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All convex bodies are in the subdifferential of some everywhere differentiable locally Lipschitz function

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Abstract

We construct a differentiable locally Lipschitz function f in $\mathbb{R}^N$ with the property that for every convex body $K \subset \mathbb{R}^N$ there exists $\bar{x} \in \mathbb{R}^N$ such that K coincides with the set $\partial_L f(\bar{x})$ of limits of derivatives $\{Df(x_n)\}_{n\geqslant 1}$ of sequences $\{x_n\}_{n\geqslant 1}$ converging to $\bar{x}$. The technique can be further refined to recover all compact connected subsets with nonempty interior, disclosing an important difference between differentiable and continuously differentiable functions. It stems out from our approach that the class of these pathological functions contains an infinite-dimensional vector space and is dense in the space of all locally Lipschitz functions for the uniform convergence.

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Contents

1 | INTRODUCTION

Given a nonempty open subset $\mathcal{U}$ of a Euclidean space $\mathbb{R}^N$, a function $f : \mathcal{U} \to \mathbb{R}$ is called Lipschitz if there exists a constant $L > 0$ such that

$$|f(x) - f(y)| \leqslant L \|x - y\|, \quad \text{for all } x, y \in \mathcal{U}. \tag{1.1}$$

We denote by $\|f\|_{\text{Lip}}$ the infimum of the above constants, so that:

$$\|f\|_{\text{Lip}} = \sup_{x,y\in\mathcal{U},\, x\neq y} \frac{|f(x) - f(y)|}{\|x - y\|}. \tag{1.2}$$

In what follows, we call a function k-Lipschitz if $\|f\|_{\text{Lip}} \leqslant k$, where $k \geqslant 0$. We also call a function locally Lipschitz, if around any point x_0 of its domain, there exists $k > 0$ and a neighborhood $\mathcal{V}$ of x_0 such that the function f is k-Lipschitz on $\mathcal{V}$.

According to the Rademacher theorem, every locally Lipschitz function is differentiable almost everywhere (see [6, Chapter 9], for example). If $\mathcal{N}$ is any null subset of $\mathcal{U} \subset \mathbb{R}^N$, then denoting by $\mathcal{D}_f$ the set of points of differentiability of f and by $Df(x)$ the derivative of f at a point $x \in \mathcal{D}_f$, the Clarke subdifferential at $x \in \mathcal{U}$ is given by the following formula (see [12, Chapter 2]):

$$\partial f(x) = \text{conv}\left\{ \lim_{x_n \to x} Df(x_n) \ : \ \{x_n\} \subseteq \mathcal{D}_f \setminus \mathcal{N} \right\}, \tag{1.3}$$

where $\text{conv}(A)$ stands for the convex envelope of a set A. It follows that the above definition is independent of the choice of $\mathcal{N}$ and that $\partial f(x)$ is a nonempty convex compact subset of the closed dual ball $\overline{B}(0, \|f\|_{\text{Lip}})$ containing the derivative $Df(x)$, whenever this latter exists.

The Clarke subdifferential admits an alternative description based on *Fréchet* subgradients, without explicit use of derivatives or the above null set. We recall that $x^* \in \mathbb{R}^N$ is a Fréchet subgradient of f at x (and denote $x^* \in \hat{\partial} f(x)$) if $x^* = \nabla \phi(x)$ for some C^1-smooth function $\phi \leqslant f$ with $\phi(x) = f(x)$. Then we say that $p \in \mathbb{R}^N$ is a *limiting* subgradient of f at x, and denote $p \in \partial_L f(x)$, if there exists a sequence $\{(x_n, x_n^*)\}_n$ in $\mathbb{R}^N \times \mathbb{R}^N$ with $x_n^* \in \hat{\partial} f(x_n)$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n^* = p$. The Clarke subdifferential can then be defined as the convex envelope of the limiting subdifferential, that is, $\partial f(x) = \text{conv}\,\{\partial_L f(x)\}$, for every locally Lipschitz function f. Therefore, $\partial_L f(x) \subset \partial f(x)$. Notice that if f is everywhere differentiable, the limiting subdifferential is given by the formula:

$$\partial_L f(x) := \left\{ \lim_{x_n \to x} Df(x_n) \right\} \tag{1.4}$$

and if f is C^1-smooth, we have $\partial f(x) = \partial_L f(x) = \{Df(x)\}$, for all $x \in \mathcal{U}$. In fact, for a Lipschitz function f, $\partial f(x)$ reduces to a singleton if and only if f is strictly differentiable at x [12, Proposition 2.2.4].

Notice that $\|\cdot\|_{\mathrm{Lip}}$ is a seminorm in the vector space $\mathrm{Lip}(\mathcal{U})$ of all real-valued Lipschitz functions on $\mathcal{U}$ and becomes a norm in the subspace $\mathrm{Lip}_{x_0}(\mathcal{U})$ of those functions that vanish at some (arbitrarily chosen) prescribed point $x_0 \in \mathcal{U}$. In particular, $(\mathrm{Lip}_{x_0}(\mathcal{U}), \|\cdot\|_{\mathrm{Lip}})$ is a Banach space (known also as the dual space of the free space of $\mathcal{U}$). Alternatively, setting $\|\cdot\|_{\mathrm{L}} := \|\cdot\|_\infty + \|\cdot\|_{\mathrm{Lip}}$ and denoting by $\mathcal{L}^\infty(\mathcal{U})$ the set of bounded functions on $\mathcal{U}$, the normed space $(\mathrm{Lip}(\mathcal{U}) \cap \mathcal{L}^\infty(\mathcal{U}), \|\cdot\|_{\mathrm{L}})$ is also complete.

If the set $\mathcal{U}$ is bounded, one can also consider the norm $\|f\|_\infty := \sup\{|f(x)| : x \in \mathcal{U}\}$ of uniform convergence. In this case $(\mathrm{Lip}(\mathcal{U}), \|\cdot\|_\infty)$ is not complete (in fact, it is dense in the Banach space $(\mathcal{C}_b(\mathcal{U}), \|\cdot\|_\infty)$ of bounded continuous functions). However, one can remedy this lack of completeness by considering the set $\mathrm{Lip}^{[k]}(\mathcal{U})$ of Lipschitz continuous functions with Lipschitz constant $\|f\|_{\mathrm{Lip}} \leqslant k$. This set is a complete metric space under the distance of uniform convergence $d_\infty(f,g) := \|f-g\|_\infty$. In this setting (where the vector structure is of course lost) and assuming that $\mathcal{U}$ is convex, a standard application of Baire's category theorem has been used by J. Borwein and X. Wang (see [9, 10], for example) to establish that the set of Lipschitz functions with maximal Clarke subdifferential (i.e., $\partial f(x) \equiv \overline{B}(0,k)$ for all $x \in \mathcal{U}$) is residual in $\mathrm{Lip}^{[k]}(\mathcal{U})$. Therefore, a generic Lipschitz function in $\mathrm{Lip}^{[k]}(\mathcal{U})$ has Lipschitz constant equal to k and *saturates* its Clarke subdifferential at every point. The first explicit construction of a Clarke-saturated function was given in [20] (in dimension one) and in [8] (in higher dimensions).

The aforementioned result of J. Borwein and X. Wang underlines the fact that uniform convergence does not entail any control on derivatives and local oscillations. The genericity is thus tightly related to the d_∞-topology: One easily sees that the set of Clarke-saturated functions (i.e., functions whose subdifferential is identically equal to the closed ball $\overline{B}(0, \|f\|_{\mathrm{Lip}})$) cannot be dense for the (more adequate) distance $d_{\mathrm{Lip}}(f,g) = \|f-g\|_{\mathrm{Lip}}$ given by the Lipschitz norm. Still, in [14] it was established that the set of Clarke-saturated functions is *spaceable* in $(\mathrm{Lip}(\mathcal{U}), \|\cdot\|_{\mathrm{L}})$, that is, it contains a closed infinite-dimensional subspace (see [1, 15] for a discussion about spaceability). The construction of this infinite-dimensional subspace of Clarke-saturated functions is explicit, but the result requires working in ℓ_1^N (rather than in the usual Euclidean space $\mathbb{R}^N$).

Let us mention for completeness that important subclasses of Lipschitz functions, such as semialgebraic (more generally, Whitney stratifiable) or finite selections of C^N-smooth functions have small Clarke subdifferentials: They often reduce to a singleton and the (generalized) critical values satisfy the conclusion of the Morse–Sard theorem, see [4, Corollary 5(ii)] and [3, Theorem 5], respectively. On the other hand, every point of a Clarke-saturated Lipschitz function is (Clarke) critical, since $0 \in \partial f(x) \equiv \overline{B}(0, \|f\|_L)$. Other pathological situations have also been detected in [13] where the authors constructed examples of Lipschitz continuous functions with finite Clarke critical values, but with pathological subgradient dynamics both in continuous and discrete time: The iterates generate bounded trajectories that fail to detect any Clarke critical point of the function. Finally, in [7] the authors constructed locally Lipschitz functions whose subdifferential assumes a prescribed set of values.

In this work, we establish the following result for the range of the Clarke subdifferential. (The term *convex body* employed below will refer to a compact convex set with nonempty interior.)

- There exists a compactly supported, differentiable 1-Lipschitz function $f : \mathbb{R}^N \to \mathbb{R}$ whose Clarke subdifferential contains all convex bodies of the closed unit ball.

The construction is different for $N = 1$ (Theorem 3.7) and for $N \geqslant 2$ (Theorem 3.12). In the first case, the function f is also *subdifferentially exhaustive* (see Definition 2.4), that is, its Clarke subdifferential takes all of its possible values. In both cases, $N = 1$ and $N \geqslant 2$, the construction reveals that the set of all such functions whose support is contained in an open bounded set $\mathcal{U}$ of $\mathbb{R}^N$ is *spaceable* in $(\mathrm{Lip}(\mathcal{U}), \|\cdot\|_{\mathrm{L}})$ and dense in $(\mathrm{Lip}^{[1]}(\mathcal{U}), d_\infty)$, see Remark 3.8 (v),(vi) and Subsection 3.2.

By enhancing the techniques employed in Subsection 3.2 we obtain, in Subsection 3.3, a more general result (Theorem 3.16), that recovers all compact connected subsets of $\mathbb{R}^N$ with nonempty interior (not only the convex bodies). The construction requires $N \geqslant 2$ (but for $N = 1$ the two notions coincide anyway). The general result reads as follows:

- There exists a compactly supported, differentiable function $f : \mathbb{R}^N \to \mathbb{R}$ whose limiting subdifferential contains all compact connected subsets of $\mathbb{R}^N$ with nonempty interior.

2 | PREREQUISITES

We recall that the term Polish space refers to any separable topological space, whose topology can be metrizable in a way that the resulting metric space is complete. We denote by $\Delta := \{0,1\}^{\mathbb{N}}$ the Cantor set and recall that every uncountable Polish space contains a homeomorphic copy of Δ, see [18, Corollary 6.5].

In this work, we consider the Euclidean space $\mathbb{R}^N$, $N \geqslant 1$, and denote by $B(0,r)$ (respectively, $\overline{B}(0,r)$) the open (respectively, closed) ball centered at $x \in \mathbb{R}^N$ with radius $r > 0$.

Given a nonempty convex compact subset C of $\mathbb{R}^N$ we set:

$$\mathcal{F}_C := \{K \subset C : K \neq \emptyset, \text{ compact}\}. \tag{2.1}$$

It is known that $\mathcal{F}_C$ is a compact metric space for the Hausdorff distance

$$D_{\mathrm{H}}(K_1, K_2) := \max\left\{\sup_{x\in K_1} d(x, K_2),\ \sup_{x\in K_2} d(x, K_1)\right\}, \tag{2.2}$$

where $d(x, A) := \inf\{\|x - a\| : a \in A\}$ for every $A \subset \mathbb{R}^N$. We further set

$$\mathcal{K}_C := \{K \subset C : K \neq \emptyset, \text{ compact convex}\}. \tag{2.3}$$

Notice that $\mathcal{K}_C$ is a closed subset of $\mathcal{F}_C$ under the Hausdorff distance, therefore $(\mathcal{K}_C, D_{\mathrm{H}})$ is also a compact metric space.

In what follows, we denote by $\mathcal{L}_N$ the Lebesgue measure on $\mathbb{R}^N$. Given an integrable function $f : \mathbb{R}^N \to \mathbb{R}$, we say that a point x is a *Lebesgue point* of f if

$$\lim_{r\searrow 0^+} \frac{1}{\mathcal{L}_N(B(x,r))} \int_{B(x,r)} |f(y) - f(x)|\, dy = 0.$$

Therefore, a Lebesgue point is a point where f does not oscillate in an average sense, see [16, section 1.7]. It is known that the set of Lebesgue points of every integrable function f is of full-measure. In particular, for a.e. $x \in \mathbb{R}^N$ it holds

$$\left| f(x) - \left(\frac{1}{\mathcal{L}_N(B(x,r))} \int_{B(x,r)} f(y)dy \right) \right| \xrightarrow[r\to 0]{} 0 \qquad \text{(Lebesgue differentiation theorem).}$$

Let us further recall the *interval splitting property* for subsets of the real line, see [19]

Definition 2.1 (splitting property).

(i). A set $A \subset \mathbb{R}$ is called everywhere positive-measured, if it intersects any nontrivial interval in a set of positive Lebesgue measure.

(ii). We say that A has the splitting property for the family of intervals of $\mathbb{R}$ if both A and $\mathbb{R}\setminus A$ are everywhere positive-measured.

The following lemma goes back to Bruckner [11] (see also [24, Lemma 4.1]).

Lemma 2.2 (countable splitting partition). *There exists a countable partition $\{A_k\}_{k\in\mathbb{N}}$ of $\mathbb{R}$, each of which splits the family of intervals.*

Let us now recall that given a nonempty open subset $\mathcal{U}$ of $\mathbb{R}^N$ and a k-Lipschitz function $f : \mathcal{U} \to \mathbb{R}$, the Clarke subdifferential operator $\partial f : \mathcal{U} \rightrightarrows \mathbb{R}^N$ has closed graph and nonempty convex compact values (in particular, $\partial f(x) \subset \overline{B}(0,k)$ for every $x \in \mathcal{U}$). We also recall that ∂f is an *upper semicontinuous* multivalued operator, in the sense that for every $\varepsilon > 0$ and $x \in \mathcal{U}$, there exists $\delta > 0$ such that for all $y \in B(x,\delta) \cap \mathcal{U}$ it holds $\partial f(y) \subset \partial f(x) + B(0,\varepsilon)$.

In what follows, $\mathcal{U}$ will denote a nonempty open subset of $\mathbb{R}^N$. We recall from [14] the following definition.

Definition 2.3 (subdifferential saturation). A Lipschitz function $f : \mathcal{U} \to \mathbb{R}$ is called *Clarke saturated* if for every $x \in \mathcal{U}$ we have $\partial f(x) = \overline{B}(0, \|f\|_{\mathrm{Lip}})$.

Therefore, a Lipschitz function f with $\|f\|_{\mathrm{Lip}} = 1$ is Clarke saturated if and only if its Clarke subdifferential at any point is equal to the unit ball of $\mathbb{R}^N$.

We shall further use the following terminology.

Definition 2.4 (subdifferential exhaustiveness). A Lipschitz function $f : \mathcal{U} \to \mathbb{R}$ is called Clarke exhaustive (respectively, almost exhaustive) if for any nonempty closed convex subset K (respectively, of nonempty interior) of the ball $\overline{B}(0, \|f\|_{\mathrm{Lip}})$, there exists $x \in \mathcal{U}$ such that $\partial f(x) = K$.

3 | MAIN RESULTS

In this section, we are going to construct an everywhere differentiable function in $\mathbb{R}^N$ with bounded derivatives (thus, in particular, a Lipschitz continuous function) whose Clarke subdifferential is almost exhaustive. This yields the result announced in the title of the paper.

The construction requires at least two dimensions (i.e., $N \geqslant 2$), but the result is also true for $N = 1$ through a different construction which will be treated first. Moreover, in the one-dimensional case the constructed function turns out to be Clarke-exhaustive, that is, the subdifferential is surjective (assuming all of its possible values).

Since we deal with functions that are everywhere differentiable, the result is rather unexpected, taking into account that the derivative is a Baire-1 function (therefore, generically continuous) and the Clarke subdifferential of a strictly differentiable function (thus, a fortiori, of a C^1-function) is singleton everywhere.

As a matter of fact, our results also hold for the (smaller) limiting subdifferential, see forthcoming Remark 3.8(i) (for $N = 1$) and Remark 3.13 (for $N \geqslant 2$). A further refinement will be performed in Section 3.3 where we eventually show that there exists a differentiable, locally Lipschitz function such that every compact connected subset of $\mathbb{R}^N$ with nonempty interior appears in the range of its limiting subdifferential.

3.1 | Subdifferentially exhaustive differentiable functions in $\mathbb{R}$

Let $f : (0,1) \to \mathbb{R}$ be 1-Lipschitz. Then for every $x \in (0,1)$, the subdifferential $\partial f(x)$ is a nonempty closed subinterval of $[-1,1]$ (possibly reducing to a singleton). We shall need the following notation:

$$\mathbb{T}^+ = \Big\{(a,b) \in \mathbb{R}^2 : \quad 0 \leqslant a \leqslant b \leqslant 1\Big\}. \tag{3.1}$$

Let us start with the following essentially known result.

Lemma 3.1. *There exists a continuous surjective curve $\gamma_1 : [0,1] \to \mathbb{T}^+$ such that $\gamma_1((0,1)) = \mathbb{T}^+$.*

Proof. It is well known that there exists a continuous surjective curve $\gamma_0 : [0,1] \to [0,1] \times [0,1]$. This map is called a Peano curve, see [22]. The function $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\varphi(a,b) = \big(\min\{a,b\}, \max\{a,b\}\big)$$

is continuous and maps $[0,1] \times [0,1]$ onto $\mathbb{T}^+$. Thus, the function $\gamma_1 = \varphi \circ \gamma_0$ satisfies the assertion of the statement. □

3.1.1 | An easy nonsmooth example

Let us first provide a straightforward construction of a 1-Lipschitz Clarke exhaustive function (omitting momentarily the additional requirement of being everywhere differentiable).

Theorem 3.2 (exhaustive Lipschitz function in $\mathbb{R}$). *There exists a Lipschitz function $f : [0,1] \to \mathbb{R}$ with $\|f\|_{\text{Lip}} = 1$ such that for every nonempty closed interval $[a,b] \subset [-1,1]$, there exists $x \in (0,1)$ such that $\partial f(x) = [a,b]$, that is, $\partial f([0,1]) = \mathcal{K}_{[-1,1]}$.*

Proof. Let $\gamma_1(t) = \big(a(t), b(t)\big)$, with $t \in [0,1]$, be the continuous curve given by Lemma 3.1 and let $A \subset (0,1)$ be a measurable set that splits the family of nonempty open intervals of [0,1] (cf. Definition 2.1(ii)). The required function f is explicitly defined as follows:

$$f(x) = \int_0^x \left[a(t)\mathbf{1}_A(t) + b(t)\mathbf{1}_{[0,1]\setminus A}(t)\right] dt.$$

Indeed, let us prove that for every $x \in [0,1]$, we have $\partial f(x) = [a(x), b(x)]$.

To this end, let us first consider a Lebesgue point $t \in A$ of the function $\mathbf{1}_A$. Since a is continuous, we have that $f'(t)$ exists and $f'(t) = a(t)$. Similarly, if $s \in (0,1) \setminus A$ is a Lebesgue point of the function $\mathbf{1}_{[0,1]\setminus A}$, then $f'(s)$ exists and $f'(s) = b(s)$. Fix now $x \in (0,1)$ (arbitrarily chosen). Since any open interval containing x meets the sets A and $[0,1]\setminus A$ on a set of positive measure, we deduce that $a(x) \in \partial f(x)$ and $b(x) \in \partial f(x)$, yielding $[a(x), b(x)] \subset \partial f(x)$.

To establish the other inclusion, let us fix $\varepsilon > 0$ and

$$\mathcal{N} := \{x \in [0,1] : x \text{ is not a Lebesgue point for } f\}.$$

Since the functions a, b are continuous, there exists $\delta > 0$ such that $|a(t) - a(x)| \leqslant \varepsilon$ and $|b(t) - b(x)| \leqslant \varepsilon$, for all $t \in (x - \delta, x + \delta)$. It follows that $\partial f(x) \subset [a(x) - \varepsilon, b(x) + \varepsilon]$. Since $\varepsilon > 0$ is arbitrarily chosen, we deduce $\partial f(x) \subset [a(x), b(x)]$ and consequently, equality holds.

Recalling that γ_1 satisfies Lemma 3.1, for every nonempty closed interval $[a,b] \subset [0,1]$, there exists $x \in (0,1)$ such that $\gamma_1(x) = (a,b) \in \mathbb{T}^+ \subset \mathbb{R}^2$, and consequently, $\partial f(x) = [a,b]$. Replacing f by the function

$$\widetilde{f}(x) := 2f(x) - x, \qquad \text{for all } x \in [0,1],$$

we obtain a function $\widetilde{f}$ which is also 1-Lipschitz: Indeed, notice that $\widetilde{f}'(x) = 2f'(x) - 1 \in [-1,1]$ whenever $f'(x)$ exists. It follows directly that

$$\partial\widetilde{f}((0,1)) = \big\{[a,b];\ -1 \leqslant a \leqslant b \leqslant 1\big\} = \mathcal{K}_{[-1,1]}.$$

The proof is complete. □

Remark 3.3. Notice that the set of bounded 1-Lipschitz Clarke exhaustive functions in $\mathbb{R}$ cannot be d_∞-residual in the (complete) metric space $(\mathrm{Lip}^{[1]}(\mathbb{R}), d_\infty)$ of all bounded 1-Lipschitz functions in $\mathbb{R}$, since it shares with the set of Clarke-saturated functions (which is known to be d_∞-residual, see [9]) only the null function $f \equiv 0$. However, we shall see later (Remark 3.8(iv)) that the set of bounded 1-Lipschitz functions in $\mathbb{R}$ that are Clarke exhaustive is dense in $(\mathrm{Lip}^{[1]}(\mathbb{R}), d_\infty)$.

3.1.2 | An involved construction ensuring differentiability

We shall now enhance the result of Theorem 3.2 by adding the requirement that the constructed function f should also be everywhere differentiable. The construction becomes more involved, but remains explicit. Before we proceed, we shall need the following preliminary results (lower integral estimations for ν-root type functions).

Lemma 3.4 (lower integral estimation I). *There exists a function* $\sigma : (0,1) \to (0,1]$ *satisfying* $\lim_{\nu\to 0} \sigma(\nu) = 0$ *such that for every* $x, h \in \mathbb{R}$ *with* $h \neq 0$*, we have:*

$$\frac{1}{h}\int_x^{x+h} |t|^\nu dt \geqslant |x|^\nu\big(1 - \sigma(\nu)\big). \tag{3.2}$$

Proof. If $x = 0$ the assertion follows trivially. Therefore, we may assume $x \neq 0$. Since the functions $x \to |x|^\nu$ are even, we can limit our attention to the case $h > 0$. We set:

$$I = \frac{1}{h} \int_x^{x+h} |t|^\nu dt.$$

We consider successively all four possible cases:

We first assume $x > 0$ and $t \in [x, x+h]$. In this case, $|t|^\nu \geqslant |x|^\nu$ and $I \geqslant |x|^\nu$, therefore (3.2) holds for any function σ with nonnegative values.

Let us now assume $x < 0 < x + h \leqslant |x|$. A direct computation gives:

$$I = \left(\frac{1 + y^{1+\nu}}{1+y}\right)\left(\frac{|x|^\nu}{1+\nu}\right), \qquad \text{where} \quad y := \frac{x+h}{|x|} \in [0,1]. \tag{3.3}$$

Consider the (continuous) functions $\Psi_\nu : [0,1] \to [0, +\infty)$, $\nu \in (0,1)$, defined by

$$\Psi_\nu(y) = \frac{1 + y^{1+\nu}}{(1+\nu)(1+y)}, \quad y \in [0,1].$$

Then the functions $\{\Psi_\nu\}_{\nu>0}$ converge pointwise to the function $\Psi \equiv 1$ as ν tends to 0. Since the above convergence is monotone, we deduce from Dini theorem that the convergence is uniform. Setting

$$\sigma(\nu) := 1 - \min_{y \in [0,1]} \frac{1 + y^{1+\nu}}{(1+\nu)(1+y)}, \tag{3.4}$$

we readily deduce that $\lim_{\nu \to 0} \sigma(\nu) = 0$. Therefore, (3.3) yields

$$I \geqslant |x|^\nu (1 - \sigma(\nu))$$

and (3.2) holds true for σ given in (3.4).

If $x < x + h \leqslant 0$, then a direct computation yields

$$I \geqslant \frac{|x|^\nu}{1+\nu} = \left(1 - \frac{\nu}{1+\nu}\right)|x|^\nu \geqslant (1 - \sigma(\nu))|x|^\nu,$$

where σ is given by (3.4).

It remains to deal with the case $x < 0 < |x| < x + h$. In this case, we have

$$I = \frac{1}{h}\int_x^{|x|} |t|^\nu dt + \frac{1}{h}\int_{|x|}^{x+h} |t|^\nu dt \geqslant \frac{1}{h}\left\{(|x| - x)|x|^\nu\big(1 - \sigma(\nu)\big) + (x + h - |x|)|x|^\nu\right\}$$
$$\geqslant (1 - \sigma(\nu))\,|x|^\nu.$$

Therefore, (3.2) is still satisfied and the proof is complete. □

We now extend (3.2) to a more general class of functions. Fixing parameters $d \in \mathbb{R}$, $m > 0$, and $\varepsilon > 0$, we set for each $\nu \in (0,1)$

$$R(t) = \min\left\{ m^{\nu}, \left(\frac{|t-d|}{\varepsilon}\right)^{\nu} \right\}, \qquad t \in \mathbb{R}. \tag{3.5}$$

The above function is continuous and nonnegative. The following result shows that R also satisfies the same lower integral estimation as in (3.2).

Lemma 3.5 (lower integral estimation II). *For every $x, h \in \mathbb{R}$ with $h \neq 0$, the function R given in (3.5) satisfies*

$$\frac{1}{h}\int_x^{x+h} R(t)dt \geqslant R(x)\big(1-\sigma(\nu)\big), \tag{3.6}$$

where $\sigma : (0,1) \to (0,1]$ is the function defined in Lemma 3.4.

(Notice that this integral estimate does not depend on the values of the parameters ε, d, and m.)

Proof. We first consider the case $\varepsilon = 1$ and $d = 0$. Let $x \in \mathbb{R}$ and $h > 0$. If $m \leqslant x < x+h$ or if $x < x+h \leqslant -m$, there is nothing to prove since in both cases the function R is constant on the interval $[x, x+h]$. The case $-m \leqslant x < x+h \leqslant m$ follows from the previous lemma, since in this case $R(t) = |t|^{\nu}$ on $[x, x+h]$.

Let us now consider the case $-m \leqslant x \leqslant m < x+h$. Then, according to the previous lemma, $\int_x^m R(t)dt \geqslant R(x)\big(1-\sigma(\nu)\big)(m-x)$. Since

$$\int_m^{x+h} R(t)dt = m^{\nu}(x+h-m) \geqslant R(x)(x+h-m),$$

we deduce

$$\frac{1}{h}\int_x^{x+h} R(t)dt = \frac{1}{h}\left(\int_x^m R(t)dt + \int_m^{x+h} R(t)dt\right) \geqslant R(x)\big(1-\sigma(\nu)\big).$$

It remains to consider the case $x < -m < x+h$. In this case,

$$\int_x^{-m} R(t)dt = R(x)(-m-x)$$

and

$$\frac{1}{h}\int_x^{x+h} R(t)dt = \frac{1}{h}\left(\int_x^{-m} R(t)dt + \int_{-m}^{x+h} R(t)dt\right) \geqslant R(x)(1-\sigma(\nu))$$

since, according to the previous case,

$$\int_{-m}^{x+h} R(t)dt \geqslant R(-m)(1-\sigma(\nu))(x+h+m) = R(x)(1-\sigma(\nu))(x+h+m).$$

This proves the validity of (3.6) for the function $R(x) = \min\{m^\nu, |x|^\nu\}$. The general case for arbitrary values of the parameters $d \in \mathbb{R}$ and $\varepsilon > 0$ in (3.5) easily follows by translation and a standard argument. □

We shall also need the following refinement of Lemma 3.1.

Lemma 3.6. *Let C be any compact subset of [0,1] which is homeomorphic to the Cantor set $\Delta := \{0,1\}^{\mathbb{N}}$. Then there exists a continuous curve $\gamma : [0,1] \to \mathbb{R}^2$, such that*

$$\gamma\big([0,1]\big) = \gamma(C) = \mathbb{T}^+ \qquad (\textit{see } (3.1)).$$

Proof. Let φ be a homeomorphism from C onto $\{0,1\}^{\mathbb{N}}$ and let $\psi : \{0,1\}^{\mathbb{N}} \to [0,1]$ be defined as follows:

$$\psi\big((x_n)\big) = \sum_{n\geqslant 1} 2^{-n} x_n.$$

It follows easily that ψ is continuous and surjective, therefore, $\gamma_2 := \psi \circ \varphi$ is a continuous function from C onto [0,1]. By Urysohn lemma, we can extend γ_2 to a continuous curve $\tilde{\gamma}_2$ from [0,1] onto [0,1]. If γ_1 denotes the function constructed in Lemma 3.1, then the continuous curve $\gamma := \gamma_1 \circ \tilde{\gamma}_2$ satisfies the assertion. □

We are now ready to construct the desired function f.

Theorem 3.7 (smooth exhaustive function in $\mathbb{R}$). *There exists a 1-Lipschitz differentiable function $f : \mathbb{R} \to \mathbb{R}$ with compact support for which the range of its Clarke subdifferential contains all closed subintervals and all singletons of* $[-1,1]$.

Proof. Let $\mathcal{D} = \{d_n\}_{n\geqslant 1}$ be a countable dense subset of [0,1]. Let $\{\varepsilon_n\}_n$ be a nonincreasing sequence of positive real numbers such that $\sum_{n\geqslant 1} \varepsilon_n < 1/2$. Let $\{\nu_n\}_n$ be a sequence in (0,1) such that $\sum_{n\geqslant 1} \sigma(\nu_n) < +\infty$, where σ is the function defined by (3.4) and evoked in Lemma 3.5. Let r_n be the function defined by

$$r_n(x) := \left(\frac{|x - d_n|}{\varepsilon_n}\right)^{\nu_n}, \qquad x \in \mathbb{R}. \tag{3.7}$$

According to our choice of ε, the set

$$F := [0,1] \diagdown \left(\bigcup_{d \in D} (d_n - \varepsilon_n, d_n + \varepsilon_n) \right)$$

is a closed subset of [0,1] of positive Lebesgue measure. Since F is an uncountable Polish space, there exists a closed subset C of F which is homeomorphic to the Cantor set $\Delta = \{0,1\}^{\mathbb{N}}$ (see [18, Corollary 6.5], for example). Notice also that

$$r_n(x) \geqslant 1, \qquad \text{for all } x \in F \text{ and } n \in \mathbb{N}.$$

We are now ready to construct our function f. Let

$$\gamma(x) = \big(\alpha(x), \beta(x)\big) \in \mathbb{T}^+, \qquad x \in [0,1],$$

be the continuous curve constructed in Lemma 3.6 with respect to the closed subset C of F evoked above. We set

$$g_0(x) = \beta(x)$$

and define inductively

$$g_n(x) = \min\left\{g_{n-1}(x), \alpha(x) + r_n(x)\right\}, \qquad \text{for } n \geqslant 1.$$

Finally, we set

$$g(x) = \inf_{n\geqslant 1} g_n(x) = \min\left\{\beta(x),\, \alpha(x) + \inf_{n\geqslant 1} r_n(x)\right\} \qquad \text{and} \qquad f(x) = \int_0^x g(t)dt. \tag{3.8}$$

Notice that the function g is upper semicontinuous (as infimum of continuous functions), hence measurable, with values in [0,1] because $\alpha \leqslant g \leqslant \beta$. Therefore, the function f is 1-Lipschitz and nondecreasing. By construction, we have

$$g(x) - \alpha(x) \leqslant r_n(x), \qquad \text{for every } x \in [0,1] \text{ and } \ n \geqslant 1.$$

Let us fix $x \in [0,1]$ and define

$$R_n(t) := \min\{g(x) - \alpha(x), r_n(t)\}, \qquad \text{for all } t \in [0,1].$$

It follows readily that $R_n(x) = g(x) - \alpha(x)$, thus $0 \leqslant R_n(x) \leqslant \beta(x) \leqslant 1$. Since

$$\max\{0, g(x) - \alpha(x) - r_n(t)\} = R_n(x) - R_n(t),$$

we obtain from (3.5)–(3.6) with $d = d_n$, $\nu = \nu_n$, and $m = \big(g(x) - \alpha(x)\big)^{1/\nu_n}$

$$\frac{1}{h}\int_x^{x+h} \max\{0, g(x) - \alpha(x) - r_n(t)\}dt = R_n(x) - \frac{1}{h}\int_x^{x+h} R_n(t)dt \leqslant R_n(x)\sigma(\nu_n) \leqslant \sigma(\nu_n). \tag{3.9}$$

□

Claim 1. The function f is differentiable at every point and $f' = g$.

Proof of Claim 1. We shall consider separately two cases:

— Case $g(x) = \alpha(x)$.

Since $g \geqslant \alpha$, $g(x) = \alpha(x)$, g is upper semicontinuous, and α is continuous, we deduce that g is continuous at x, and consequently f is differentiable at x with $f'(x) = g(x)$. Notice that the level set

$$[g-\alpha=0]:=\{x\in[0,1]:\ g(x)=\alpha(x)\}$$

of the function $g-\alpha$ is dense $\mathcal{G}_\delta$ in [0,1]: indeed, it contains the dense set $\mathcal{D}=\{d_n\}_{n\geqslant 1}$ (notice that $r_n(d_n)=0$ and consequently, by (3.8), $g(d_n)=\alpha(d_n)$, for every $n\geqslant 1$) and it is $\mathcal{G}_\delta$ since the strict sublevel sets

$$\left[g-\alpha<\frac{1}{n}\right]=\left\{x\in[0,1]:\ g(x)-\alpha(x)<\frac{1}{n}\right\}$$

are open (thanks to the upper semicontinuity of g and the continuity of α) and

$$[g-\alpha=0]=\bigcap_{n\geqslant 1}[g-\alpha<1/n].$$

— Case $g(x)>\alpha(x)$.

Since g is upper semicontinuous, we always have

$$\limsup_{h\to 0}\frac{f(x+h)-f(x)}{h}=\limsup_{h\to 0}\frac{1}{h}\int_x^{x+h}g(t)dt\leqslant g(x).$$

It remains to prove that for fixed $\varepsilon>0$, there exists $h_1>0$ such that, if $|h|\leqslant h_1$, then

$$\frac{f(x+h)-f(x)}{h}\geqslant g(x)-5\varepsilon. \tag{3.10}$$

Without loss of generality, we may assume

$$\kappa:=g(x)-\alpha(x)-\varepsilon>0.$$

Thus, for any $n\in\mathbb{N}$, x does not belong to the closed set $r_n^{-1}(\{\kappa\})$, which yields that

$$\operatorname{dist}\left(x,r_n^{-1}(\{\kappa\})\right)\ =\ \operatorname{dist}\left(x,r_n^{-1}([0,\kappa])\right)\ >\ 0.$$

Moreover, up to a subsequence,

$$\operatorname{dist}\left(x,r_n^{-1}(\{\kappa\})\right)\underset{n\to+\infty}{\longrightarrow}0.$$

Therefore, setting

$$N(x,h):=\min\left\{n\geqslant 1:\ r_n^{-1}(\{\kappa\})\cap[x-h,x+h]\neq\emptyset\right\},\qquad \text{for } h>0,$$

we deduce easily that

$$\lim_{h\to 0}N(x,h)=+\infty.$$

Let us fix $h_0 > 0$ such that $N := N(x, h_0)$ satisfies both

$$\sum_{n>N} \sigma(\nu_n) < \varepsilon \qquad \text{and} \qquad |g_N(x) - g(x)| < \varepsilon.$$

Then, we fix $0 < h_1 \leqslant h_0$ such that, if $t \in [x - h_1, x + h_1]$, then

$$|g_N(x) - g_N(t)| \leqslant \varepsilon \qquad \text{and} \qquad |\alpha(x) - \alpha(t)| \leqslant \varepsilon.$$

Consequently, if $|h| \leqslant h_1$, we have

$$\frac{1}{h}\int_x^{x+h} g_N(t)dt \geqslant g_N(x) - \varepsilon \geqslant g(x) - \varepsilon.$$

Therefore, in order to prove (3.10), it is enough to prove that

$$\frac{1}{h}\int_x^{x+h} \big(g_N(t) - g(t)\big)dt \leqslant 4\varepsilon, \qquad \text{whenever } |h| \leqslant h_1.$$

Since $g(t) = \min\left\{g_N(t), \alpha(t) + \inf_{n>N} r_n(t)\right\}$, we obtain

$$g_N(t) - g(t) = \max\left\{0,\ \sup_{n>N}\{g_N(t) - \alpha(t) - r_n(t)\}\right\}.$$

If $|t - x| \leqslant h_1$, then we also have $g_N(t) - \alpha(t) \leqslant g(x) - \alpha(x) + 3\varepsilon$. Hence,

$$\begin{aligned} g_N(t) - g(t) &\leqslant \max\left\{0,\ \sup_{n>N}\{g(x) - \alpha(x) + 3\varepsilon - r_n(t)\}\right\} \\ &\leqslant \max\left\{0,\ \sup_{n>N}\{g(x) - \alpha(x) - r_n(t)\}\right\} + 3\varepsilon \\ &\leqslant \sum_{n>N} \max\{0,\ \{g(x) - \alpha(x) - r_n(t)\}\} + 3\varepsilon. \end{aligned}$$

Integrating the above inequality, we obtain thanks to (3.9)

$$\begin{aligned} \frac{1}{h}\int_x^{x+h} \big(g_N(t) - g(t)\big)dt &\leqslant \sum_{n>N} \frac{1}{h}\int_x^{x+h} \max\{0,\ g(x) - \alpha(x) - r_n(t)\}dt + 3\varepsilon \\ &\leqslant \sum_{n>N} \sigma(\nu_n) + 3\varepsilon \leqslant 4\varepsilon. \end{aligned} \tag{3.11}$$

Thus, we have shown that f is differentiable at each point and that $f' = g$. □

Claim 2. $\mathrm{Im}(\partial f) = \partial f((0,1)) = \mathcal{K}_{(0,1)} := \{[a, b] :\ 0 \leqslant a \leqslant b \leqslant 1\}$.

Proof of Claim 2. For every $x \in (0,1)$, we have $0 \leqslant f'(x) = g(x) \leqslant 1$, whence $\partial f(x) \subset [0,1]$. Let us now fix $x \in C$. Since $C \subset F$, we have $r_n(x) \geqslant 1$ for all $n \geqslant 1$ and consequently

$$f'(x) = g(x) = \beta(x) \in \partial f(x).$$

Since the set $[f' = \alpha] = [g = \alpha]$ is dense in [0,1] and α is continuous, we deduce that $\alpha(x) \in \partial f(x)$, hence $[\alpha(x), \beta(x)] \subset \partial f(x)$. The reverse inclusion follows easily from (1.3), since $f'(x) = g(x) \in [\alpha(x), \beta(x)]$ and the functions α and β are continuous.

Let us finally recall that the curve $\gamma = (\alpha, \beta)$ satisfies the conclusion of Lemma 3.6. This ensures that

$$\partial f(C) = \{[a,b] : \ 0 \leqslant a \leqslant b \leqslant 1\}.$$

We conclude that $\partial f((0,1)) = \partial f(C) = \mathcal{K}_{[0,1]}$ as asserted.

Replacing again f by $\widetilde{f} := 2f - I$, where I is the identity on [0,1], we obtain a differentiable function $\widetilde{f}$ with derivatives in $[-1,1]$. It easily follows that $\widetilde{f}$ is 1-Lipschitz and satisfies

$$\partial \widetilde{f}((0,1)) = \mathcal{K}_{[-1,1]} = \{[a,b] : \ -1 \leqslant a \leqslant b \leqslant 1\}.$$

The proof is complete. □

Remark 3.8.

(i) In the above construction, $\partial f(x)$ is a singleton if and only if x belongs to the ($\mathcal{G}_\delta$ dense) subset $[g = \alpha]$ of [0,1]. Moreover, since $g = f'$ has the Darboux property, we can easily deduce that $\partial f(x) = \partial_L f(x)$, for all $x \in (0,1)$ and consequently, the conclusion also holds for the limiting subdifferential.

(ii) We can assume that C is contained in (0,1) and that $\alpha(0) = \beta(0) = \alpha(1) = \beta(1) = 0$. This allows to extend f to a differentiable function on $\mathbb{R}$ satisfying $f'(0) = f'(1) = 0$.

(iii) We can also assume $\mathcal{L}_1(C) = 0$. In this case, we have a negligible set C satisfying

$$\partial f(C) := \{\partial f(x) : x \in C\} = \mathcal{K}_{[-1,1]}.$$

(iv) It is clear from the above construction that the domain of f can be any nontrivial interval of arbitrarily small length and that the range of f can be taken inside $[c - \varepsilon, c + \varepsilon]$ for any choice of $c \in \mathbb{R}$ and $\varepsilon > 0$. It follows easily, by a standard argument, that for any nonempty open interval $\mathcal{J}$ of $\mathbb{R}$, the set of bounded, differentiable, Clarke exhaustive k-Lipschitz functions in $\mathcal{J}$ is d_∞-dense in the (complete) metric space $\left(\mathrm{Lip}^{[k]}(\mathcal{J}), d_\infty\right)$ of all bounded Lipschitz functions in $\mathcal{J}$ with $\|f\|_{\mathrm{Lip}} \leqslant k$.

(v) Let $I = (a,b)$ be a nonempty (possibly unbounded) interval. Then the set $\mathcal{E}$ of all real-valued Lipschitz functions in I which are everywhere differentiable and Clarke exhaustive is spaceable when equipped with the semidistance $d_{\mathrm{Lip}}(f,g) := \|f - g\|_{\mathrm{Lip}}$, for all f, g in $\mathcal{E}$.

Indeed, it is sufficient to consider a sequence of disjoint intervals $\{(a_n, b_n)\}_n$ such that

$$a < a_n < b_n < a_{n+1} < b\,, \quad \text{for every } n \in \mathbb{N}\,,$$

a sequence of Clarke exhaustive functions $\{f_n\}_n$ such that $\|f_n\|_{\mathrm{Lip}} = 1$ and $\operatorname{supp} f_n \subset (a_n, b_n)$ for all $n \in \mathbb{N}$, and the operator $T : c_0(\mathbb{N}) \to \mathcal{E}$ defined by

$$T(\{x_n\}_n) := \sum_{n=1}^{\infty} x_n f_n(\cdot).$$

Since the supports of the functions f_n are pairwise disjoint, it follows easily that the operator T is well defined and establishes a linear isometry between $c_0(\mathbb{N})$ and its image. Therefore, the metric space $(\mathcal{E}, d_{\mathrm{Lip}})$ contains an isometric copy of $c_0(\mathbb{N})$. Similar constructions of operators T can be found in [2, 14].

(vi) The set of all Lipschitz functions in [0,1] which are everywhere differentiable and Clarke exhaustive cannot be $\|\cdot\|_\infty$-spaceable in $(\mathrm{Lip}([0,1]), \|\cdot\|_\infty)$ (the latter being seen as a dense subspace of the Banach space $(\mathcal{C}([0,1], \|\cdot\|_\infty))$. This is a straightforward consequence of the classical fact that every subspace Y of Lipschitz functions which is $\|\cdot\|_\infty$-closed in $\mathcal{C}([0,1])$ is necessarily finite dimensional. Let us sketch a proof for reader's convenience: We consider the family of linear operators $\{T_{x,y} : x, y \in [0,1],\ x \neq y\}$ defined by $T_{x,y}(f) = \frac{f(x)-f(y)}{|x-y|}$, for all $f \in Y \subset \mathrm{Lip}([0,1])$. Since $T_{x,y}(f) \leqslant ||f||_{\mathrm{Lip}}$ for all $x, y \in [0,1]$, $x \neq y$, and $(Y, \|\cdot\|_\infty)$ is complete, applying the Banach–Steinhaus theorem, we deduce that for some $M > 0$ and all $x, y \in [0,1]$, $x \neq y$, it holds $||T_{x,y}|| \leqslant M$. It follows from Arzelà–Ascoli theorem that every $||\cdot||_\infty$-bounded sequence $\{f_n\}_n$ in Y has a converging subsequence, and consequently, the closed unit ball $\overline{B}_Y(0,1)$ of Y is compact, ensuring that Y is finite dimensional.

3.2 | Subdifferential containing all convex bodies in $\mathbb{R}^N$ ($N \geqslant 2$)

We shall now deal with the higher dimensional case and construct a differentiable Lipschitz function f which is almost exhaustive, that is, its Clarke subdifferential contains all nonempty convex compact subsets of $\overline{B}(0, \|f\|_{\mathrm{Lip}})$ of nonempty interior. The question of whether it is possible to obtain a Lipschitz Clarke exhaustive function in dimension $N \geqslant 2$ remains open.

Let us stress the fact that the forthcoming construction cannot be applied in one dimension. Roughly speaking, our approach occupies one dimension to code the family of convex bodies in $\overline{B}(0, \|f\|_{\mathrm{Lip}})$ (based on the fact that any compact geodesic metric space can be represented as a continuous surjective image of [0,1]) and requires at least one extra dimension to make an efficient use of this coding. Although the overall construction is less explicit and more involved, the reader can possibly trace some analogies between the aforementioned surjection and the curve obtained in Lemma 3.1 which was used to recover all closed intervals in [0,1].

In order to keep notation simple, $\mathbb{R}^N$ will be considered with its natural Euclidean structure (despite the fact that our results Lemma 3.9 and Theorem 3.12 hold true in any finite-dimensional normed space). Therefore, by Riesz representation theorem, the dual space of $\mathbb{R}^N$ will be identified to itself. We shall also identify 1-forms $Df(x)$ with gradients $\nabla f(x)$, for any differentiable function $f : \mathbb{R}^N \to \mathbb{R}$. In what follows we are going to construct the following:

(I) for every $n \geqslant 1$, a compactly supported differentiable n-Lipschitz function $f_n : \mathbb{R}^N \to \mathbb{R}$ whose Clarke subdifferential contains in its range every compact convex subset of nonempty interior that lies in the closed ball $\overline{B}(0, n)$.

Similarly to the one-dimensional case, the method of construction will directly yield that the set of all functions as above is d_∞-dense in $(\mathrm{Lip}(\mathcal{U}), \|\cdot\|_\infty)$ (for $\mathcal{U} \subset \mathbb{R}^N$ open and bounded) and $\|\cdot\|_{\mathrm{L}}$-spaceable in $(\mathrm{Lip}(\mathcal{U}), \|\cdot\|_{\mathrm{L}})$.

(II) a differentiable locally Lipschitz function $f : \mathbb{R}^N \to \mathbb{R}$ whose Clarke subdifferential contains in its range all compact convex bodies of $\mathbb{R}^N$.

Notice that the second assertion follows directly from the first: It is enough to consider a family of differentiable Lipschitz functions $f_n : \mathbb{R}^N \to \mathbb{R}$ with $\mathrm{Lip}(f_n) = n$ and disjoint supports (for instance, $\mathrm{supp}(f_n) \subset B(3n\, e_1, 1)$ where $e_1 = (1, 0, \dots, 0)$), satisfying statement (I) and define the function

$$f(x) = \sum_{n \geqslant 1} f_n(x), \quad \text{for all } x \in \mathbb{R}^N. \tag{3.12}$$

One readily gets that f is everywhere differentiable, locally Lipschitz, and satisfies assertion (II).

Let us now proceed to the construction evoked in (I). It clearly suffices to do it for the case $n = 1$ and construct a 1-Lipschitz function.

This will be done in two stages: We first fix a compact convex subset C in $\mathbb{R}^N$ that contains 0 and construct an L-Lipschitz function (with $C \subset \overline{B}(0, L)$) whose Clarke subdifferential contains all compact convex subsets K of C that contain 0. The general case will follow using separability arguments, by considering an adequate sequence $\{C_n\}_n$ of compact convex sets with $0 \in \mathrm{int}\, C_n$, then gluing adequate translations of the corresponding constructed functions.

3.2.1 | An intermediate construction

For a nonempty compact convex subset C of $\mathbb{R}^N$ with $0 \in C$, recalling from (2.3)–(2.2) the definition of $(\mathcal{K}_C, D_{\mathrm{H}})$, we denote by

$$\mathcal{K}_C^0 := \{K \in \mathcal{K}_C \; : \; 0 \in K\} \tag{3.13}$$

the set of all convex compact subsets of C containing 0. Notice that $\mathcal{K}_C^0$ is closed in $\mathcal{K}_C$, therefore $(\mathcal{K}_C^0, D_{\mathrm{H}})$ is a compact metric space. Moreover, it is a *geodesic space* (see [23, p. 72], for example). Indeed, for any two elements $K_0, K_1 \in \mathcal{K}_C^0$ and $\lambda \in (0, 1)$, we have:

$$K_\lambda := (1 - \lambda)K_0 + \lambda K_1 \in \mathcal{K}_C^0 \qquad \text{and} \qquad D_{\mathrm{H}}(K_0, K_\lambda) = \lambda D_{\mathrm{H}}(K_0, K_1). \tag{3.14}$$

We shall show, as an application of the next lemma, that there exists a differentiable 1-Lipschitz function $f : \mathbb{R}^N \to \mathbb{R}$ such that $\mathcal{K}^0_{\overline{B}(0,1)}$ is contained in the image of the subdifferential of f.

Lemma 3.9. *Let $C \subset \mathbb{R}^N$ be a convex compact set such that $0 \in C$ and $L := \max_{x \in C}\{\|x\|\}$. Then:*

(i) *There is a differentiable L-Lipschitz continuous and compactly supported function $f : \mathbb{R}^N \to \mathbb{R}$ such that:*

$$\textit{for every } K \in \mathcal{K}_C^0, \textit{ there exists } x \in \mathbb{R}^N \textit{ such that } \partial f(x) = K. \tag{3.15}$$

(ii) *Let us further assume* $0 \in \mathrm{int}(C)$. *Then in addition to (3.15) we get:*

$$\partial f(x) \subset C, \text{ for all } x \in \mathbb{R}^N. \tag{3.16}$$

Proof.

(i) If $C = \{0\}$, then the function $f \equiv 0$ satisfies trivially the conclusion. Therefore, we may assume $\{0\} \subsetneq C$. Since $\left(\mathcal{K}_C^0, D_{\mathrm{H}}\right)$ is a compact metric space, there exists a continuous surjective map from the Cantor set Δ to $\mathcal{K}_C^0$ (see [18, Theorem 4.18]). Since $\mathcal{K}_C^0$ is also geodesic, a standard argument shows that this map can be extended to a continuous surjective map

$$h : [0,1] \to \mathcal{K}_C^0 \qquad \text{(coding the elements of } \mathcal{K}_C^0\text{)}.$$

Let

$$\mathcal{D} = \{d_n \ : \ n \in \mathbb{N}\}$$

be a countable dense subset of (0,1).

Let us define, by induction, two sequences $\{\alpha_n\}_n$ and $\{\varepsilon_n\}_n$, satisfying $\alpha_n > \varepsilon_n > 0$, for all $n \geqslant 1$, satisfying

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \varepsilon_n = 0$$

and the following property: Setting

$$Q_n = (d_n, \alpha_n, 0, \dots, 0) \in \mathbb{R}^N, \quad \text{where } n \in \mathbb{N}, \tag{3.17}$$

the balls $\overline{B}(Q_n, \varepsilon_n)$ are pairwise disjoint and contained in $(0,1)^N$.

Indeed, assuming that $\alpha_1, \dots, \alpha_n, \varepsilon_1, \dots, \varepsilon_n$ have already been constructed accordingly, pick

$$0 < \alpha_{n+1} < m_n := \min\{\alpha_i - \varepsilon_i;\ 1 \leqslant i \leqslant n\}$$

and then choose $0 < \varepsilon_{n+1} < \alpha_{n+1}$ such that $\alpha_{n+1} + \varepsilon_{n+1} < m_n$ and $\varepsilon_{n+1} < \min\{d_{n+1}, 1 - d_{n+1}\}$. Notice that we can also assume the extra condition $\lim\limits_{n\to\infty} \varepsilon_n/\alpha_n = 0$ (which will be needed later). A concrete choice of such sequences is given by $\alpha_n = 1/2^n$, $\varepsilon_n = 1/n2^{n+2}$ and $\{d_n\}_{n\in\mathbb{N}}$ be a standard enumeration of the dyadics in (0,1) given by $d_1 = 1/2$ and

$$d_n = \frac{2\, i(n) - 1}{2^{m(n)+1}},$$

where for every $n \geqslant 2$, we denote by $m(n)$ the unique $m \in \{1, \dots, n\}$ such that

$$s_m := \sum_{k=0}^{m-1} 2^k < n \leqslant s_{m+1} := \sum_{k=0}^{m} 2^k$$

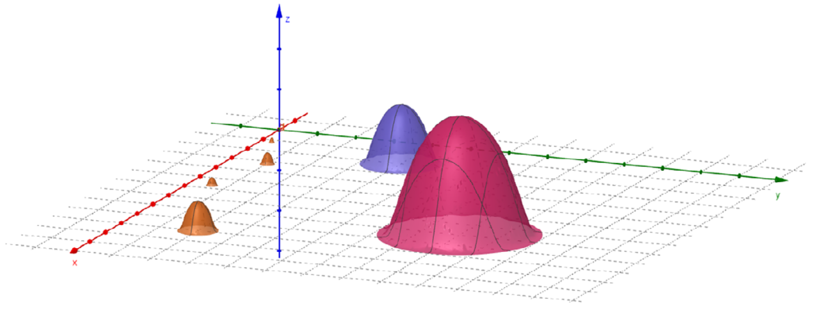

FIGURE 1 Sketch of the function constructed in Lemma 3.9.

and we set $i(n) := n - s_{m(n)} \in \{1, \dots, 2^{m(n)-1}\}$. Further, for every $n \in \mathbb{N}$, we define

$$H_n := \Big(h(d_n) + \overline{B}(0, \gamma_n) \Big) \bigcap \overline{B}(0, L), \tag{3.18}$$

where $\{\gamma_n\}_n$ is an arbitrary sequence of positive numbers converging to 0. Therefore, for every $n \in \mathbb{N}$, we have

$$B(0, \min\{\gamma_n, L\}) \subset H_n \subset h(d_n) + \overline{B}(0, \gamma_n). \tag{3.19}$$

Since H_n is a convex compact subset of $\mathbb{R}^N$ such that $0 \in \operatorname{int}(C)$, according to a consequence of a result of J. Borwein, M. Fabian, I. Kortezov, and P. Loewen [5, Theorem 12] (see also T. Gaspari [17]), for every $n \in \mathbb{N}$, there exists a C^1-smooth function $b_n : \mathbb{R}^N \to \mathbb{R}$, with support in the unit ball, such that $\nabla b_n(\mathbb{R}^N) = H_n$ and $\|b_n\|_\infty \leqslant 1$. We set

$$\phi_n(x) := \varepsilon_n \cdot b_n\left(\frac{x - Q_n}{\varepsilon_n}\right) \tag{3.20}$$

and observe that ϕ_n is L-Lipschitz and satisfies:

$$\|\phi_n\|_\infty \leqslant \varepsilon_n \qquad \text{and} \qquad \operatorname{supp}(\phi_n) \subset \overline{B}(Q_n, \varepsilon_n).$$

It follows that the elements of the family $\mathcal{S} = \{\operatorname{supp}(\phi_n) : \ n \in \mathbb{N}\}$ are pairwise disjoint and contained in $[0,1]^N$ (see Figure 1). Moreover, for any $x \in \mathbb{R}^N$ and $n \in \mathbb{N}$, we have

$$\nabla \phi_n\big(\overline{B}(Q_n, \varepsilon_n)\big) = H_n. \tag{3.21}$$

Notice further that if $x \notin \mathbb{R} \times \{0\}^{N-1}$, then $B(x, \delta)$ intersects at most one element of the family $\mathcal{S}$ for $\delta > 0$ sufficiently small.

We are ready to define the function f that satisfies our assertion:

$$\begin{cases} f : \mathbb{R}^N \to \mathbb{R} \\ f(x) = \sum_{n=1}^{\infty} \phi_n(x). \end{cases} \tag{3.22}$$

Since $\mathrm{supp}(f) \subset [0,1]^N$, the function f is compactly supported. It follows easily that f is L-Lipschitz and coincides with ϕ_n in a neighborhood of Q_n. Therefore, $\partial f(x) \subset \overline{B}(0,L)$, for all $x \in \mathbb{R}^N$. Moreover, since $\|\phi_n\|_\infty \underset{n\to\infty}{\longrightarrow} 0$, f vanishes and is continuous on $\mathbb{R} \times \{0\}^{N-1}$. The next claim yields directly (3.15). □

Claim 1. For every $K \in \mathcal{K}_C^0$ there exists $x \in [0,1] \times \{0\}^{N-1}$ with $\partial f(x) = K$.

Proof of the Claim 1. Fix $\widehat{K} \in \mathcal{K}_C^0$ and pick any $\widehat{t} \in [0,1]$ such that $h(\widehat{t}) = \widehat{K}$. Set

$$\hat{x} = (\hat{t}, 0, \dots, 0) \in [0,1] \times \{0\}^{N-1}.$$

We first show $\widehat{K} \subset \partial f(\hat{x})$. Indeed, by continuity of the function h we have

$$\lim_{t \to \widehat{t}} D_H(h(t), \widehat{K}) = 0.$$

Take a sequence $d_{k(n)} \in \mathcal{D}$ converging to $\widehat{t}$ so that $\widehat{x} = \lim_{n\to\infty} Q_{k(n)}$. Recalling (3.19) we deduce that:

$$\lim_{n\to\infty} H_{k(n)} = \widehat{K}. \tag{3.23}$$

Thus, if $p \in \widehat{K}$, there exist points $x_n \in \overline{B}(Q_{k(n)}, \varepsilon_n)$, $n \geqslant 1$, such that the sequence $\{\nabla f(x_n)\}_n$ converges to p. Since $\widehat{x} = \lim_{n\to\infty} x_n$, we obtain $p \in \partial f(\widehat{x})$. This proves that $\widehat{K} \subset \partial f(\widehat{x})$.

Let us now prove $\partial f(\hat{x}) \subset \widehat{K}$. Fix $\varepsilon > 0$. Since h is continuous, there exists $\delta > 0$ such that

$$h(t) \subset \widehat{K} + B(0, \varepsilon/2), \qquad \text{for all } \ t \in (\widehat{t} - \delta, \widehat{t} + \delta) \cap [0,1]. \tag{3.24}$$

For $\rho > 0$ sufficiently small (the exact value of ρ will be fixed later), we set:

$$\mathcal{U}_\rho := \left[(\widehat{t} - \frac{\delta}{2}, \widehat{t} + \frac{\delta}{2}) \times (-\rho, \rho)^{N-1} \right] \setminus \left[\{0\} \times \mathbb{R}^{N-1} \right].$$

Since for every $x \in \mathcal{U}_\rho$ there is at most one $n \in \mathbb{N}$ such that $x \in \mathrm{supp}(\phi_n)$, it follows that either $\nabla f(x) = 0$ (if x does not belong to any element of the family $\mathcal{S}$) or in view of (3.21),

$$\nabla f(x) = \nabla \phi_n(x) \in H_n.$$

In this latter case, since $d_n \in (\widehat{t} - \delta, \widehat{t} + \delta)$ it follows from (3.24) and (3.18) that

$$\nabla f(x) \in H_n \subset h(d_n) + \overline{B}(0, \gamma_n) \subset \left(\widehat{K} + B(0, \frac{\varepsilon}{2}) \right) + \overline{B}(0, \gamma_n).$$

We can take $\rho > 0$ sufficiently small to ensure that $\gamma_n < \varepsilon/2$, whenever $\mathrm{supp}(\phi_n) \cap \mathcal{U}_\rho \neq \emptyset$. Choosing $\rho > 0$ in this way, we infer that

$$\partial f(x) \subset \widehat{K} + B(0, \varepsilon), \text{ for all } x \in \mathcal{U}_\rho.$$

Since the set $\mathcal{N} := \{0\} \times \mathbb{R}^{N-1}$ is negligible for the Lebesgue measure, we deduce easily from the formula (1.3) of the Clarke subdifferential that

$$\partial f(\hat{t} \times \{0\}^{N-1}) \subset \hat{K} + B(0, \varepsilon).$$

Since $\varepsilon > 0$ can be chosen arbitrary small, we obtain the desired conclusion. □

Claim 2. The function f is differentiable on $\mathbb{R}^N$.

Proof of the Claim 2. Since the compact sets $\mathrm{supp}(\phi_n)$ are disjoint subsets $\mathbb{R}^N$ and do not intersect the closed subset $[0,1] \times \{0\}^{N-1}$ of $\mathbb{R}^N$, the function f is C^1-smooth on $\mathbb{R}^{\mathbb{N}} \setminus \left([0,1] \times \{0\}^{N-1}\right)$. Let us now treat the case where $x \in [0,1] \times \{0\}^{N-1}$. In this case, $f(x) = 0$. Take any $y \in \mathbb{R}^N$. If the point y does not belong to $\mathrm{supp}(\phi_n)$ for any n, then $f(y) = 0$, while if $y \in \mathrm{supp}(\phi_n)$ for some $n \in \mathbb{N}$, then we deduce from (3.20) that $|f(y) - f(x)| = |f(y)| \leqslant \varepsilon_n \ll \|y - x\|$ because $\|y - x\| \geqslant \alpha_n - \varepsilon_n$ and $\lim_{n\to\infty} \varepsilon_n / \alpha_n = 0$. Since $\mathrm{supp}(\phi_n)$ is compactly contained in $(0,1)^{\mathbb{N}}$, we conclude that f is differentiable at x and $\nabla f(x) = 0$.

This completes the proof of (i).

(ii) We now assume that there exists $\lambda > 0$ such that $B(0, \lambda) \subset C$. To construct a function f that satisfies (3.15)–(3.16), we replace the definition of H_n in (3.18) by

$$H_n := \left(h(d_n) + \overline{B}(0, \gamma_n)\right) \cap C,$$

and we proceed as before. It follows easily that $\partial f(x) \subset C \subset B(0, L)$, for all $x \in \mathbb{R}^N$ (in particular f is L-Lipschitz) and (3.15) follows as in (i). □

Remark 3.10.

(i) A more elementary (and self-contained) proof of Lemma 3.9 can be provided if the assumption of differentiability of f is dropped. Indeed, following the lines of the above proof, once the sets H_n in (3.18) are defined, we can consider the functions $\hat{b}_n : \mathbb{R}^N \to \mathbb{R}$ defined by

$$\hat{b}_n(x) := \min\{0, \sup_{p \in H_n} \langle p, x\rangle - c_n\},$$

where $c_n > 0$ is chosen such that $\mathrm{supp}(\hat{b}_n) \subset \overline{B}(0,1)$ and $\|\hat{b}_n\|_\infty \leqslant 1$. Note that, in a neighborhood of 0, the function $\hat{b}_n + c_n$ is the support function of H_n. The function $\hat{b}_n$ is nondifferentiable and $\partial \hat{b}_n(x) \subset H_n = \partial \hat{b}_n(0)$ for all $x \in X$. Then we define ϕ_n as in (3.20), using the functions $\{\hat{b}_n : n \in \mathbb{N}\}$ (instead of b_n) and the function $\hat{f} : \mathbb{R}^N \to \mathbb{R}$ as in (3.22). Proceeding as in the above proof and using the fact that the Clarke subdifferential $\partial \hat{f}$ is outer semicontinuous and $\partial \hat{f}(Q_n) = H_n$ for all $n \in \mathbb{N}$, we deduce that $\partial \hat{f}((t, 0, \dots 0)) = h(t)$, for any $t \in [0,1]$.

(ii) We can also use this idea to construct an everywhere differentiable function satisfying Lemma 3.9. Indeed, fixing a positive mollifier $\rho : \mathbb{R}^N \to \mathbb{R}$, we set $\rho_n(\cdot) := \nu_n^{-N} \rho(\cdot/\nu_n)$, $\nu_n \geqslant 1$, and consider the convolution $b_n^* := \hat{b}_n * \rho_n$. Then taking $\nu_n > 0$ sufficiently small, we ensure that b_n^* is a good approximation of $\hat{b}_n$, which becomes better and better as $\nu \to 0$. (The interested reader is invited to work out the details of this construction.)

(iii) A careful inspection of the proof of Lemma 3.9 reveals that one can work directly with the continuous surjective map $h : \Delta \mapsto \mathcal{K}_C^0$ by simply replacing [0,1] by Δ in the proof and by taking a countable dense subset $\mathcal{D}$ of $\Delta \setminus \{0,1\}$. The coding over Δ does not use the fact that the space $(\mathcal{K}_C^0, D_{\mathrm{H}})$ is a geodesic space. This remark will be particularly relevant in Section 3.3.

3.2.2 | Main result: recovering convex bodies

Based on Lemma 3.9 (which recovers all convex bodies containing 0), we can now deduce the general case. We shall also need the following lemma.

Lemma 3.11. *Let $x^* \in \mathbb{R}^N$ be such that $\|x^*\| < 1$. Then, there exists a continuously differentiable and 1-Lipschitz function $h : \mathbb{R}^N \to \mathbb{R}$ with support in the unit ball $B(0,1)$ and $\delta > 0$ such that*

$$\nabla h(x) = x^*(x), \qquad \textit{for all } x \in B(0,\delta).$$

The proof of the above lemma is straightforward. It is sufficient to set $x \mapsto x^*(x)$ on a small ball centered at 0, consider an affine interpolation outside this ball which brings to the value to 0, and finally use a mollifier with a sufficiently small support.

We are now ready to state the main result of this section.

Theorem 3.12 (almost exhaustive function in $\mathbb{R}^N$). *There exists a differentiable 1-Lipschitz compactly supported function $f : \mathbb{R}^N \to \mathbb{R}$ such that for every convex body K of $\overline{B}(0,1)$, there exists $x \in \mathbb{R}^N$ such that $\partial f(x) = K$.*

Proof. Let $\{q_n^*\}_n \subset B(0,1)$ be a dense sequence in $\overline{B}(0,1)$. We claim that there exists a differentiable, 1-Lipschitz, and compactly supported function $g : \mathbb{R}^N \to \mathbb{R}$ satisfying that for each $n \in \mathbb{N}$, there exists a set $\mathcal{U}_n \subset \mathbb{R}^N$ with nonempty interior, such that

$$\nabla g(x) = q_n^* \qquad \text{if } x \in \mathcal{U}_n.$$

Let us present a quick construction of the function g. First, applying Lemma 3.11, for any $n \in \mathbb{N}$, there exists a continuously differentiable and 1-Lipschitz function $g_n : \mathbb{R}^N \to \mathbb{R}$ with support in the unit ball such that $\nabla g_n(x) = q_n^*(x)$, for all x in a neighborhood of 0. Take any sequence $\{x_n\}_n$ of distinct points of $B(0,1)$ that converges to some point ℓ of the open unit ball, with $\ell \neq x_n$, for all $n \in \mathbb{N}$. Choose further $\{\varepsilon_n\}_n \subset (0,1)$ such that $\left\{\overline{B}(x_n, \varepsilon_n)\right\}_n$ is a sequence of disjoint closed balls contained in $B(0,1)$. The required function g is defined by

$$g(x) := \sum_n \varepsilon_n g_n\left(\frac{x - x_n}{\varepsilon_n}\right).$$

The function g is the sum of disjointly supported functions, hence g is 1-Lipschitz, the support of g is contained in the unit ball, $\nabla g(x) = q_n^*$ in a neighborhood $\mathcal{U}_n$ of x_n, and g is differentiable at every point $x \in \mathbb{R}^N \setminus \{\ell\}$.

Let us now show that the function g is also differentiable at ℓ, provided the sequence $\{\varepsilon_n\}_n$ satisfies

$$\lim_{n\to\infty} \frac{\varepsilon_n}{\|x_n-\ell\|} = 0 \qquad \text{(by shrinking the values of } \varepsilon_n \text{ we can always guarantee this).}$$

Indeed, for n sufficiently large and for any $x \in B(x_n, \varepsilon_n)$, we have

$$g(x) = \varepsilon_n\, g_n\Big(\frac{x-x_n}{\varepsilon_n}\Big) \leqslant \varepsilon_n \quad \text{and} \quad \frac{g(x)-g(\ell)}{\|x-\ell\|} \leqslant \frac{\varepsilon_n}{\|x-\ell\|} \leqslant \frac{\varepsilon_n}{\|x_n-\ell\|}\underbrace{\left(\frac{\|x_n-\ell\|}{\|x-\ell\|}\right)}_{\geqslant 1/2} \xrightarrow[n\to\infty]{} 0,$$

yielding that g is differentiable at ℓ with $\nabla g(\ell) = 0$.

For each $n \in \mathbb{N}$, let $x_n \in \mathbb{R}^N$ and $\lambda_n > 0$ be such that $\overline{B}(x_n, \lambda_n) \subset \operatorname{int} \mathcal{U}_n$. Set $C_n := \overline{B}(-q_n^*, 1)$ and notice that $0 \in \operatorname{int} C_n$. Applying Lemma 3.9(ii) for $C = C_n$, we obtain a differentiable 1-Lipschitz function $f_n : \mathbb{R}^N \to \mathbb{R}$ satisfying (3.15)–(3.16). Up to a suitable re-scaling, namely replacing f_n by $\delta_n f_n(\cdot/\delta_n)$, we can assume $\operatorname{supp} f_n \subset \operatorname{int} B(0, \lambda_n)$. We define the function

$$\begin{cases} f : \mathbb{R}^N \to \mathbb{R} \\ f(x) = g(x) + \sum_{n=1}^{\infty} f_n(x - x_n). \end{cases}$$

Notice that, for any $x \in \mathbb{R}^N$, there is at most one $n \in \mathbb{N}$ such that $x - x_n \in \operatorname{supp} f_n$. Moreover, for any $n \in \mathbb{N}$ and $x \in \mathcal{U}_n$, we deduce that

$$\nabla f(x) = q_n^* + \nabla f_n(x - x_n) \in B(0,1),$$

and if x is not in any $\mathcal{U}_n$, then $\nabla f(x) = \nabla g(x) \in B(0,1)$. It follows easily that f is 1-Lipschitz. Let us now verify that f satisfies the property asserted in the statement of the theorem. To this end, let $K \subset B(0,1)$ be a convex compact set with nonempty interior. Since $\{q_n^*\}_n$ is dense in $B(0,1)$, there exists $n \in \mathbb{N}$ such that $q_n^* \in \operatorname{int} K$. Therefore, $K - q_n^* \subset \overline{B}(-q_n^*, 1) = C_n$. From property (3.15) of Lemma 3.9, there exists $y \in \operatorname{supp}(f_n) \subset B(0, \lambda_n)$ such that $\partial f_n(y) = K - q_n^*$. Recalling that $x_n \in \mathcal{U}_n$, setting $x_K := y + x_n \in B(x_n, \lambda_n) \subset \mathcal{U}_n$, we obtain

$$\partial f(x_K) = \nabla g(x_K) + \partial f_n(y) = K.$$

The proof is complete. □

Remark 3.13. A careful inspection of the proof of Theorem 3.12 reveals that for the constructed function $f : \mathbb{R}^N \to \mathbb{R}$, the Clarke subdifferential $\partial f(x)$ and the limiting subdifferential $\partial_L f(x)$ coincide at every point. Let us recall that the same situation occurred in Theorem 3.7, for the case $N = 1$, based on the fact that differentiable real-valued functions on the real line have the Darboux property (cf. Remark 3.8(i)). Consequently, the main results of this paper apply equally well for the limiting subdifferential.

3.3 | Recovering compact connected sets with nonempty interior

In the current subsection, we refine the previous construction to obtain an everywhere differentiable, compactly supported, 1-Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ such that its subdifferential contains every closed connected subset of the unit ball with nonempty interior.

We shall work with the limiting subdifferential $\partial_L f$ which at a given point $x \in \mathbb{R}^N$ consists of all accumulation points of sequences of derivatives $\{\nabla f(x_n)\}_{n\geqslant 1}$ as $x_n \to x$. In strong contrast with the case of strictly differentiable functions (where the only possible limit is $\nabla f(x)$), we show that we can recover all compact connected sets (even completely irregular fractal-type sets) provided they have nonempty interior.

To start, let C be any convex compact set containing 0 and consider the set

$$\widetilde{\mathcal{K}}_C^0 := \{K \subset C : K \text{ is compact connected and } 0 \in K\}. \tag{3.25}$$

We first show that, similarly to $\mathcal{K}_C^0$, the above set can also be coded on the Cantor set $\Delta \subset [0,1]$.

Lemma 3.14. $(\widetilde{\mathcal{K}}_C^0, D_{\mathrm{H}})$ *is a compact metric space (therefore, it can be seen as continuous surjective image of the Cantor set* Δ*).*

Proof. Let us first show that $\widetilde{\mathcal{K}}_C^0$ is closed in $\mathcal{F}_C^0$ (see (2.1)) for the Hausdorff distance. To this end, let $\{K_n\}_n$ be a sequence in $\widetilde{\mathcal{K}}_C^0$ that converges to a compact set $K \in \mathcal{F}_C^0$. It is straightforward to see that $0 \in K \subset C$. If K is not connected, then there would exist two nonempty disjoint open subsets U_1 and U_2 in $\mathbb{R}^N$ such that $K^i = K \cap U_i$ is nonempty, for $i \in \{1,2\}$ and $K = K^1 \cup K^2$. Then the convergence $D_{\mathrm{H}}(K_n, K) \longrightarrow 0$ forces K_n to be disconnected for n sufficiently large, which is a contradiction. This shows that $(\widetilde{\mathcal{K}}_C^0, D_{\mathrm{H}})$ is a compact metric space and there exists a continuous surjective function h that maps the Cantor set Δ onto $\widetilde{\mathcal{K}}_C^0$ (see [18, Theorem 4.18]). □

Based on Remark 3.10(iii), we can now refine the proof of Lemma 3.9 and enhance the conclusion. This is done in the following lemma, whose proof follows closely the proof of Lemma 3.9. We present a sketch of the proof, highlighting the main changes.

Before we proceed, let us recall that a closed set $C \subset \mathbb{R}^d$ is called *strictly convex* if for any two distinct points $x, y \in C$, the open segment (x, y) joining x and y lies in the interior of C. (In particular, a strictly convex set is either singleton or has nonempty interior.)

Lemma 3.15. *Let* $C \subset \mathbb{R}^N$ *be a convex compact set such that* $0 \in C$ *and* $L := \max_{x\in C}\{\|x\|\}$*. Then:*

(i) *There is a differentiable L-Lipschitz continuous and compactly supported function* $f : \mathbb{R}^N \to \mathbb{R}$ *such that:*

$$\text{for every } K \in \widetilde{\mathcal{K}}_C^0, \text{ there exists } x \in \mathbb{R}^N \text{ such that } \partial_L f(x) = K. \tag{3.26}$$

(ii) *Let us further assume* $0 \in \mathrm{int}(C)$ *and* C *is strictly convex. Then in addition to the above conclusion we get:*

$$\partial_L f(x) \subset C, \text{ for all } x \in \mathbb{R}^N. \tag{3.27}$$

Proof (Sketch).

(i) Let $h : \Delta \to \widetilde{\mathcal{K}}_C^0$ be a continuous surjective map (which will be used to code the elements of $\widetilde{\mathcal{K}}_C^0$). Let $\mathcal{D} = \{d_n \,:\, n \in \mathbb{N}\}$ be a countable dense subset of $\Delta \setminus \{0,1\}$ and consider two sequences $\{\alpha_n\}_n$ and $\{\varepsilon_n\}_n$ of positive real numbers as in the proof of Lemma 3.9. In particular, we have $\alpha_n > \varepsilon_n > 0$, for all $n \geqslant 1$, $\lim_{n\to\infty} \alpha_n = 0$, and $\lim_{n\to\infty} \varepsilon_n/\alpha_n = 0$.

Define $\{Q_n\}_n$ by (3.17). Then, the sets $\{B(Q_n, \varepsilon_n)\}_n$ are pairwise disjoint and are contained in $[0,1]^N$. Let $\{\gamma_n\}_n$ be an arbitrary sequence of positive numbers converging to 0. For every $n \in \mathbb{N}$, since $h(d_n)$ is totally bounded, there exists a finite γ_n-net A_n of $h(d_n)$, containing 0, that is,

$$0 \in A_n \subset h(d_n) \subset \bigcup_{a\in A_n} \overline{B}(a, \gamma_n).$$

We then define

$$\widetilde{H}_n := \Big(A_n + \overline{B}(0, 2\gamma_n) \Big) \bigcap \overline{B}(0, L) = \bigcup_{a\in A_n} \overline{B}(a, 2\gamma_n) \cap \overline{B}(0, L). \tag{3.28}$$

Notice that $\widetilde{H}_n$ is a finite union of strictly convex sets and $0 \in B\,(0, \min\{2\gamma_n, L\})$. Moreover, for every $n \in \mathbb{N}$, we have

$$h(d_n) \subset \operatorname{int}(\widetilde{H}_n) \subset h(d_n) + \overline{B}(0, 2\gamma_n). \tag{3.29}$$

Therefore, $\operatorname{int}(\widetilde{H}_n)$ is connected, therefore, according to [5, Theorem 8], for every $n \in \mathbb{N}$, there exists a C^1-smooth function $b_n : \mathbb{R}^N \to \mathbb{R}$, with support in the unit ball, such that $\nabla b_n(\mathbb{R}^N) = \widetilde{H}_n$ and $\|b_n\|_\infty \leqslant 1$. We set

$$\phi_n(x) := \varepsilon_n \cdot b_n\Big(\frac{x - Q_n}{\varepsilon_n}\Big). \tag{3.30}$$

We are ready to define the function f that satisfies our assertion:

$$\begin{cases} f : \mathbb{R}^N \to \mathbb{R} \\ f(x) = \sum_{n=1}^{\infty} \phi_n(x). \end{cases} \tag{3.31}$$

Since $\operatorname{supp}(f) \subset [0,1]^N$, the function f is compactly supported.

Claim. For every $K \in \widetilde{\mathcal{K}}_C^0$ there exists $x \in [0,1] \times \{0\}^{N-1}$ with $\partial_L f(x) = K$.

Proof of the Claim. It follows as in the proof of Lemma 3.9 by noticing that (3.29) gives us that

$$D_{\mathrm{H}}(\widehat{H}_n, h(d_n)) \leqslant 2\gamma_n, \text{ for all } n \in \mathbb{N},$$

and that the Cantor set Δ is a perfect set.

Finally, the differentiability of f follows exactly as in the proof of Lemma 3.9. This completes the proof of (i).

(ii) We now assume that there exists $\lambda > 0$ such that $B(0,\lambda) \subset C$ and that C is strictly convex. To construct a function f that satisfies (3.26)–(3.27), we replace the definition of $\widetilde{H}_n$ in (3.28) by

$$\widetilde{H}_n := \Big(A_n + \overline{B}(0, 2\gamma_n)\Big) \cap C = \bigcup_{a\in A_n} \overline{B}(a, 2\gamma_n) \cap C.$$

Thus, $\widetilde{H}_n$ is a finite union of strictly convex sets and $0 \in \mathrm{int}(\widetilde{H}_n)$. Proceeding as before, it easily follows that $\partial_L f(x) \subset C \subset B(0, L)$, for all $x \in \mathbb{R}^N$ (in particular f is L-Lipschitz) and (3.26) follows as in (i). □

Similarly to the proof of Theorem 3.12, we can now use Lemma 3.15 to obtain the existence of a compactly supported differentiable 1-Lipschitz function $f : \mathbb{R}^N \to \mathbb{R}$ such that the range of its limiting subdifferential $\partial_L f$ contains all compact, connected subsets of the closed unit ball $\overline{B}(0, 1)$ with nonempty interior. (Notice that Lemma 3.15 uses the fact that the Euclidean balls are strictly convex.) Then by a standard argument, already evoked in the beginning of Subsection 3.2, see (3.12) we deduce the following result.

Theorem 3.16. *There exists a differentiable locally Lipschitz function $f : \mathbb{R}^N \to \mathbb{R}$ such that for every compact, connected subset K of $\mathbb{R}^N$ with nonempty interior, there exists $x \in \mathbb{R}^N$ such that $\partial_L f(x) = K$. Moreover, given $\varepsilon > 0$, f can be taken to satisfy $\|f\|_\infty < \varepsilon$.*

Let us mention the following interesting consequence of the above result. Denoting by

$$\mathrm{gph}(\nabla f) := \{(x, \nabla f(x)) : x \in \mathbb{R}^N\} \subset \mathbb{R}^N \times \mathbb{R}^N$$

the graph of the derivative ∇f of a differentiable function $f : \mathbb{R}^N \to \mathbb{R}$, we have the following:

Corollary 3.17. *There exists a differentiable locally Lipschitz function $f : \mathbb{R}^N \to \mathbb{R}$ with the property that for every compact, connected subset K of $\mathbb{R}^N$ with nonempty interior, there exists $\bar{x} \in \mathbb{R}^N$ such that*

$$(\bar{x}, y) \in \overline{\mathrm{gph}(\nabla f)} \iff y \in K. \tag{3.32}$$

This illustrates the gap between mere differentiability versus C^1-smoothness, since in the latter case, only a singleton set K (namely, $K = \{\nabla f(\bar{x})\}$) satisfies (3.32).

Let us finally notice that Theorem 3.16 can be seen as a result of *almost exhaustiveness* for the limiting subdifferential of a *differentiable*, locally Lipschitz function. Indeed, Malý [21] established a Darboux-type property for the gradient ∇f of a differentiable function f in $\mathbb{R}^N$, namely, that

$$\nabla f(B) := \{\nabla f(x) : x \in B\}$$

is connected, for any convex body B of $\mathbb{R}^N$. It follows that if f is differentiable and locally Lipschitz, then the above set is bounded and the limiting subdifferential is also given by the formula

$$\partial_L f(\bar{x}) = \bigcap_{\varepsilon>0} \mathrm{cl}\Big(\{\nabla f(x) : x \in \overline{B}(\bar{x}, \varepsilon)\}\Big).$$

Therefore, $\partial_L f(\bar{x})$ contains $\{\nabla f(\bar{x})\}$ and is always a compact connected set (as intersection of nested compact connected sets). It follows that the differentiable, locally Lipschitz function f of the statement of Theorem 3.16 is almost exhaustive for the limiting subdifferential (compare with Definition 2.4).

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