.$$

Then for $0 < \varepsilon < 1 < M$ we readily obtain

$$\widehat{f}^{M}(x) \leq \widehat{f}(x) \leq \widehat{f}^{\varepsilon}(x) \leq f^{\times\times}(x) \leq f(x), \quad \text{for } x \in K,$$

with equality if and only if $f \in \Gamma_\Phi(K)$. Consequently, the trace-convexification of a continuous function f is generally not unique, and not equal to its Choquet-convexification $f^{\times\times}$ (this latter is always the biggest possible convexification). Notice that if f is already Choquet-convex, then all of the above convexifications coincide with f.

In the following result, we prove that a supremum of Choquet convex functions is a lsc convex-trace function.

Proposition 15. *Let $\{f_i\}_{i\in I} \subset \Gamma_\Phi(K)$ be a nonempty family of uniformly bounded Choquet-convex functions on a compact space K. Then the (bounded) function $f := \sup_{i\in I} f_i$ is lower semicontinuous convex-trace on K.*

Proof. Let us assume that $\|f_i\|_\infty \leq M$ for all $i \in I$. Then by Corollary 13 and Theorem 11(ii), for each $i \in I$, there exists a convex w^*-continuous function $F_i : K(\Phi) \to \mathbb{R}$ such that $f_i = F_i \circ \delta^\Phi$ and $-(M+1) \leq F_i(Q) \leq M$, for all $Q \in K(\Phi)$. Set $F := \sup_{i\in I} F_i$. Then F is w^*-lsc convex function. On the other hand, for every $x \in K$, we have

$$F(\delta_x^\Phi) \;=\; \sup_{i\in i} F_i(\delta_x^\Phi) \;=\; \sup_{i\in I} f_i(x) = f(x),$$

which, in view of Definition 8(ii), yields that $f \in T\mathcal{C}^{<}(K)$. Finally, notice that both functions F and f take their values on $[-(M+1), M]$, therefore they are real-valued. □

3.4 Convex-trace sets.

In this subsection we define the notion of trace-convexity for subsets of a compact topological space. This notion inevitably relates tightly to the definition of trace-convexity of functions and, as we shall see, it coincides with the notion of Φ-convexity given by Ky Fan (see [10], [14] or [2]). In fact, the theory can be naturally developed in a more general framework, that of a completely regular (Hausdorff) topological space. We recall that such spaces admit the so-called Stone-Čech compactification. (We refer to [13] for definition and properties of these spaces.)

More precisely, throughout this section we assume that X is a completely regular topological space, which is dense to some compact set K. If the compact set K is not explicitely given, then we can always consider $K = \beta X$ (the Stone-Čech compactification of X).

Let further Φ be a closed subspace of $\mathcal{C}(K)$ containing the constant functions and separating points in X. If $K = \beta X$, then $\mathcal{C}(K) \equiv \mathcal{C}_b(X)$ (Banach space of all continuous bounded real-valued functions on X). We also recall from (6) the definition of the set $K(\Phi) \subset \Phi^*$ and from (7) the definition of the canonical injection $\delta^\Phi : K \to K(\Phi)$.

Remark 16 (Other cases)**.** The forthcoming definition of Φ-trace-convexity as well as all results of this section remain true if X is in particular compact. In this case we have $X = \beta X = K$ and we can simply replace X by K in all statements. Another interesting special case is when the set X is open (and dense) into some given compact set K, see Subsection 5.2 (cf. $X = \mathbb{D}$ is the open complex disk).

We are ready to give the following definition (see also [16, Proposition 8.22]).

Definition 17 (Convex-trace sets)**.** A closed set $C \subset X$ is called convex-trace with respect to Φ, if there exists a closed convex subset $\widehat{C}$ of $(K(\Phi), w^*)$ such that

$$\delta^\Phi(C) = \delta^\Phi(X) \cap \widehat{C}. \tag{18}$$

The set of all convex-trace subsets of X will be denoted by $\mathcal{P}_{TC}(X)$.

In other words,

$$C \in \mathcal{P}_{TC}(X) \quad \Longleftrightarrow \quad C = X \cap \left(\delta^\Phi\right)^{-1}(\widehat{C}), \text{ for some } w^*\text{-closed convex set } \widehat{C} \subset K(\Phi) \tag{19}$$

Remark 18. Notice that the set $\widehat{C}$ in (18) and (19) can be taken to be the w^*-closed convex hull of $\delta^\Phi(C)$ in $K(\Phi)$. Notice further that X satisfies trivially (18), therefore it is convex-trace.

3.4.1 Trace-convexification of a set

Based on the equivalence given in (19) we obtain an alternative characterization of trace-convexity. In what follows, we denote by $[g \le r] := \{x \in X : g(x) \le r\}$ the sublevel set of the function $g \in \mathbb{R}^X$ at $r > 0$.

Proposition 19 (Characterization of convex-trace sets)**.** *Let C be a closed subset of X. The following assertions are equivalent:*

(i). $C \in \mathcal{P}_{TC}(X)$ *;*

(ii). *There exists a family $\{\phi_i\}_{i\in I} \subset \Phi$ which is uniformly bounded on K and a bounded sequence $\{\lambda_i\}_{i\in I} \subset \mathbb{R}$ such that*

$$C = \bigcap_{i\in I} \{x \in X : \phi_i(x) \leq \lambda_i\}. \tag{20}$$

(iii). *There exists $f \in TC^{<}(K)$ (lsc, convex-trace) such that*

$$C = \{x \in X : f(x) \leq 0\}. \tag{21}$$

Proof. (i)⇒(ii). Let $C \in \mathcal{P}_{TC}(X)$. Then there exists a w^*-closed convex subset $\widehat{C}$ of $K(\Phi) \subset \Phi^*$ such that (19) holds. By the Hahn-Banach separation theorem (in the locally convex space (Φ^*, w^*)) we deduce that $\widehat{C}$ is the intersection of closed half-spaces $H_i := \{Q \in \Phi^* : \langle Q, \phi_i\rangle \leq \lambda_i\}$, $i \in I$. Each half-space is defined by a linear functional ϕ_i. Since $K(\Phi)$ is $||\cdot||_{\Phi^*}$-bounded in Φ^*, we may take $||\phi_i||_\Phi = ||\phi_i||_\infty = 1$ for all $i \in I$ and deduce that $\{\lambda_i\}_{i\in I} \subset (-M, M)$, for some $M > 0$. Then we deduce from (19) that

$$C = X \cap \left(\delta^\Phi\right)^{-1}(\widehat{C}) = X \bigcap \bigcap_{i\in I} \left(\delta^\Phi\right)^{-1}(H_i) = \bigcap_{i\in I} \{x \in X : \phi_i(x) \leq \lambda_i\}.$$

(ii)⇒(iii). Let us assume that (20) holds for some uniformly bounded family $\{\phi_i\}_{i\in I}$. Since Φ contains the constant functions, we can replace ϕ_i by $\phi_i - \lambda_i$ (cf. Remark 10) and observe that $[\phi \leq \lambda_i] = [\phi - \lambda_i \leq 0]$. Then we set $g = \sup_{i\in I}(\phi_i - \lambda_i)$. It follows readily that $\bigcap_{i\in I}\{x \in X : \phi_i(x) \leq \lambda_i\} = [g \leq 0]$, while by Proposition 15, we deduce that $g \in TC^{<}(\beta X)$.

(iii)⇒(i). Let us now assume that (21) holds and let $F \in \Gamma^{<}(K(\Phi))$ (lsc convex) such that $f = F \circ \delta^\Phi$. Then $C = X \cap \left(\delta^\Phi\right)^{-1}(\widehat{C})$ where $\widehat{C} = [F \leq 0]$ is obviously closed and convex in $K(\Phi)$, and the result follows from (19). □

Using (20) of the above proposition, we can easily deduce the following corollary.

Corollary 20 (Separation theorem)**.** *Let C be a nonempty subset of X. The following assertions are equivalent:*

(i). *$C \in \mathcal{P}_{TC}(X)$ (i.e. C is (closed) convex-trace in X) ;*

(ii). *For every $\bar{x} \in K \setminus C$, then there exists $\phi \in \Phi$ such that*

$$\sup_{x\in C} \phi(x) < \phi(\bar{x}). \tag{22}$$

Given a nonempty subset $S \subset X$ and a closed subspace Φ of $\mathcal{C}(K)$ as above, we define the Φ-trace convexification $\overline{\text{co}}_\Phi(S)$ of S as follows:

$$\overline{\text{co}}_\Phi(S) = \bigcap_{S\subset C} \{C : C \in \mathcal{P}_{TC}(X)\}$$

The following result follows easily from the definitions (see also Definition 8.15 and Proposition 8.22 in [16].)

Proposition 21 (Characterization of convexification)**.** *Let $S \subset X$ and consider the usual closed convexification of $\delta^\Phi(S)$ in $K(\Phi)$, that is,*

$$\overline{\mathrm{co}}^{w^*}(\delta^\Phi(S)) = \bigcap_{\delta^\Phi(S)\subset\widehat{C}} \left\{ \widehat{C} :\ \widehat{C} \subset K(\Phi) \ \ w^*\text{-closed and convex} \right\}.$$

Then

$$\overline{\mathrm{co}}_\Phi(S) = X \cap \left(\delta^\Phi\right)^{-1} (\overline{\mathrm{co}}^{w^*}(\delta^\Phi(S)).$$

3.4.2 Abstract Krein-Milman theorem and relation with Ky Fan convexity

If $X = K$ is compact, property (ii) of Proposition 19 corresponds to the definition of Φ-convexity given by Ky Fan [10]. Therefore Proposition 19 shows that:

– A set $C \subset K$ is Φ-convex-trace if and only if it is Φ-convex in the sense of Ky Fan.

Before we proceed, let us recall the classical Krein-Milman theorem in a locally convex space E. For $x, y \in E$, we define the open segment $(x, y) := \{tx + (1-t)y : t \in (0,1)\} \subset E$. We first recall the definition of an extreme point.

Definition 22 (Extreme point)**.** Let S be a nonempty subset of a locally convex space E. We say that $\bar{p} \in S$ is an *extreme* point of S if whenever $\bar{p} \in (p_1, p_2)$, with $p_1, p_2 \in C$, it holds $p_1 = p_2 = \bar{p}$. We denote by $\mathrm{Ext}(S)$ the set of all extreme points of S.

Recall that the Krein-Milman theorem asserts that if C is a convex compact subset of a locally convex space, then C is the closed convex hull of its extreme points, that is, $C = \overline{\mathrm{co}}(\mathrm{Ext}(C))$. A more precise version asserts that for any nonempty subset A of C it holds:

$$\overline{\mathrm{co}}(A) = \overline{\mathrm{co}}(\mathrm{Ext}(\bar{A})).$$

The Krein-Milman theorem has a partial converse known as Milman's theorem (see [18] eg.) which states that if A is a subset of C and the closed convex hull of A is all of C, then every extreme point of C belongs to the closure of A, that is,

$$(A \subset C;\ \ C = \overline{\mathrm{co}}(A)) \Longrightarrow \mathrm{Ext}(C) \subset \overline{A}.$$

We shall now see that the above results can be naturally stated for Φ-convex-trace subsets of a completely regular topological space X. To this end, let us start with the following definition which extends the notion of an extreme point in this topological setting.

Definition 23 (Φ-extreme point)**.** Let X be completely regular topological space and $S \subset X$. A point $x \in S$ is called Φ-extreme in S if $\delta_x^\Phi = \delta^\Phi(x)$ is an extreme point of $\overline{\mathrm{co}}^{w^*}(\delta^\Phi(S))$. We denote by $\mathrm{Ext}_\Phi(S)$ the set of all Φ-extreme points of S.

We now show that the Krein-Milman theorem holds true in our abstract setting. An alternative proof of the same result can be found in [16, Chapter 8].

Theorem 24 (Abstract Krein-Milman theorem)**.** *Let $S \subset X$ be a compact set. Then,*

$$\overline{\mathrm{co}}_\Phi(S) = \overline{\mathrm{co}}_\Phi(\mathrm{Ext}_\Phi(S)).$$

Therefore, if $C \subset X$ is compact and Φ-convex-trace, then $C = \overline{\mathrm{co}}_\Phi(\mathrm{Ext}_\Phi(C))$.
In other words, a compact Φ-convex-trace set is the Φ-convex hull of its Φ-extreme points.

Proof. Applying the Krein-Milman theorem in the locally convex space (Φ^*, w^*) for the convex compact set $C := \overline{\text{co}}^{w^*}(\delta^\Phi(S)) \subset K(\Phi)$ we have that

$$\overline{\text{co}}^{w^*}(\delta^\Phi(S)) = \overline{\text{co}}^{w^*}(\text{Ext}(\overline{\text{co}}^{w^*}(\delta^\Phi(S))).$$

On the other hand, by the partial converse of the Krein-Milman theorem (Milman's theorem), setting $A = \delta^\Phi(S)$ we deduce that

$$\text{Ext}(\overline{\text{co}}^{w^*}(\delta^\Phi(S)) \subset \overline{\delta^\Phi(S)}^{w^*} = \delta^\Phi(S).$$

It follows from the definition of Φ-extreme points that

$$\text{Ext}(\overline{\text{co}}^{w^*}(\delta^\Phi(S)) = \delta^\Phi(\text{Ext}_\Phi(S)).$$

Using Proposition 21, the Krein-Milman theorem and the above equality, we have

$$\begin{aligned}
\overline{\text{co}}_\Phi(S) &= X \cap \left(\delta^\Phi\right)^{-1}(\overline{\text{co}}^{w^*}(\delta^\Phi(S))) \\
&= X \cap \left(\delta^\Phi\right)^{-1}(\overline{\text{co}}^{w^*}(\text{Ext}(\overline{\text{co}}^{w^*}(\delta^\Phi(S))))) \\
&= X \cap \left(\delta^\Phi\right)^{-1}(\overline{\text{co}}^{w^*}(\delta^\Phi(\text{Ext}_\Phi(S)))) \\
&= \overline{\text{co}}_\Phi(\text{Ext}_\Phi(S)).
\end{aligned}$$

This gives the first part of the theorem. For the second part, if C is assumed compact and Φ-convex trace, then $C = \overline{\text{co}}_\Phi(C)$ and the conclusion follows from the first part. ☐

The next result can be found also in [16, Corollary 8.19]

Corollary 25. *Let K be a compact Hausdorff topological space and Φ be a closed subspace of $\mathcal{C}(K)$ containing the constant functions and separating points in K. Then, we have*

$$K = \overline{\text{co}}_\Phi(\partial_\Phi(K)).$$

Proof. Comparing Definition 23 with (9) we easily see that $x \in \text{Ext}_\Phi(K)$ if and only if $x \in \partial_\Phi(K)$. In other words, the Φ-extreme points of K and the elements of the Choquet boundary of K are the same. On the other hand, the set K is trivially Φ-trace convex, since $K(\Phi)$ is convex compact and $\delta^\Phi(K) = \delta^\Phi(K) \cap K(\Phi)$. The conlusion is straightforward from Theorem 24. ☐

Remark 26 (Comparison with the Ky Fan theory)**.** According to the Ky Fan theory ([10], [14]), given $y, z \in K$, the Φ-segment $[y, z]_\Phi$ is defined to be the set of all $x \in K$ such that for any $\phi \in \Phi$ the following implication holds:

$$\phi(x) \leq \min\{\phi(y), \phi(z)\} \Longrightarrow \phi(x) = \phi(x) = \phi(y). \tag{23}$$

Then a point $x \in K$ is called Φ-extreme (in the sense of Ky Fan) for the compact set K if whenever $x \in [y, z]_\Phi$ for $y, z \in K$, it holds $x = y = z$. We now prove the following claim.

Claim. Every Φ-extreme point of K (cf. Definition 23) is Φ-extreme in the sense of Ky Fan.

Proof of the Claim. Indeed, by (9) we have $x \in \text{Ext}_\Phi(K) \Leftrightarrow \delta_x^\Phi \in \text{Ext}(K(\Phi))$ (recall that $K(\Phi) = \overline{\text{co}}^{w^*}(\delta^\Phi(K)))$. Let us assume, towards a contradiction, that $x \in \text{Ext}_\Phi(K)$ and $x \in [y, z]_\Phi$ for some $y, z \in K$ with $y \neq x$. Then if $z = x$, since Φ separates points in K we get

$\phi(x) = \phi(z) < \phi(y)$ for some $\phi \in \Phi$, contradicting (23). If now both y, z are different than x, then in view of (9) δ_x^Φ is extreme in $K(\Phi)$ and consequently $\delta_x^\Phi \notin [\delta_y^\Phi, \delta_z^\Phi]$ (the usual segment in the w^*-compact convex set $K(\Phi) \subset \Phi^*$). By Hahn-Banach theorem (for the $\sigma(\Phi^*, \Phi)$-topology of Φ^*) we deduce that for some $\phi \in \Phi$,

$$\min\{\phi(y), \phi(z)\} = \min\{\langle \delta_y^\Phi, \phi\rangle, \langle \delta_z^\Phi, \phi\rangle\} \geq \min_{Q \in [\delta_y^\Phi . \delta_z^\Phi]} \langle Q, \phi\rangle > \langle \delta_x^\Phi, \phi\rangle = \phi(x),$$

which again contradicts (23). This completes the proof of the claim.

The converse of the claim is not true in general, since Φ-extreme points in the sense of Ky Fan may be numerous. To see this, take for instance $X = \mathbb{D}$ to be the unit disk of the complex plane and Φ be the class of harmonic functions of the open disk, which are continuous on the closed disk $\bar{\mathbb{D}}$ (see Subsection 5.2). Then we easily see that all Φ-segments are trivial (singletons) and consequently all points of $\bar{\mathbb{D}}$ are extreme (whereas the Choquet boundary of $\mathbb{D}$ coincides with the usual topological boundary).

Therefore, Theorem 24 is an enhanced version of the Ky Fan result in [10] (see also [14]).

4 Maximum principles for convex-trace functions.

In this section we establish a general version of maximum principle, that goes beyond Choquet convexity, and is adapted to the setting of Definition 8. In particular:

In Subsection 4.1 we establish a maximum principle for upper semicontinuous convex-trace functions on a compact topological space (Theorem 28), generalizing the maximum principle obtained in [17, Section 3.2] in a twofold aspect: the function f is not necessarily continuous, and the compact K is not assumed to be metrizable.

In Subsection 4.2 we consider the metrizable case and establish enhanced versions of the maximum principle evoking a family of functions as well as a genericity result.

4.1 Maximum principle in topological spaces.

We recall from Corollary 13 that the class of Choquet convex functions coincides with the class of continuous convex-trace functions, while our results are formulated in $T\mathcal{C}^{>}(K)$ (upper semicontinuous convex-trace functions). Our results are based on the classical Bauer maximum principle.

Before we proceed, let us introduce the following notation: for a nonempty set C and a function $f : C \to \mathbb{R}$, we denote by

$$C_{\max}(f) := \{\bar{x} \in C : f(\bar{x}) = \max_{x \in C} f(x)\},$$

the set of maximizers of f on C. We also denote by $C_{\min}(f) := C_{\max}(-f)$ the set of minimizers of f on C. Under this notation we have the following result:

Proposition 27 (Maximizing a convex function on $K(\Phi)$)**.** *Let K be a compact space. Let $F : (K(\Phi), w^*) \to \mathbb{R}$ be an upper semicontinuous convex function. Then, we have that*

$$\max_{Q \in K(\Phi)} F(Q) \;=\; \max_{x \in K} (F \circ \delta^\Phi)(x),$$

and consequently,

$$\delta^\Phi(K_{\max}(F \circ \delta^\Phi)) \subset [K(\Phi)]_{\max}(F).$$

Proof. Using the classical Bauer theorem, we have that

$$\max_{Q\in K(\Phi)} F(Q) \;=\; \max_{Q\in \mathrm{Ext}(K(\Phi))} F(Q).$$

Since $\mathrm{Ext}(K(\Phi)) = \delta^{\Phi}(\partial_{\Phi}(K)) \subset \delta^{\Phi}(K)$, it follows that

$$\max_{Q\in K(\Phi)} F(Q) \;\leq\; \max_{x\in K}(F\circ\delta^{\Phi})(x).$$

On the other hand, since $\delta^{\Phi}(K) \subset K(\Phi)$, the above inequality is in fact an equality. Therefore we have that $\delta^{\Phi}(K_{\max}(f)) \subset [K(\Phi)]_{\max}(F)$ as asserted. $\square$

We now establish the following result, which extends [17, Maximum principle (page 241)] from the class of Choquet convex functions (which coincides with $T\mathcal{C}(K)$) to the class of usc convex-trace functions $T\mathcal{C}^{>}(K)$.

Theorem 28 (Bauer maximum principle for usc convex-trace functions)**.** *Let K be a compact space and $f : K \to \mathbb{R}$ be an usc convex-trace function. Then, there exists $\bar{x} \in \partial_{\Phi}(K)$ such that $f(\bar{x}) = \max_{x\in K} f(x)$.*

Proof. By definition, there exists an upper semicontinuous convex function $F : (K(\Phi), w^*) \to \mathbb{R}$ such that $f = F\circ\delta^{\Phi}$. Applying the classical Bauer theorem to F, there exists $\widehat{Q} \in \mathrm{Ext}(K(\Phi))$ such that

$$\max_{Q\in K(\Phi)} F(Q) = F(\widehat{Q}).$$

Since $\mathrm{Ext}(K(\Phi)) = \delta^{\Phi}(\partial_{\Phi}(K))$, there exists $\bar{x} \in \partial_{\Phi}(K)$ such that $\widehat{Q} = \delta^{\Phi}_{\bar{x}}$. It follows that

$$\max_{Q\in K(\Phi)} F(Q) = f(\bar{x}).$$

Since $\delta^{\Phi}(K) \subset K(\Phi)$, the inequality $\max_{Q\in K(\Phi)} F(Q) \geq \max_{x\in K} f(x)$ holds trivially.
The proof is complete. $\square$

Theorem 28 yields directly the following result.

Corollary 29. *Let K be a compact space and $f : K \to \mathbb{R}$ be an upper semicontinuous convex-trace function. If $f(x) \leq 0$ for all $x \in \partial_{\Phi}(K)$, then $f(x) \leq 0$, for all $x \in K$.*

Recalling from (13) the definition of $\widehat{f}$, and combining Theorem 28 with Theorem 11, we obtain the following corollary.

Corollary 30. *Let K be a compact space and $f \in \mathcal{C}(K)$. Then, there exists $\bar{x} \in \partial_{\Phi}(K)$ such that $\widehat{f}(\bar{x}) = \max_{x\in K} \widehat{f}(x)$.*

4.2 The maximum principle for compact metric space.

In this subsection we focus on the case where the compact space K is metrizable, which in fact, is the usual setting for the notion of Choquet convexity, and the framework considered in [17]. In this case, making use of the metric structure of K and of the metrizability of the w^*-topology of $K(\Phi)$, and using an adequate version of variational principle, we are going to establish extensions of the Bauer maximum principle in two directions:

- We shall deal with the class $T\mathcal{C}^{>}(K)$ of upper semicontinuous convex-trace functions (this class contains strictly the class of Choquet convex functions).
- We establish a multi-maximum result evoking a family of functions (Theorem 35), as well as an abstract generic result (Theorem 37).

Let us start by recalling from [2, Lemma 3] the following version of variational principle that we shall use in the sequel.

Lemma 31 (Variational Principle). *Let (K,d) be a compact metric space and $(\Phi, \|.\|_\Phi)$ be a Banach space such that $\Phi \subset \mathcal{C}(K)$, Φ separates points in K and for some $\alpha > 0$ it holds:*

$$\alpha \, \|\phi\|_\Phi \, \geq \, \|\phi\|_\infty, \quad \textit{for all } \phi \in \Phi.$$

Let $f : (K,d) \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. Then, the set

$$N(f) = \big\{ \phi \in \Phi : \; K_{\min}(f - \phi) \textit{ is not a singleton } \big\}$$

is of first Baire category in Φ.

Before we proceed, let us recall the following definition.

Definition 32 (w^*-exposed points). Let E be a locally convex space and S a nonempty w^*-closed subset of the dual space E. We say that $\bar{p} \in S$ is w^*-*exposed* in S, and denote $\bar{p} \in w^*\text{-Exp}(S)$, if there exists $x \in X$ such that

$$\langle \bar{p}, x \rangle > \langle p, x \rangle, \quad \text{for all } p \in S \setminus \{\bar{p}\}.$$

It is straightforward from Definition 22 and Definition 32 that every w^*-exposed point is extreme, that is $w^*\text{-Exp}(S) \subset \text{Ext}(S)$. This inclusion might in general be strict.

The classical Krein-Milman theorem ensures the existence of extreme points for convex compact sets. However, in absence of convexity, we cannot in general conclude that $\text{Ext}(S) \neq \emptyset$. Still, the conclusion holds true if E is a Banach space and $S \subset E$ is compact for either the norm or the weak topology. But if $E = X^*$ is a dual Banach space and S is w^*-compact, the conclusion could fail. This being said, using Lemma 31 we deduce an important instance of w^*-compact sets with extreme points by establishing the existence of w^*-exposed points.

Lemma 33 (existence of extreme points). *Let Φ be a Banach space and $S \subset \Phi^*$ be w^*-compact and metrizable. Then $w^*\text{-Exp}(S) \neq \emptyset$ and consequently $\text{Ext}(S) \neq \emptyset$.*

Proof. Every $\phi \in \Phi$ can be seen as a continuous function on the compact metric space (S, w^*). Set: $\alpha := \max_{p\in S} \|p\| + 1$. Then

$$\|\phi\|_\Phi \;=\; \sup_{p\in\Phi^*} |\langle \frac{p}{||p||}, \phi\rangle| \;\geq\; \sup_{p\in S} |\langle \frac{p}{||p||}, \phi\rangle| \;\geq\; \frac{1}{\alpha} \sup_{p\in S} |\langle p, \phi\rangle| \;=\; \frac{1}{\alpha}\, \|\phi\|_\infty,$$

where the last inequality is based on the linearity of ϕ. Since obviously Φ separates points in S, we can apply Lemma 31 to the function $f \equiv 0$ to deduce that for a generic $\phi \in \Phi$, $-\phi$ attains a unique minimum on S at some point $\bar{p} \in S$. This yields that $\bar{p} \in w^*\text{-Exp}(S) \subset \text{Ext}(S)$. Therefore, both $w^*\text{-Exp}(S)$ and $\text{Ext}(S)$ are nonempty. $\square$

4.2.1 A multi-maximum principle.

Let us first establish the following result, which has an independent interest.

Lemma 34 (Common extreme maximizer)**.** *Let Φ be a Banach space and $C \subset \Phi^*$ be convex w^*-compact and metrizable. Let $\{F_i\}_{i\in I}$ be a nonempty family of real-valued w^*-usc, convex functions on (C, w^*) with a common maximizer, that is,*

$$C_{\max}(I) := \bigcap_{i\in I} C_{\max}(F_i) \neq \emptyset.$$

Then

$$w^*\text{-Exp}(C_{\max}(I)) \neq \emptyset \quad \textit{and} \quad \text{Ext}(C_{\max}(I)) \subset \text{Ext}(C).$$

In particular, there exists $\bar{p} \in \text{Ext}(C)$ such that

$$F_i(\bar{p}) = \max_{p\in C} F_i(p), \qquad \textit{for all } i \in I.$$

Proof. Since $F_i : (C, w^*) \longrightarrow \mathbb{R}$ is usc, the set $C_{\max}(F_i)$ is nonempty and w^*-compact in Φ^*. By hypothesis,

$$C_{\max}(I) = \bigcap_{i\in I} C_{\max}(F_i) \neq \emptyset.$$

Since $S = C_{\max}(I)$ is nonempty w^*-compact and metrizable in Φ^*, applying Lemma 33 we deduce that

$$w^*\text{-Exp}(S) \neq \emptyset.$$

It remains to show that $\text{Ext}(S) \subset \text{Ext}(C)$. To this end, let $\bar{p} \in \text{Ext}(S)$ and assume, towards a contradiction, that there exists $p_1, p_2 \in C \setminus \{\bar{p}\}$ such that $\bar{p} \in (p_1, p_2)$. Since $\bar{p} \in \text{Ext}(S)$, we may assume with no loss of generality that $p_1 \in C \setminus S$. Therefore, there exists $i_0 \in I$ such that $p_1 \notin C_{\max}(F_{i_0})$. It follows that

$$F_{i_0}(p_1) < \max_{p\in C} F_{i_0}(p) = F_{i_0}(\bar{p}) \quad \text{ and } \quad F_{i_0}(p_2) \leq \max_{p\in C} f_{i_0}(p) = F_{i_0}(\bar{p}),$$

which contradicts the fact that F_{i_0} is convex. Thus, $w^*\text{-Exp}(S) \subset \text{Ext}(S) \subset \text{Ext}(C)$ and the conclusion follows. $\square$

We are now ready to establish the following result which is a generalized version of Bauer's maximum principle. Roughly speaking, whenever a family of usc convex-trace functions on K has at least one common maximizer, then a common maximizer can be found among the points of the Choquet boundary of K.

Theorem 35. *Let K be a compact metric space and Φ a closed subspace of $\mathcal{C}(K)$ that separates points in K and contains the constant functions. Let further $\{f_i\}_{i\in I} \subset T\mathcal{C}^{>}(K,\Phi)$ be such that*

$$K_{\max}(I) := \bigcap_{i\in I} K_{\max}(f_i) \neq \emptyset. \tag{24}$$

Then, there exists $\bar{x} \in \partial_\Phi(K)$ (Choquet boundary of K) such that

$$f_i(\bar{x}) = \max_{x\in K} f_i(x), \quad \textit{for every } i \in I.$$

Proof. Since K is compact metric space, $\mathcal{C}(K)$ is separable and so is its closed subspace Φ. It follows that the convex w^*-compact subset $K(\Phi)$ of Φ^* is metrizable. By Definition 8, for each $i \in I$ there exists an usc convex function $F_i : (K(\Phi), w^*) \to \mathbb{R}$ such that $f_i = F_i \circ \delta^\Phi$. Set $C := (K(\Phi), w^*)$. By Proposition 27, $\delta^\Phi(K_{\max}(I)) \subset \bigcap_{i\in I} C_{\max}(F_i)$, therefore by (24)

$$C(I) := \bigcap_{i\in I} C_{\max}(F_i) \neq \emptyset.$$

Then Lemma 34 yields the existence of a common maximizer

$$\bar{Q} \in \mathrm{Ext}(K(\Phi)) = \delta^\Phi(\partial_\Phi(K))$$

for all usc convex functions F_i, $i \in I$. Therefore, there exists $\bar{x} \in \partial_\Phi(K)$ such that $\bar{Q} = \delta^\Phi_{\bar{x}}$. Since

$$\max_{x\in K} f_i(x) \leq \max_{Q\in K(\Phi)} F_i(Q) = F_i(\delta^\Phi_{\bar{x}}) = f_i(\bar{x}), \quad \text{for all } i \in I,$$

we conclude that $\bar{x} \in \bigcap_{i\in I} K_{\max}(f_i)$. Therefore we conclude that $\bar{x} \in \partial_\Phi(K)$ is a common maximizer of all functions f_i, $i \in I$. $\square$

We obtain the following characterization of the Choquet boundary of a compact metric space.

Corollary 36 (Characterization of the Choquet boundary)**.** *Let K be a compact metric space and Φ as in Theorem 35. Then*

$$\bar{x} \in \partial_\Phi(K) \Longleftrightarrow \{\bar{x}\} = \bigcap \left\{K_{\max}(f) : \bar{x} \in K_{\max}(f), f \in T\mathrm{C}^{>}(K)\right\}.$$

Proof. Let us first assume that

$$\bigcap \left\{K_{\max}(f) : \bar{x} \in K_{\max}(f), f \in TC^{>}(K)\right\} = \{\bar{x}\}. \tag{25}$$

Then setting

$$\mathcal{F} := \left\{f \in TC^{>}(K) :\ \bar{x} \in K_{\max}(f)\right\}$$

we have $\bigcap_{f\in\mathcal{F}} K_{\max}(f) \neq \emptyset$. Therefore, by Theorem 35, $\bigcap_{f\in\mathcal{F}} K_{\max}(f) \cap \partial_\Phi(K) \neq \emptyset$ and consequently, $\bar{x} \in \partial_\Phi(K)$.

Conversely, let $\bar{x} \in \partial_\Phi(K)$, that is, $\delta_{\bar{x}} \in \mathrm{Ext}(K(\Phi))$. We define the function $F_{\bar{x}} : (K(\Phi), w^*) \to \mathbb{R}$ by $F_{\bar{x}}(\delta_{\bar{x}}) = 1$ and $F_{\bar{x}}(Q) = 0$ on $K(\Phi) \setminus \{\delta_{\bar{x}}\}$. Clearly $F_{\bar{x}}$ is convex and upper semicontinuous on $(K(\Phi), w^*)$. It follows that $f_{\bar{x}} := F_{\bar{x}} \circ \delta^\Phi \in TC^{>}(K)$ and $K_{\max}(f_{\bar{x}}) = \{\bar{x}\}$, which readily yields (25). $\square$

4.2.2 Generic maximum principle.

In this section we shall establish a generic version of the maximum principle. Similarly to Subsection 4.2.1, we consider a compact metric space (K,d) and we assume that Φ is a closed subspace of $\mathcal{C}(K)$ that separates points in K and contains the constant functions. In this setting, the Banach space $\mathcal{C}(K)$ (and a fortiori Φ) is separable and the convex set $K(\Phi) \subset \Phi^*$ is w^*-compact metrizable. We shall denote by d_Φ a metric on $K(\Phi)$ compatible with its w^*-topology. Then $(K(\Phi), d_\Phi)$ is also a compact metric space.

For any nonempty set X we denote by $\mathbb{R}^X$ the space of all real-valued functions on X and by ρ_∞ the (complete) metric;

$$\rho_\infty(f,g) := \sup_{x\in X} \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}, \quad \text{for all } f,g \in \mathbb{R}^X. \tag{26}$$

In what follows, we set

$$\widehat{\Phi} := \{\widehat{\phi} \in \Phi^{**} : \phi \in \Phi\}, \quad \text{where } \widehat{\phi}(Q) = \langle Q, \phi\rangle, \text{ for all } Q \in \Phi^*.$$

Recalling terminology from Definition 8, and dropping dependence on Φ in the notation of convex-trace functions, we have:

$$\Phi \subset \Gamma_\Phi(K) = T\mathcal{C}(K) \subset T\mathcal{C}^>(K) \quad \text{and} \quad \widehat{\Phi} \subset \Gamma(K(\Phi)) \subset \Gamma^>(K(\Phi)).$$

We also recall from Remark 10 that a subset $\mathcal{B}$ of $T\mathcal{C}^>(K)$ is called Φ-stable, if $\Phi + \mathcal{B} \subset \mathcal{B}$. Examples of Φ-stable subsets of $T\mathcal{C}^>(K)$ are Φ itself, the class of Choquet convex functions $\Gamma_\Phi(K) = T\mathcal{C}(K)$ and the set of usc convex-trace functions $T\mathcal{C}^>(K)$.

Let us finally notice that Theorem 35 (applied to a family of one element) yields that if a function $f \in T\mathcal{C}^>(K)$ has a unique maximizer, then this maximizer belongs to the Choquet boundary of K. We are now ready to state the main result of this section.

Theorem 37 (Genericity of unique maximizer)**.** *Let (K,d) be a compact metric space and Φ be a closed subspace of $C(K)$ which separates the points of K. Let $\mathcal{B}$ be a Φ-stable subset of $T\mathcal{C}^>(K)$. Then, the set*

$$\mathcal{G} := \{f \in \mathcal{B} : K_{\max}(f) \textit{ is a singleton}\}$$

is a G_δ dense subset in $(\mathcal{B}, \rho_\infty)$.

Proof. Let us denote by ρ_∞ the metric of uniform convergence on both $\mathbb{R}^{K(\Phi)}$ and $\mathbb{R}^K$. Since uniform limits maintain upper semicontinuity and convexity, it follows that the metric space $(\Gamma^>(K(\Phi)), \rho_\infty)$ is complete (as a closed subspace of $(\mathbb{R}^{K(\Phi)}, \rho_\infty)$). A standard argument now shows that $(T\mathcal{C}^>(K), \rho_\infty)$ is closed in $(\mathbb{R}^K, \rho_\infty)$ and consequently, it is also complete.

We now consider the following canonical map $\mathcal{S} : (\Gamma^>(K(\Phi)), \rho_\infty) \longrightarrow (T\mathcal{C}^>(K), \rho_\infty)$ defined by $\mathcal{S}(\widehat{F}) = \widehat{F} \circ \delta^\Phi$. It is easily seen that $\mathcal{S}$ is surjective and 1-Lipschitz. Moreover, it is easy to see that there exists $\mathcal{A} \subset \Gamma^>(K(\Phi))$ such that $\mathcal{B} = \mathcal{S}(\mathcal{A})$ and $\widehat{\Phi} + \mathcal{A} \subset \mathcal{A}$ (that is, $\mathcal{A}$ is a $\widehat{\Phi}$-stable subset of $\Gamma^>(K(\Phi))$).

Claim. The set $\mathcal{D} := \left\{\widehat{F} \in \mathcal{A} : \widehat{F} \text{ has unique maximum on } K(\Phi)\right\}$ is G_δ dense in $(\mathcal{A}, \rho_\infty)$.

Proof of the Claim. For $n \geq 1$, we set:

$$\widehat{\mathcal{U}}_n := \left\{\widehat{F} \in \mathcal{A};\ \exists Q_n \in K(\Phi) \ \text{ with } \widehat{F}(Q_n) > \sup_{Q\in K(\Phi):\, d_\Phi(Q,Q_n)\geq \frac{1}{n}} \widehat{F}(Q)\right\}.$$

It is easy to see that $\widehat{\mathcal{U}}_n$ is an open subset of $(\mathcal{A}, \rho_\infty)$ for all $n \geq 1$, and $\mathcal{D} := \bigcap_{n\geq 1} \widehat{\mathcal{U}}_n$. Thanks to Lemma 31 (applied to the compact metric space $(K(\Phi), d_\Phi)$ and the subspace $(\widehat{\Phi}, \|\cdot\|_\infty)$ of $\mathcal{C}(K(\Phi))$), for every $0 < \varepsilon < 1$ and $\widehat{F} \in \mathcal{A}$, there exists a function $\phi \in \Phi$ such that $\rho_\infty(\widehat{\phi}, 0) < \varepsilon$ and $-\widehat{F} - \widehat{\phi}$ attains a unique minimum on $K(\Phi)$. Let us denote by $Q_0 \in K(\Phi)$ this unique minimizer. Then we deduce that $\widehat{G} := \widehat{F} + \widehat{\phi} \in \bigcap_{n\geq 1} \widehat{\mathcal{U}}_n$ (we take $Q_n = Q_0$, for all $n \geq 1$) and $\rho_\infty(\widehat{G}, \widehat{F}) < \varepsilon$. Thus the G_δ-set $\mathcal{D}$ is dense in $(\mathcal{A}, \rho_\infty)$ and the claim follows.

From the classical Bauer theorem, the unique maximizer of every usc convex function $\widehat{F} \in \mathcal{D} \subset \Gamma^{>}(K(\Phi))$ is necessarily an extreme point of $K(\Phi)$. Then by Proposition 27 we deduce that for every $\widehat{F} \in \mathcal{D}$, the usc convex-trace function $\mathcal{S}(\widehat{F}) = \widehat{F} \circ \delta^\Phi$ has a unique maximum on K, which is necessarily attained at a point in the Choquet boundary $\partial_\Phi(K)$. Since

$$\mathcal{D} \subset \left\{ \widehat{F} \in \mathcal{A} : K_{\max}\left(\mathcal{S}(\widehat{F})\right) \text{ singleton} \right\},$$

we obtain readily that $\mathcal{S}(\mathcal{D}) \subset \mathcal{G}$. Since $\mathcal{S}$ is a Lipschitz surjective map and $\mathcal{D}$ is dense in $(\mathcal{A}, \rho_\infty)$, we deduce that $\mathcal{G}$ is dense in $(\mathcal{B}, \rho_\infty)$. Moreover $\mathcal{G}$ is a G_δ subset of $(T\mathcal{C}^{>}(K), \rho_\infty)$ since it can be written as $\mathcal{G} = \bigcap_{n\geq 1} \mathcal{U}_n$, where

$$\mathcal{U}_n := \left\{ f \in \mathcal{B};\ \exists x_n \in K\ f(x_n) > \sup_{x\in K:\, d(x,x_n)\geq \frac{1}{n}} f(x) \right\},$$

is open in $(\mathcal{B}, \rho_\infty)$ for each $n \geq 1$. This completes the proof. □

It follows from the above result, by taking $\mathcal{B} = T\mathcal{C}^{>}(K)$, that a generic upper semicontinuous convex-trace function on K attains a unique maximum (necessarily at a point of the Choquet boundary). By taking $\mathcal{B} = \Gamma_\Phi(K)$, we obtain the same conclusion for a generic Choquet-convex function. Both results are new and together with Theorem 35 provide generalized version of the classical Bauer maximum principle.

5 Examples.

In this section we provide three examples-schemes to illustrate this theory. In the first example (Subsection 5.1) we show how the Choquet boundary of the closed interval $[0,1]$ may change depending on the choice of Φ. In particular, every closed subset of $[0,1]$ that contains the extreme points $\{0,1\}$ can be identified to the Choquet boundary of $[0,1]$ under an adequate choice of the space $\Phi \subset \mathcal{C}([0,1])$. In the second example (Subsection 5.2) we deal with the unit disk of the complex plane. Then the class of convex-trace functions consists of the subharmonic functions, if Φ is taken to be the harmonic functions. We describe the convex-traces subsets of the disk using Runge's approximation theorem as well as the maximum principle for harmonic functions. Finally, in Subsection 5.3 we illustrate trace-convexity for subsets of the set of natural numbers $\mathbb{N}$ (with its discrete topology).

5.1 Choquet boundaries of $[0,1]$.

Let us first notice that in view of Definition 23 and relation (9), the set $\mathrm{Ext}_\Phi(K)$ of Φ-extreme points of K coincides with the Φ-Choquet boundary of K (see also the proof of Corollary 25).

We shall now describe the Φ-Choquet boundary of the closed interval $K = [0,1]$, under various choices of closed subspaces $\Phi \subset \mathcal{C}([0,1])$.

(i). Let us first consider the case $\Phi = \mathrm{Aff}([0,1])$. In this case we have (cf. Remark 6)

$$\partial_\Phi([0,1]) = \partial([0,1]) = \{0,1\} \quad \text{and} \quad \Gamma_\Phi([0,1]) = \Gamma([0,1]),$$

that is, the Choquet boundary coincides with the usual boundary and the class of Choquet functions coincides with the class of convex continuous functions on $[0,1]$.

(ii). Let $\Delta \subset [0,1]$ be the usual Cantor set. Then $[0,1] \setminus \Delta = \bigcup_{n\geq 1}(a_n, b_n)$ with $\{a_n\}_n$, $\{b_n\}_n \subset \Delta$. Then defining

$$\Phi_\Delta := \left\{\phi \in \mathcal{C}([0,1] : \ \phi|_{[a_n,b_n]} \text{ affine, for all } n \geq 1\right\},$$

we obtain that the Cantor set Δ becomes the Φ_Δ-Choquet boundary of $[0,1]$, that is,

$$\partial_{\Phi_\Delta}([0,1]) = \Delta.$$

Indeed, if $x \in [0,1] \setminus \Delta$, then there exists $n_0 \geq 1$ such that $x \in (a_{n_0}, b_{n_0})$ and consequently $x = ta_{n_0} + (1-t)b_{n_0}$, for some $t \in (0,1)$. We set

$$\mu := t\delta_{a_{n_0}} + (1-t)\delta_{b_{n_0}} \in \mathcal{M}^1([0,1]).$$

Since every $\phi \in \Phi_\Delta$ is affine on $[a_n, b_n]$ we obtain

$$\langle \mu.\phi\rangle = t\phi(a_{n_0}) + (1-t)\phi(b_{n_0}) = \phi(x) = \langle \delta_x.\phi\rangle,$$

yielding $\mu \sim \delta_x$ and consequently $x \notin \partial_{\Phi_\Delta}([0,1])$. On the other hand, if $x \in \Delta$, then x is the unique maximum of the function $\phi \in \Phi_\Delta$ defined by $\phi(t) = -|t-x|$, $t \in [0,1]$. Therefore, $x \in \partial_{\Phi_\Delta}([0,1])$.

Remark. The above proof works in the same way for any closed subset F of $[0,1]$ that contains the extreme points $\{0,1\}$. Therefore, any such closed subset is the Φ-Choquet boundary of $[0,1]$ under an adequate choice of Φ.

(iii). Let us now take $\Phi = \mathcal{C}[0,1]$. Then every point of $[0,1]$ belongs to the Choquet-boundary (i.e. $\partial_\Phi([0,1]) = [0,1]$) and every continuous function is Φ-Choquet convex.

5.2 Harmonic functions on the disk.

In this subsection we shall deal with harmonic functions on the unit disk. For prerequisites in complex analysis, as well as for notions and results that will be evoked in this section we refer the reader to [11].

Let $\mathbb{D} := \{z \in \mathbb{C} : \ |z| < 1\}$ denote the open disk of the complex plane and $K = \bar{\mathbb{D}}$ its closure. We set:

$$\Phi = \{u : \bar{\mathbb{D}} \to \mathbb{R} \text{ continuous, } u|_{\mathbb{D}} \text{ harmonic}\}.$$

In this case, the class of Choquet-convex functions $\Gamma_\Phi(K) = T\mathcal{C}(K)$ coincides with the continuous subharmonic functions on $\mathbb{D}$, while $T\mathcal{C}^>(K)$ corresponds to the class of upper semicontinuous subharmonique functions.

Let us now recall that subharmonique functions satisfy the *maximum principle*, that is, the maximum of any subharmonic function u over any set is attained at the boundary of the set. Combining this with Corollary 36 we deduce that the Choquet boundary $\partial_{\Phi}\bar{\mathbb{D}}$ of $\bar{\mathbb{D}}$ is contained in $\partial\bar{\mathbb{D}} := \{z \in \mathbb{C} : |z| = 1\}$. On the other hand, any $\bar{z} \in \partial\mathbb{D}$ is the unique maximizer of some $\phi \in \Phi$ (to see this, it suffices to take any $h \in \mathcal{C}(\partial\bar{\mathbb{D}})$ with strict maximum at $\bar{z}$ and obtain ϕ as the unique solution of the Laplace equation $\Delta u = 0$ on $\mathbb{D}$ with $u|_{\partial\bar{\mathbb{D}}} = h$. Therefore, in view of Theorem 35 (applied to one function) we deduce:

$$\partial_{\Phi}\bar{\mathbb{D}} = \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}.$$

We shall now describe the convex-trace subsets of $\mathbb{D}$. We shall need the following lemma.

Lemma 38. *Let $\mathcal{U}$ be a nonempty open simply connected subset of $\mathbb{D}$ such that $\bar{\mathcal{U}} \subset\mathbb{D}$. Then the (closed) set $\bar{\mathcal{U}}$ is convex-trace in $\mathbb{D}$.*

Proof. We shall use the characterization of trace-convexity given in Corollary 20. To this end, let $\bar{x} \in \bar{\mathbb{D}}\diagdown\bar{\mathcal{U}}$. We distinguish two cases.

Case 1. $\bar{x} \in \partial\mathbb{D}$.

In this case, using the same standard argument that we evoked before, we deduce that $\bar{x}$ is the unique maximizer of some $\phi \in \Phi$ and consequently $\sup_{x\in\bar{\mathcal{U}}} \phi(x) < \phi(\bar{x})$.

Case 2. $\bar{x} \in \mathbb{D}\diagdown\bar{\mathcal{U}}$.

In this case, there exists $r > 0$ such that $\bar{B}(\bar{x}, r) \subset \mathbb{D}$ and $\bar{B}(\bar{x}, r) \cap \bar{\mathcal{U}} = \emptyset$. Then $\bar{B}(\bar{x}, r) \cup \bar{\mathcal{U}}$ is a compact simply connected subset of $\mathbb{D}$. Let $\mathcal{V}_1, \mathcal{V}_2$ be disjoint open simply connected subsets of $\mathbb{D}$ such that $\bar{B}(\bar{x}, r) \subset \mathcal{V}_1$ and $\bar{\mathcal{U}} \subset \mathcal{V}_2$. Then the function

$$\varphi(z) = \begin{cases} 0, & \text{if } z \in \mathcal{V}_1 \\ 1, & \text{if } z \in \mathcal{V}_2 \end{cases}$$

is (trivially) holomorphic on $\mathcal{V}_1 \cup \mathcal{V}_2$. Applying Runge's approximation theorem we deduce that there exists a complex polynomial $f(z)$ such that

$$||f - \varphi||_{\infty} := \sup_{z\in\bar{B}(\bar{x},r)\cup\bar{\mathcal{U}}} |f(z) - \varphi(z)| < \frac{1}{3}.$$

Taking $u = \mathrm{Re}(f)$ we deduce easily that $u \in \Phi$ (harmonic) and $\rho_{\infty}(u, \varphi) < 1/3$. It follows that $\sup_{x\in\bar{\mathcal{U}}} u(x) < 1/3$ and $u(\bar{x}) > 2/3$. Therefore (22) holds and $\bar{\mathcal{U}} \in \mathcal{P}_{TC}(\mathbb{D})$. $\square$

We shall now provide a nice description of convex-trace sets of $\mathbb{D}$.

Proposition 39 (convex-trace sets of the complex disk)**.** *Let C be a nonempty, compact subset of $\mathbb{D}$ with finitely many connected components. Then*

$$C \in \mathcal{P}_{TC}(X) \quad \Longleftrightarrow \quad C \textit{ is simply connected}$$

Proof. Let $C \neq \emptyset$ be compact in $\mathbb{D}$. We shall first show that "being simply connected" is a necessary condition for trace-convexity. Indeed, assume towards a contradiction that $\mathbb{D}\diagdown C$ is not path connected and let $\bar{x} \in \mathbb{D}\diagdown C$ be a point that cannot be joined to the boundary $\partial\mathbb{D}$ via a continuous path lying in $\mathbb{D}\setminus C$. Then $\bar{x}$ is surrounded by a curve lying in C and (22) fails for all $\phi \in \Phi$ in view of the maximum principle for harmonic functions.

Let us now assume that C is simply connected and let $\bar{x} \in \bar{\mathbb{D}}\setminus C$. Similarly to the proof of Lemma 38 (Case 1) we may assume that $\bar{x} \in \mathbb{D}$. Since C is compact and is contained in $\mathbb{D}$, there exists $r > 0$ sufficiently small, such that the r-enlargement $C_r := C + B(0,r)$ is open, simply connected and its closure $\bar{C}_r := C + \bar{B}(0,r)$ remains in $\mathbb{D}$. Shrinking further r if necessary, we may assume that $B(\bar{x},r) \subset \mathbb{D}$ and $B(\bar{x},r) \cap C_r = \emptyset$. Notice that (the open set) C_r has finitely many components and that each connected component is open and simply connected. We denote by $\{\mathcal{V}_i\}_{i=1}^k$ the connected components of C_r and we define $\varphi : C_r \cup B(\bar{x},r) \to \mathbb{C}$ by $\varphi|_{\mathcal{V}_i} \equiv i$ for $i \in \{1,\ldots,k\}$ and $\varphi|_{B(\bar{x},r)} \equiv k+1$. Then φ is trivially a holomorphic functions on C_ε for any $\varepsilon \in (0,r)$. Then by Runge's approximation theorem we deduce the existence of a complex polynomial $f \in H(\mathbb{D})$ with

$$||f - \varphi||_\infty = \sup_{z \in \bar{B}(\bar{x},\varepsilon) \cup C_\varepsilon} |f(z) - \varphi(z)| < \frac{1}{3}.$$

Taking $u = \mathrm{Re}(f) \in \Phi$ we conclude that

$$\sup_{x \in C} u(x) < k + \frac{1}{3} \quad \text{and} \quad u(\bar{x}) > k + \frac{2}{3}$$

which yields the result. $\square$

5.3 Trace-convexity for subsets of $\mathbb{N}$.

Let $X = \mathbb{N}$ be the set of natural numbers, viewed as a completely regular topological space with its discrete topology. Let further $K = \beta\mathbb{N}$. Then

$$C_b(\mathbb{N}) = C(\beta\mathbb{N}) = \ell^\infty(\mathbb{N}) = \left\{ y = \{y_n\}_n : \sup_{n \geq 1} |y_n| := ||y||_\infty < +\infty \right\}.$$

Let $\mathbf{1}$ denote the constant sequence with all coordinates equal to 1, and $\boldsymbol{b} = \{b_n\}_n$ with $b_n = 1/n$ for all $n \in \mathbb{N}$. We take

$$\Phi = \mathrm{span}\{\mathbf{1}, \boldsymbol{b}\} \quad \text{(2-dimensional subset of } \ell^\infty(\mathbb{N})).$$

Then Φ obviously contains the constant functions (constant sequences). It also separates points in $\mathbb{N}$ since the sequence $\boldsymbol{b}$ is injective. Notice that Φ can be isometrically identified to $\mathbb{R}^2$ by identifying $c = (c_1, c_2)$ to the (bounded) sequence $\widehat{c} = \left(c_1 + \frac{c_2}{n}\right)_{n \geq 1}$ and by equipping $\mathbb{R}^2$ with the following norm:

$$||c||_\Phi := ||\widehat{c}||_\infty = \max\{|c_1|, |c_1 + c_2|\}.$$

The positive cone of Φ (cone of positive sequences of Φ) corresponds to the cone

$$\Phi_+ = \left\{ c = (c_1, c_2) \in \mathbb{R}^2 : \, c_1 \geq 0, c_1 + c_2 \geq 0 \right\} \subset (\mathbb{R}^2, ||.||_\Phi),$$

with polar cone

$$\Phi_+^* := \left\{ Q = (Q_1, Q_2) \in \mathbb{R}^2 : \, Q_1 \geq Q_2 \geq 0 \right\} \subset (\mathbb{R}^2, ||.||_{\Phi^*})$$

where

$$||Q||_{\Phi^*} = \begin{cases} \max\{|Q_1|, |Q_2|\}, & \text{if } Q_1Q_2 > 0 \\ |Q_1| + |Q_2|, & \text{if } Q_1Q_2 < 0. \end{cases}$$

Then (6) yields:

$$K(\Phi) = \Phi_+^* \cap \{Q \in \Phi^* : ||Q||_{\Phi^*} = 1\} = \{Q \in \Phi_+^* : \langle Q, \mathbf{1}\rangle = 1\} = \{\, (1, 1+t) : \, t \in [0,1]\}\,.$$

Thus, $\operatorname{Ext} K(\Phi) = \{(1,0),(1,1)\}$. According to (7), the canonical injection $\delta^\Phi : \mathbb{N} \to K(\Phi)$ gives for $k \in \mathbb{N}$:

$$\delta^\Phi(k) = \widehat{k} \quad \text{with} \quad \widehat{k}(\widehat{c}) = \widehat{c}(k) = c_1 + \frac{c_2}{k} = \Big\langle (1, \frac{1}{k}), (c_1, c_2) \Big\rangle\,.$$

Therefore $\delta^\Phi(\mathbb{N}) = \{(1, 1/k) : k \geq 1\} \subset K(\Phi)$ and the Φ-Choquet boundary of $\mathbb{N}$ is $\partial_\Phi \mathbb{N} = \{1, \infty\}$ where $\infty \in \beta X$ is the ultrafilter generated by the sets $A_n = \{k : k \geq n\}$. It follows directly from Definition 17 that a subset $A \subset \mathbb{N}$ is convex-trace if and only if it is an *interval* with respect to the order of $\mathbb{N}$, that is, if it is of the form

$$\{k : n_1 \leq k \leq n_2\}, \; \{k : k \leq n_2\} \quad \text{or } \{k : k \geq n_1\}$$

for some $n_1, n_2 \in \mathbb{N}$. Finally, a function (sequence) $y = \{y_n\}_{n\geq 1} \in \ell^\infty$ is Choquet convex (according to Corollary 13 and Definition 8) if and only if $y_n = f(1/n)$ for some convex function $t \mapsto f(t)$ defined on $[0,1] \approx K(\Phi)$.

Let us finally mention that different interesting notions of trace-convexity can be obtained by taking $\boldsymbol{b} = (b_n)_n$ to be (injective and) non-monotone (for instance, $b_n = 1 + (-1)^n/n$, for all $n \geq 1$) or by completely different choices of Φ (of dimension either 2 or more). The interesting reader might see how these choices modify the Choquet boundary of $\mathbb{N}$ in relation with the results of Section 4.2.

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