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A slope generalization of Attouch theorem

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Abstract

A classical result of variational analysis, known as Attouch theorem, establishes an equivalence between epigraphical convergence of a sequence of proper convex lower semicontinuous functions and graphical convergence of the corresponding subdifferential maps up to a normalization condition which fixes the integration constant. In this work, we show that in finite dimensions and under a mild boundedness assumption, we can replace subdifferentials (sets of vectors) by slopes (scalars, corresponding to the distance of the subdifferentials to zero) and still obtain the same characterization: namely, the epigraphical convergence of functions is equivalent to the epigraphical convergence of their slopes. This surprising result goes in line with recent developments on slope determination (Boulmezaoud et al. in SIAM J Optim 28(3):2049–2066, 2018; Pérez-Aros et al. in Math Program 190(1–2):561-583, 2021) and slope sensitivity (Daniilidis and Drusvyatskiy in Proc Am Math Soc 151(11):4751-4756, 2023) for convex functions.

Keywords Attouch theorem · Convex function · Slope · Epi-convergence · Sensitivity analysis

Mathematics Subject Classification Primary 26B25 · 46J52; Secondary 37C10 · 46N10 · 49K40

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1 Introduction

In 1977, Hédy Attouch showed that a sequence of proper convex lower semicontinuous (in short lsc) functions $\{f_n\}_{n\geq 1}$ epi-converges to (a lsc convex function) f if and only if the sequence $\{\partial f_n\}_{n\geq 1}$ of the corresponding subdifferentials converges graphically to the subdifferential ∂f of f and a normalization condition (fixing the constant of integration) holds, see [2, 3]. Epi-convergence of a sequence of functions refers to the set-convergence of the sequence of epigraphs of the functions, while the graph convergence of a sequence of set-valued maps involves the set-convergence of their graphs. In finite dimensions, both convergences are in the Painlevé–Kuratowski sense. The result remains valid in a reflexive Banach space, provided the convergence of $\{\text{epi}\, f_n\}_{n\geq 1}$ to $\text{epi}\, f$ is taken in the Mosco sense (see [26]).

Attouch theorem has been further extended in [4, 12, 21] to any Banach space, using the notion of slice convergence which is shown to be equivalent to the Mosco epi-convergence of both functions and their convex conjugates. Further extensions cover more general classes of functions, as for instance the class of primal lower nice functions (see [12, 24, 29] e.g.).

The importance of the Attouch theorem can be measured by its numerous applications: it has been used to establish strong solutions in parabolic variational inequalities [6], stability results in numerical optimization [23], stochastic optimization [18] as well as theoretical results on generalized second order derivatives of convex functions [31] or in relation with the differentiability of Lipschitz set-valued maps [16]. It also meets applications in non-regular mechanics and in subgradient evolution systems, see [3, 5] and references therein.

The original proofs of the Attouch theorem (see [2–4]) are based on the integration formula of Rockafellar [30] for the class of maximal cyclically monotone operators, which is a characteristic property of the subdifferential map of a convex function. The approach of [12] is different, but still relies on the *subdifferential determination* of any convex function. Indeed, it is well-known that the equality $\partial f = \partial g$ for any two convex lsc functions f, g guarantees that the functions are equal up to a constant.

Quite recently, the following intriguing result has been established: convex lsc functions are fully determined by the slope mapping $x \mapsto s_f(x) := \text{dist}(0, \partial f(x))$ (rather than the whole subdifferential), up to an additive constant, provided they are bounded from below. In other words, knowledge of the remoteness of the subdifferential (which is a scalar) at every point, gives in this case, enough information for the full determination of the subdifferential and consequently, of the function, that is,

$$s_f = s_g \quad \iff \quad \partial f = \partial g \quad \iff \quad f = g + \text{cst}. \tag{1.1}$$

This result has first been established in Hilbert spaces for the class of smooth (convex and bounded from below) functions [11] and has then been extended to the class of (nonsmooth) convex continuous and bounded from below functions [27]. Although it is not relevant for our purposes, let us mention for completeness that (1.1) was ultimately established in [33] for convex functions defined in an arbitrary Banach space. Further extensions to the class of Lipschitz functions in metric spaces, upon knowledge of the critical set, have been done in [17].

Very recently, a study of robustness of the *slope determination* result has been carried out in [14], motivated by the following question:

If the slopes of two convex functions are close, are the function values close?

In finite dimensions, the main result of [14] reads, roughly speaking, as follows: if f, g are two convex continuous functions that attain their minimum value, then:

$$\|g - f\|_{\mathcal{U}} \lesssim \|s_g - s_f\|_{\mathcal{U}} + \|g - f\|_{C_f \cup C_g},$$

where $\mathcal{U}$ is any bounded set, $\|\cdot\|_{\mathcal{U}}$ is the sup-norm over $\mathcal{U}$, $C_f := \operatorname{argmin} f$ and $C_g := \operatorname{argmin} g$. In particular, the quantity $\|g - f\|_{\mathcal{U}}$ is controlled in a Lipschitz manner by the slope deviation $\|s_g - s_f\|_{\mathcal{U}}$, yielding the following convergence result:

Theorem 1.1 ([14], Corollary 3.3) *Let $\{f_n\}_{n\geq 0}$ be convex continuous functions such that*

$$\mathcal{C}_{f_n} := \operatorname{argmin} f_n \neq \emptyset \;\; \textit{for all } n \geq 0 \;\; \textit{and} \;\; \mathcal{C} := \left(\cup_{n\geq 0} \mathcal{C}_{f_n}\right) \;\; \textit{is bounded.}$$

Assume further that:
(i). $\{s_{f_n}\}_n$ *converges to* s_{f_0} *uniformly on bounded sets;*
(ii). $\{f_n\}_n$ *converges to* f_0 *uniformly on* $\mathcal{C}$.
Then $\{f_n\}_n$ *converges to* f_0 *uniformly on bounded sets.*

The assumption of existence of (global) minima in the above result is suboptimal, since it is stronger than mere boundedness from below, which was the main assumption in (1.1), see also [14, Remark 3.4]. In addition, Theorem 1.1 does not cover variational deviations, which is the natural framework of the Attouch theorem.

In this work we generalize the result of [27] (slope determination) and complement the result of [14] (slope sensitivity), establishing a slope version of the Attouch theorem in finite dimensions, under the condition that the limiting function f is bounded from below. Since graphical convergence of subdifferentials is ostensibly much stronger than epi-convergence of the slopes (see Sect. 2 for a formal proof of this implication), the converse implication is the core of our main result (see Sect. 4). Therefore, in a sensitivity framework, our main theorem (*c.f.* Theorem 1.6) generalizes the Attouch theorem, in a similar way that the slope determination generalizes subdifferential determination.

1.1 Basic setting and notation

We consider the d-dimensional Euclidean space $\mathbb{R}^d$ endowed with its usual inner product $\langle\cdot,\cdot\rangle$ and its Euclidean norm $\|\cdot\|$. For a subset $A \subset \mathbb{R}^d$, we denote by $\operatorname{int}(A)$, $\operatorname{cl}(A)$, ∂A and $\operatorname{ri}(A)$ its interior, closure, boundary and relative interior, respectively. Given $x \in \mathbb{R}^d$, we write $B(x,r)$ and $\overline{B}(x,r)$ to denote the open and closed r-balls centered at x, and we define its distance to the set A as follows:

$$\operatorname{dist}(x, A) := \inf_{a\in A} \|x - a\|.$$

For a function $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, we denote its effective domain and respectively its epigraph by:

$$\operatorname{dom} f := \{x \in \mathbb{R}^d : f(x) < +\infty\} \quad \text{and} \quad \operatorname{epi} f = \{(x,\alpha) \in \mathbb{R}^d \times \mathbb{R} : \alpha \geq f(x)\}.$$

The (Moreau–Rockafellar) subdifferential of f is then defined as follows:

$$\partial f(x) = \{x^* \in \mathbb{R}^d : f(y) \geq f(x) + \langle x^*, y - x\rangle, \ \forall y \in \mathbb{R}^d\}, \tag{1.2}$$

if $x \in \operatorname{dom} f$ and empty otherwise. Note that f may not be a convex function and $\partial f(x)$ may be empty even if $x \in \operatorname{dom} f$. If f is proper (i.e., $\operatorname{dom} f \neq \emptyset$), we denote by $f^* : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ its Fenchel conjugate, that is,

$$f^*(x^*) = \sup_{x \in \mathbb{R}^d} \left\{\langle x^*, x\rangle - f(x)\right\}. \tag{1.3}$$

It is easy to check from the definition of f^* that Young-Fenchel inequality holds true: for all $(x, x^*) \in \mathbb{R}^d \times \mathbb{R}^d$ one has that $f(x) + f^*(x^*) \geq \langle x^*, x\rangle$. Moreover, the subdifferential of f can be characterized in terms of its conjugate function as follows:

$$x^* \in \partial f(x) \iff f(x) + f^*(x^*) = \langle x^*, x\rangle. \tag{1.4}$$

Following [19] we define the (metric or local) slope of a function $f : \mathbb{R}^d \to \mathbb{R}$ at a point $x \in \mathbb{R}^d$ as follows:

$$s_f(x) = \begin{cases} \limsup\limits_{y \to y} \frac{\{f(x) - f(y)\}^+}{d(y.x)}, & \text{if } x \in \operatorname{dom} f \\ +\infty, & \text{otherwise,} \end{cases}$$

where $\{\alpha\}^+ = \max\{0, \alpha\}$ and $d(y, x) = \|x - y\|$. This notion has been extensively studied in the framework of metric analysis, see [1, 8, 9, 20, 22] and references therein. In the special case that the function f is convex and lsc, for every $x \in \mathbb{R}^d$ one has:

$$s_f(x) = \operatorname{dist}(0, \partial f(x)) \qquad \text{(distance of the subdifferential to 0).} \tag{1.5}$$

Whenever f is a proper convex lsc function and $x \in \operatorname{dom} f$, it is well-known that $\partial f(x)$ is a convex closed set. Notice that $s_f(x) = +\infty$ if and only if $\partial f(x) = \emptyset$. Whenever $\partial f(x)$ is nonempty, we denote by $\partial^\circ f(x)$ the (unique) element of minimal norm of $\partial f(x)$, that is,

$$\partial^\circ f(x) = \operatorname{proj}(0; \partial f(x)), \quad \forall x \in \operatorname{dom} \partial f, \tag{1.6}$$

where $\operatorname{proj}(\cdot; A)$ stands for the projection to a set $A \subset \mathbb{R}^d$ and

$$\operatorname{dom} \partial f = \{x \in \mathbb{R}^d \ \ \partial f(x) \neq \emptyset\}$$

is the effective domain of the subdifferential of f. Notice that (1.2) yields

$$0 \in \partial f(x) \iff s_f(x) = 0 \iff x \in \arg\min f \quad \text{(set of global minimizers of } f) \tag{1.7}$$

As already mentioned in the introduction, the slope determines, up to a constant, the class of convex lsc functions that are bounded from below. In a Hilbert space setting an important intermediate result, the so-called comparison principle, was established in [27]. This is recalled below.

Theorem 1.2 (Comparison principle) *Let $f, g : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be two convex lsc functions that are bounded from below. Assume that*

(i). $\inf f \geq \inf g$*; and*
(ii). $s_f(x) \geq s_g(x)$*, for all* $x \in \mathbb{R}^d$*.*

Then it holds: $f \geq g$*.*

In what follows we identify the subdifferential ∂f (which is a set-valued map from $\mathbb{R}^d$ to $\mathbb{R}^d$) with its graph $\mathrm{gph}(\partial f)$ (which is the subset of $\mathbb{R}^d \times \mathbb{R}^d$ given just below) and we indistinctly switch from the notation $x^* \in \partial f(x)$ to the notation $(x, x^*) \in \partial f$. Under this slight abuse of notation, we have:

$$\partial f := \{(x, x^*) \in \mathbb{R}^d \times \mathbb{R}^d \,:\, x^* \in \partial f(x)\} \left(\subset \mathbb{R}^d \times \mathbb{R}^d\right),$$
$$\triangle f := \{(x, x^*, \alpha) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} : \, x^* \in \partial f(x),\ \alpha = f(x)\} \left(\subset \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}\right).$$

For a proper convex lsc function $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ and a point $x_0 \in \mathrm{cl}(\mathrm{dom}\, f)$, we say that an absolutely continuous curve $\gamma : [0, +\infty) \to \mathbb{R}^d$ is a *(maximal) steepest descent curve* for f emanating from x_0 if it solves the differential inclusion

$$\begin{cases} \dot{\gamma}(t) \in -\partial f(\gamma(t)), & \forall t \in [0, +\infty), \\ \gamma(0) = x_0. \end{cases} \tag{1.8}$$

It is well known (see, e.g., [5, Chapter 17]) that for any initial point $x_0 \in \mathrm{cl}(\mathrm{dom}\, f)$, there exists a unique steepest descent curve emanating from x_0. In addition, the functions $t \mapsto f(\gamma(t))$ and $t \mapsto s_f(\gamma(t))$ are decreasing and satisfy

$$\lim_{t \to +\infty} f(\gamma(t)) = \inf f \qquad \text{and} \qquad \lim_{t \to +\infty} s_f(\gamma(t)) = 0. \tag{1.9}$$

If γ is bounded (that is, $\gamma([0, +\infty)) \subset B(0, M)$ for some $M > 0$), then it has finite length (see [13, 25] e.g.). This happens exactly when $\arg\min f \neq \emptyset$ and in this case, $\gamma(t) \underset{t \to +\infty}{\longrightarrow} \gamma_\infty$, with $s_f(\gamma_\infty) = 0$. (Notice here that it is possible to have convergence in finite time, i.e. $\gamma(T) = \gamma_\infty$ for some $T > 0$, case in which γ becomes stationary afterwards: think for example of the function $f(x) = \|x\|$, for all $x \in \mathbb{R}$.)

1.2 Notions of convergence and Attouch theorem

Let $\{S_n\}_n$ be a sequence of subsets of $\mathbb{R}^d$. We consider the inferior and superior limits of $\{S_n\}_n$ in the sense of Painlevé–Kuratowski as

$$\operatorname*{Liminf}_{n\to\infty} S_n := \left\{x \in \mathbb{R}^d \ : \ \limsup_{n\to\infty} \operatorname{dist}(x, S_n) = 0\right\},$$
$$\operatorname*{Limsup}_{n\to\infty} S_n := \left\{x \in \mathbb{R}^d \ : \ \liminf_{n\to\infty} \operatorname{dist}(x, S_n) = 0\right\}.$$

We say that $\{S_n\}_n$ converges to a set S in the sense of Painlevé-Kuratowski, which we denote by $S_n \xrightarrow{PK} S$, if both, the inferior and superior limits of $\{S_n\}_n$ coincide with S. Noting that $\operatorname{Liminf} S_n \subset \operatorname{Limsup} S_n$, one can write

$$S_n \xrightarrow{PK} S \iff \operatorname*{Limsup}_{n\to\infty} S_n \subset S \subset \operatorname*{Liminf}_{n\to\infty} S_n. \tag{1.10}$$

In what follows, given a sequence of functions $\{\phi_n\}_n$ from $\mathbb{R}^d$ to $\mathbb{R}\cup\{+\infty\}$, we define the functions $\phi_l, \phi_u : \mathbb{R}^d \to \mathbb{R}\cup\{\pm\infty\}$ as the lower and, respectively, the upper epigraphical limits of $\{\phi_n\}_n$, given as follows:

$$\phi_l(x) = (\text{e}-\operatorname*{liminf}_{n\to\infty} \phi_n)(x) := \inf_{x_n\to x}\ \liminf_{n\to\infty}\ \phi_n(x_n), \tag{1.11}$$
$$\phi_u(x) = (\text{e}-\operatorname*{limsup}_{n\to\infty} \phi_n)(x) := \inf_{x_n\to x}\ \limsup_{n\to\infty}\ \phi_n(x_n)$$

where, in both cases, the infimum is taken over all sequences $\{x_n\}_n \subset \mathbb{R}^d$ converging to x. Given an increasing sequence of natural numbers $\{k(n)\}_n$ (which we indistinctly also denote by $\{k_n\}_n$) we denote by $\phi_{l,k(n)}$ (respectively, $\phi_{u,k(n)}$) the lower (respectively, upper) epigraphical limit of the subsequence $\{\phi_{k(n)}\}_n$ of $\{\phi_n\}_n$.

Remark 1.3 (attainability of infimum and lower semicontinuity) The infima that define ϕ_l and ϕ_u in (1.11) are attained, that is, for every $x \in \mathbb{R}^d$ there exist (infimizing) sequences $\{x_n^l\}_n$ and $\{x_n^u\}_n$, converging to x, satisfying:

$$\phi_u(x) = \limsup_{n\to\infty}\ \phi_n(x_n^u) \quad \text{and} \quad \phi_l(x) = \liminf_{n\to\infty}\ \phi_n(x_n^l).$$

Based on the above remark and using a diagonal argument, we easily deduce that the functions ϕ_l and ϕ_u are lsc.

Finally, we say that a sequence of functions $\{\phi_n\}_n$ *converges epigraphically* to a function ϕ and denote $\phi_n \xrightarrow{e} \phi$, if the sequence of epigraphs $\{\operatorname{epi}\phi_n\}_n$ converges to $\operatorname{epi}\phi$ in the sense of Painlevé-Kuratowski, that is:

$$\phi_n \xrightarrow{e} \phi \qquad \iff \qquad \operatorname{epi}\phi_n \xrightarrow{PK} \operatorname{epi}\phi.$$

It is well-known that

$$\phi_n \xrightarrow{e} \phi \iff \phi_u = \phi = \phi_l \iff \phi_l \geq \phi \geq \phi_u. \tag{1.12}$$

Let us finally recall, in the finite dimensional setting, the following celebrated variational approximation result due to H. Attouch [2]:

Theorem 1.4 (Attouch theorem) *Let* f, $\{f_n\}_n : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ *be proper convex lsc functions. The following assertions are equivalent:*

(i). $\operatorname{epi} f_n \xrightarrow{PK} \operatorname{epi} f$ *(that is,* $f_n \xrightarrow{e} f$*).*

(ii). $\partial f_n \xrightarrow{PK} \partial f$ *and:*

$$\exists (x, x^*) \in \partial f \text{ and a sequence } (x_n, x_n^*) \in \partial f_n, \quad (x_n, x_n^*, f_n(x_n)) \to (x, x^*, f(x)). \tag{NC}$$

(iii). $\triangle f_n \xrightarrow{PK} \triangle f$.

The normalization condition (NC) is necessary in order to fix a reference point. Without this condition, simple counterexamples can be constructed: indeed, consider the functions $f_n(x) \equiv n$, for all $n \geq 1$ and the function $f(x) \equiv 0$. Then $\partial f_n(x) = \partial f(x) = \{0\}$, for all $x \in \mathbb{R}^d$ and $n \geq 1$, but $f_n(x) \to \infty$, for all $x \in \mathbb{R}^d$. We would also like to point out that, since our setting is finite dimensional, the equivalence $(i) \leftrightarrow (iii)$ can be found e.g. in [10, 12].

Our objective in this work is to provide a version of the Attouch theorem which is based on the epigraphical convergence of the slope mappings $s_{f_n} : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ (rather than the graphical convergence of the subdifferential maps $\partial f_n : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$). Before we proceed to this, let us extract the following consequence of Theorem 1.4 (i)⇒(ii) for future use.

Remark 1.5 Let $f_n \xrightarrow{e} f$. Then for every strictly increasing sequence $\{k_n\}_{n\geq 1}$ and for every $\{(x_{k_n}, x_{k_n}^*)\}_n \subset \mathbb{R}^d \times \mathbb{R}^d$ such that $x_{k_n} \to x$, $x_{k_n}^* \to x^*$ and $x_{k_n}^* \in \partial f_{k_n}(x_{k_n})$, we have $x^* \in \partial f(x)$.

1.3 Our contribution

The goal of this work is to establish that epigraphical convergence of convex functions can be characterized by epigraphical convergence of the slopes.

Our approach relies on the determination result of [27] and naturally inherites the restriction that the limit function should be bounded from below. As in the Attouch theorem, a normalization condition will also be required. In this work we can either use the same condition (NC) as in Theorem 1.4 or an alternative condition over the infimum of the epigraphical lower and upper limits. Concretely, our main result is as follows:

Theorem 1.6 (main result) *Let* $f, \{f_n\}_n : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ *be proper convex lsc functions. Assume that* $\inf f \in \mathbb{R}$. *Then, the following assertions are equivalent:*

(i). $f_n \xrightarrow{e} f$.
(ii). $s_{f_n} \xrightarrow{e} s_f$ *and* (NC) *holds.*
(iii). $s_{f_n} \xrightarrow{e} s_f$ *and* $\inf f_l = \inf f = \inf f_u$.

The rest of the manuscript is organized as follows: in Sect. 2 we show that implications (ii) and (iii) of the statement of Theorem 1.6 follow easily from (i) and Theorem 1.4 (Attouch theorem). Then, Sect. 3, is devoted to a preliminary study of the functions f_l and f_u. Finally, in Sect. 4, we show that either one of (ii) or (iii) implies (i). The approach is divided into two parts: we first show that $f_u \leq f$ in Sect. 4.1, and then in Sect. 4.2 we prove that $f \leq f_l$. The main result and final comments are given at the end (Sect. 5).

2 From epigraphical convergence to slope convergence

In this section, we show the "easy" implications of Theorem 1.6, namely, (i)⇒(ii),(iii). The proof consists of studying the upper and the lower epigraphical limits of the slope sequence $\{s_{f_n}\}_n$ then combine with (1.12) to deduce the result. A standard argument that will repeatedly appear in this work, is to study separately the points where the limit function (in this case s_f) is finite from those where the limit is infinite.

Theorem 2.1 *Let* $f, \{f_n\}_n : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ *be proper convex lsc functions. Assume that* $f_n \xrightarrow{e} f$. *Then*

$$s_{f_n} \xrightarrow{e} s_f, \qquad \inf f = \inf f_l = \inf f_u \qquad \textit{and} \qquad \textit{(NC) holds.}$$

Proof Our assumption yields $f = f_u = f_l$, thus, $\inf f = \inf f_l = \inf f_u$. Condition (NC) follows from Theorem 1.4 (i)⇒(ii). It remains to prove the epiconvergence of the sequence $\{s_{f_n}\}_n$ to s_f. To this end, let $x \in \operatorname{dom} f$ and consider separately two cases:

- *Case 1*: $\partial f(x) = \emptyset$ (that is, $s_f(x) = +\infty$)

In this case, we need to show that $(\text{e}-\operatorname{liminf} s_{f_n})(x) = +\infty$. Let us assume, towards a contradiction, that there exists a sequence $\{x_n\}_n \subset \mathbb{R}^d$ converging to x, such that $\liminf_{n\to\infty} s_{f_n}(x_n) < +\infty$. Then, for an adequate subsequence $\{x_{k_n}\}_n$ we would have

$$\liminf_{n\to\infty} s_{f_n}(x_n) = \lim_{n\to\infty} s_{f_{k_n}}(x_{k_n}) < \infty,$$

and (up to a new subsequence) $x^*_{k_n} \to x^*$, for some $x^* \in \mathbb{R}^d$, where $x^*_{k_n} := \partial^\circ f_{k_n}(x_{k_n})$ is the element of minimal norm in $\partial f_{k_n}(x_{k_n})$, as in (1.6). By Remark 1.5 we infer that $x^* \in \partial f(x)$, which is a contradiction. Therefore, $(\text{e}-\operatorname{liminf} s_{f_n})(x) = +\infty = s_f(x)$.

- *Case 2*: $\partial f(x) \neq \emptyset$ (that is, $s_f(x) < +\infty$)

Let $x \in \operatorname{dom} s_f$ and $\bar{x}^* \in \partial f(x)$ such that $\|\bar{x}^*\| = s_f(x)$. Since $(x, \bar{x}^*, f(x)) \in \triangle f$ and since $\triangle f_n \xrightarrow{PK} \triangle f$ (*c.f.* Theorem 1.4), there exists a sequence $(x_n, x_n^*, f_n(x_n)) \in \triangle f_n$ converging to $(x, \bar{x}^*, f(x))$. Thus, using (1.5), we deduce

$$\limsup_{n\to\infty} s_{f_n}(x_n) \le \limsup_{n\to\infty} \|x_n^*\| = \|\bar{x}^*\| = s_f(x),$$

which yields $(\text{e}-\limsup s_{f_n})(x) \le s_f(x)$. It remains to show that

$$\inf_{x_n\to x} \liminf_{n\to\infty} s_{f_n}(x_n) \ge s_f(x).$$

To this end, we consider an arbitrary sequence $\{x_n\}_n \subset \mathbb{R}^d$ converging to x. For a suitable subsequence $\{k_n\}_n$ we have:

$$\lim_{n\to\infty} s_{f_{k_n}}(x_{k_n}) = \liminf_{n\to\infty} s_{f_n}(x_n) = \rho$$

and we need to show that $\rho \ge s_f(x)$. We can obviously assume that $\rho < +\infty$. For each $n \in \mathbb{N}$, let $x_{k_n}^* = \partial^\circ f_{k_n}(x_{k_n})$ be the element of minimal norm of $\partial f_{k_n}(x_{k_n})$, that is, $\|x_{k_n}^*\| = s_{f_{k_n}}(x_{k_n})$. Since $s_{f_{k_n}}(x_{k_n}) \to \rho$, the subsequence $\{x_{k_n}^*\}_n$ is bounded and converges (up to a new subsequence) to some $x^* \in \mathbb{R}^d$, with $\|x^*\| = \rho$. By Remark 1.5 we have $x^* \in \partial f(x)$ and consequently

$$s_f(x) \le \|x^*\| = \lim_{n\to\infty} \|x_{k_n}^*\| = \lim_{n\to\infty} s_{f_{k_n}}(x_{k_n}) = \rho.$$

Since the sequence $\{x_n\}_n$ is arbitrary, we have $s_f(x) \le (\text{e}-\liminf s_{f_n})(x)$.

The proof is complete. □

3 Some intermediate results

In this part, we obtain some preliminary results, which are needed for the proof of the "difficult" implication of our main theorem. Some of the forthcoming results are essentially known, other are less obvious and require a careful analysis.

3.1 General results from convex analysis

The first result is essentially known.

Proposition 3.1 *Let $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a proper convex lsc function. Then* $\text{ri}(\text{dom}\, s_f) = \text{ri}(\text{dom}\, f)$ *and therefore it is a convex set.*

Proof Let us first notice that $\text{dom}\, s_f = \text{dom}\, \partial f$. Since f is convex, ∂f is nonempty on $\text{ri}(\text{dom}\, f)$. Thus, $\text{ri}(\text{dom}\, f) \subset \text{dom}\, s_f \subset \text{dom}\, f$. Without loss of generality, we may assume that $0 \in \text{dom}\, f$. Set $V = \text{span}(\text{dom}\, f)$, that is, the subspace of $\mathbb{R}^d$ generated by $\text{dom}\, f$. Notice that $\text{ri}(\text{dom}\, f)$ generates the same subspace V, therefore $\text{span}(\text{dom}\, s_f) = V$. Since the relative interiors of $\text{dom}\, s_f$ and $\text{dom}\, f$ are taken with respect to the same space V, we have $\operatorname{ri}(\operatorname{dom} s_f) \subset \operatorname{ri}(\operatorname{dom} f)$. The conclusion follows from the convexity of $\operatorname{dom} f$. □

The following result is also quite intuitive.

Proposition 3.2 *Let $f, g : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be two functions, with f convex and g lsc. Let $A \subset \operatorname{dom} f$ be a nonempty convex set. Assume that $f \geq g$ on A. Then, $f \geq g$ on* $\operatorname{cl}(A)$.

Proof Let $\bar{x} \in \operatorname{cl}(A)$ and $x \in \operatorname{ri}(A)$. Then $(\bar{x}, x] \subset A$. Note that $f(x) \geq g(x) \in \mathbb{R}$. Since f is convex and g is lsc, we have that

$$
\begin{aligned}
g(\bar{x}) &\leq \liminf_{t\to 0^+} g(tx+(1-t)\bar{x}) \\
&\leq \liminf_{t\to 0^+} f(tx+(1-t)\bar{x}) \leq \liminf_{t\to 0^+} \{tf(x)+(1-t)f(\bar{x})\} = f(\bar{x}).
\end{aligned}
$$

Since $\bar{x}$ was arbitrarily chosen in $\operatorname{cl}(A)$, we conclude that $f \geq g$ on $\operatorname{cl}(A)$.

Let $K \subset \mathbb{R}^d$ be a nonempty convex set. We denote by $\sigma_K : \mathbb{R}^d \to \mathbb{R}$ the support function of K, that is, for any $x \in \mathbb{R}^d$ we have

$$\sigma_K(x) := \sup_{y\in K} \langle x, y\rangle.$$

Additionally, for $x \in K$, we denote by $N_K(x)$ the normal cone of K at x. It is well known that

$$N_K(x) = \{x^* \in \mathbb{R}^d \ : \ \sigma_K(x^*) \leq \langle x^*, x\rangle\}.$$

With this in mind, the following proposition establishes a density characterization for the subdifferential of convex functions.

Proposition 3.3 *Let $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a proper convex lsc function and let $(x, x^*) \in \operatorname{dom} f \times \mathbb{R}^d$. Assume there exists a dense subset D of* $\operatorname{ri}(\operatorname{dom} f)$ *such that*

$$\forall y \in D,\ \exists y^* \in \partial f(y) \text{ such that } \langle y^* - x^*, y - x\rangle \geq 0. \tag{3.1}$$

Then $x^* \in \partial f(x)$.

Proof Without loss of generality, we may assume $0 \in \operatorname{ri}(\operatorname{dom} f)$ and set $V = \operatorname{span}(\operatorname{dom} f)$. We consider two cases.

- *Case 1*: $V = \mathbb{R}^d$ and consequently $\operatorname{ri}(\operatorname{dom} f) = \operatorname{int}(\operatorname{dom} f)$

In this case, ∂f is locally bounded on $\operatorname{int}(\operatorname{dom} f)$ and upper semicontinuous (in the sense of set-valued maps). Therefore (3.1) entails that

$$\forall y \in \operatorname{int}(\operatorname{dom} f),\ \exists y^* \in \partial f(y) \text{ such that } \langle y^* - x^*, y - x\rangle \geq 0.$$

In particular, for every differentiability point y of f, it holds that $\langle \nabla f(y) - x^*, y - x\rangle \geq 0$. Moreover, since for every $y \in \mathrm{int}(\mathrm{dom}\, f)$ we have that $\partial f(y)$ can be recovered as the convex hull of limits of gradients of f (see, e.g., [32, Theorem 9.61]), we get that

$$\langle y^* - x^*, y - x\rangle \geq 0, \quad \forall (y, y^*) \in \mathcal{D},$$

where

$$\mathcal{D} = \mathrm{cl}(\{(z, z^*) : \ z \in \mathrm{int}(\mathrm{dom}\ f),\ z^* \in \partial f(z)\}).$$

Now, take $(\bar{y}, \bar{y}^*) \in \partial f \setminus \mathcal{D}$. Clearly, $\bar{y} \in \partial(\mathrm{dom}\ f)$ and $f(\bar{y}) \in \mathbb{R}$. Consider the convex body $K = \mathrm{cl}(\mathrm{dom} f)$ and set

$$K_n = \left(1 - \frac{1}{n}\right) K \quad \text{and} \quad f_n = f + I_{K_n}, \quad \forall n \in \mathbb{N},$$

where I_{K_n} is the indicator function of K_n, that is, $I_{K_n}(x) = 0$, if $x \in K_n$ and $+\infty$ elsewhere. Notice that $K_n \subset \mathrm{int}(K)$ and that by construction $f_n \xrightarrow{e} f$. Applying Theorem 1.4 (Attouch theorem), we deduce that $\Delta f_n \xrightarrow{PK} \Delta f$. Therefore, there exists a sequence $\{(z_n, z_n^*, f_n(z_n))\}_n$ converging to $(\bar{y}, \bar{y}^*, f(\bar{y}))$ such that $z_n \in K_n$ and $z_n^* \in \partial f_n(z_n)$. Since f and I_{K_n} are proper convex lsc functions, $z_n \in \mathrm{dom}\ f \cap \mathrm{dom}\ I_{K_n}$ and f is continuous at z_n (because $z_n \in \mathrm{int}(\mathrm{dom}\ f)$), we can apply the sum rule for subdifferentials given in [28, Theorem 3.16] to get: $y_n^* \in \partial f(z_n)$ and $v_n^* \in \partial I_{K_n}(z_n) \equiv N_{K_n}(z_n)$ such that $z_n^* = y_n^* + v_n^*$. In particular, $(z_n, y_n^*) \in \mathcal{D}$ and therefore $\langle y_n^* - x^*, z_n - x\rangle \geq 0$.

Notice further that $f_n(z_n) = f(z_n) \to f(\bar{y})$ and that $\sigma_{K_n} = (1 - \frac{1}{n})\sigma_K$. Therefore, for every $z \in K_n$ we have that

$$\begin{aligned} N_{K_n}(z) &= \left\{ v^* \in \mathbb{R}^d \ : \ \sigma_{K_n}(v^*) \leq \langle v^*, z\rangle \right\} \\ &= \left\{ v^* \in \mathbb{R}^d \ : \ \sigma_K(v^*) \leq \langle v^*, (1 - \tfrac{1}{n})^{-1} z\rangle \right\} = N_K\left((1 - \tfrac{1}{n})^{-1} z \right). \end{aligned}$$

In particular, since $v_n^* \in N_{K_n}(z_n) = N_K((\frac{n}{n-1})z_n)$, for every $x \in K$ we have

$$\langle v_n^*, x - (\tfrac{n}{n-1})z_n\rangle \leq 0$$

and combining with the definition of the subdifferential map given in (1.2), we deduce that

$$\begin{aligned} \langle z_n^* - x^*, z_n - x\rangle &= \langle y_n^* - x^*, z_n - x\rangle + \langle v_n^*, z_n - x\rangle \geq \langle v_n^*, z_n - x\rangle \\ &= \langle v_n^*, (\tfrac{n}{n-1}) z_n - x\rangle - \frac{1}{n-1}\langle v_n^*, z_n\rangle \geq -\frac{1}{n-1}\langle v_n^*, z_n\rangle \\ &= \frac{1}{n-1}(\langle y_n^*, z_n - 0\rangle - \langle z_n^*, z_n\rangle) \geq \frac{1}{n-1}\left(f(z_n) - f(0) - \langle z_n^*, z_n\rangle\right). \end{aligned}$$

Since $(z_n, z_n^*, f(z_n)) \to (\bar{y}, \bar{y}^*, f(\bar{y}))$, we can take limit at both sides of the obtained inequality, deducing that $\langle \bar{y}^* - x^*, \bar{y} - x\rangle \geq 0$. Since $(\bar{y}, \bar{y}^*)$ was arbitrarily chosen in ∂f, we obtain:

$$\langle y^* - x^*, y - x\rangle \geq 0, \quad \forall (y, y^*) \in \partial f,$$

and we conclude that $(x, x^*) \in \partial f$ by maximal monotonicity of the subdifferential (see, e.g., [28, Theorem 3.24]).

- *Case 2*: (general case)

Let $\pi_V : \mathbb{R}^d \to V$ be the orthogonal projection onto V and let us denote by $g : V \to \mathbb{R} \cup \{+\infty\}$ the restriction of f on V. Note that for every $z \in \text{dom}\, g$, every $z^* \in V$ and every $v^* \in V^\perp$, we have:

$$\begin{aligned} z^* \in \partial g(z) &\iff \forall z' \in V,\ \langle z^*, z' - z\rangle \leq g(z') - g(z) \\ &\iff \forall z' \in V,\ \langle z^* + v^*, z' - z\rangle \leq f(z') - f(z) \\ &\iff \forall z' \in \mathbb{R}^d,\ \langle z^* + v^*, z' - z\rangle \leq f(z') - f(z) \\ &\iff z^* + v^* \in \partial f(z), \end{aligned}$$

that is,

$$\partial f(z) = \pi_V^{-1}(\partial g(z)), \quad \forall z \in \text{dom}\ f.$$

Therefore, it is enough to verify that $\pi_V(x^*) \in \partial g(x)$. Note that, by projecting the subgradients of f onto V, (3.1) entails that

$$\forall y \in D, \exists y^* \in \partial g(y) \text{ such that } \langle y^* - \pi_V(x^*), y - x\rangle \geq 0,$$

where D is a dense subset of $\text{int}(\text{dom}\, g)$ and where the interior is taken with respect to the space V. Using the same reasoning as in Case 1 above, we conclude that $\pi_V(x^*) \in \partial g(x)$. The proof is complete. □

3.2 Two key lemmas

We now state and prove two important technical lemmas. The first one states that for a sequence of proper convex lsc functions $\{f_n\}_n$ from $\mathbb{R}^d$ to $\mathbb{R} \cup \{+\infty\}$, if a sequence of points $\{x_n\}_n$ converges and the sequence of slopes $\{s_{f_n}(x_n)\}_n$ is bounded, then the sequence $\{x_n\}_n$ automatically infimizes the expressions of f_u and f_l given in (1.11) evaluated at its point of convergence.

Lemma 3.4 *Let $f_n : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, $n \geq 1$, be a sequence of proper lsc convex functions. Let $\{x_n\}_n \subset \mathbb{R}^d$ be such that $\{s_{f_n}(x_n)\}_n$ is bounded. Assume that $\{x_n\}_n$ converges to some $\bar{x}$. Then*

$$f_l(\bar{x}) = \liminf_{n\to\infty} f_n(x_n) \quad \text{and} \quad f_u(\bar{x}) = \limsup_{n\to\infty} f_n(x_n).$$

Proof Let $\{y_n\}_n \subset \mathbb{R}^d$ be an arbitrary sequence converging to $\bar{x}$. Then

$$\liminf_{n\to\infty} f_n(y_n) \geq \liminf_{n\to\infty} \left\{ f_n(x_n) - s_{f_n}(x_n)\,\|x_n - y_n\| \right\} = \liminf_{n\to\infty} f_n(x_n).$$

It follows readily that $\liminf_{n\to\infty} f_n(x_n) = f_l(\bar{x})$.

Let further $\{k_n\}_n$ be a strictly increasing sequence such that

$$\limsup_{n\to\infty} f_n(x_n) = \lim_{n\to\infty} f_{k_n}(x_{k_n}).$$

Then for every sequence $\{y_n\}_n$ converging to $\bar{x}$ we have:

$$\begin{aligned}\limsup_{n\to\infty} f_n(y_n) &\geq \limsup_{n\to\infty} f_{k_n}(y_{k_n}) \geq \limsup_{n\to\infty} \left\{ f_{k_n}(x_{k_n}) - s_{f_{k_n}}(x_{k_n})\,\|x_{k_n} - y_{k_n}\| \right\} \\ &= \lim_{n\to\infty} f_{k_n}(x_{k_n}) = \limsup_{n\to\infty} f_n(x_n).\end{aligned}$$

Therefore we conclude that $\limsup_{n\to\infty} f_n(x_n) = f_u(\bar{x})$ and the proof is complete.

The previous result will be now used to establish our second important technical lemma:

Lemma 3.5 *Let $f, \{f_n\}_n : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be proper convex lsc functions such that the sequence of slope functions $\{s_{f_n}\}_{n\geq 1}$ epigraphically converges to s_f. Assume further that there exists a sequence $\{x_n\}_n \subset \mathbb{R}^d$, $\bar{x} \in \operatorname{dom} s_f$ and $\alpha \in \mathbb{R}$ such that*

$$\{s_{f_n}(x_n)\}_n \text{ is bounded} \quad \text{and} \quad \lim_{n\to\infty} (x_n, f_n(x_n)) = (\bar{x}, \alpha). \tag{3.2}$$

Then,

$$dom\, s_f \subset dom\, f_l \cap dom\, f_u \subset dom\, f_l \cup dom\, f_u \subset \operatorname{cl}(dom\, s_f) = \operatorname{cl} dom\, f).$$

Proof The equality $\operatorname{cl}(\operatorname{dom} s_f) = \operatorname{cl}(\operatorname{dom} f)$ follows easily from the fact that $\operatorname{dom} s_f = \operatorname{dom} \partial f$ is dense in $\operatorname{dom} f$ (see, e.g., [28, Theorem 3.17]). In addition, thanks to Lemma 3.4, we have $f_l(\bar{x}) = f_u(\bar{x}) = \alpha \in \mathbb{R}$.

Let $y \in \operatorname{dom} s_f$ and let us assume, towards a contradiction, that $f_l(y) = +\infty$. Then for any sequence $\{y_n\}_n \subset \mathbb{R}^d$ that converges to y we have $\liminf_{n\to\infty} f_n(y_n) = +\infty$. Note that the inequality

$$f_n(x) \geq f_n(y) - s_{f_n}(y)\,\|x - y\| \tag{3.3}$$

is valid for all $n \geq 1$ and all $x, y \in \operatorname{dom} f_n$. Take the sequence $x_n \to \bar{x}$ given by (3.2), which verifies that $\liminf_{n\to\infty} f_n(x_n) = f_l(\bar{x})$, and choose $\{y_n\}_n \subset \mathbb{R}^d$ such that $\big(y_n, s_{f_n}(y_n)\big) \to (y, s_f(y))$ (*c.f.* Remark 1.3). Then replacing x by x_n and y by y_n in (3.3) above, we easily deduce $f_l(\bar{x}) = +\infty$, which is a contradiction.

Therefore, $f_l(y) < +\infty$.

On the other hand, setting $\sigma = \sup_{n\to\infty} s_{f_n}(x_n) < +\infty$, we easily see that for any sequence $\{y_n\}_n$ converging to y we have:

$$\liminf_{n\to\infty} f_n(y_n) \geq \liminf_{n\to\infty} \left\{ f_n(x_n) - s_{f_n}(x_n)\, \|x_n - y_n\| \right\} \geq f(\bar{x}) - \sigma\, \|\bar{x} - y\| > -\infty,$$

yielding $f_l(y) \in \mathbb{R}$. Since y is an arbitrary vector in $\operatorname{dom} s_f$, we have that $\operatorname{dom} s_f \subset \operatorname{dom} f_l$.

Let us now show that $f_u(y) < +\infty$. Indeed, assuming the contrary, for any sequence $\{y_n\}_n \subset \mathbb{R}^d$ that converges to y, we would have $\limsup_{n\to\infty} f_n(y_n) = +\infty$. Evoking again Remark 1.3, we can take $\{y_n\}_n$ such that $\big(y_n, s_{f_n}(y_n)\big) \to (y, s_f(y))$. Therefore, for every $n \geq 1$ we would have:

$$f_n(x_n) \geq f_n(y_n) - s_{f_n}(y_n)\, \|x_n - y_n\|$$

and taking limsup at both sides of the above inequality, we would obtain $\alpha = f_u(\bar{x}) = +\infty$, which is a contraction. Therefore,recalling that $-\infty < f_l(y)$ and that $f_l \leq f_u$, we deduce the inclusion

$$\operatorname{dom} s_f \subset \operatorname{dom} f_l \cap \operatorname{dom} f_u.$$

Let us now show that $\operatorname{dom} f_l \cup \operatorname{dom} f_u \subset \operatorname{cl}(\operatorname{dom} s_f)$. To this end, let $y \notin \operatorname{cl}(\operatorname{dom} s_f)$ and let $\varepsilon > 0$ be such that $\overline{B}(y, \varepsilon) \subset \mathbb{R}^d \setminus \operatorname{cl}(\operatorname{dom} s_f)$.

We claim that $s_{f_n} \to \infty$ uniformly on $B(y, \varepsilon)$. Indeed, otherwise, there would exist $M > 0$, a strictly increasing sequence $\{k_n\}_n \subset \mathbb{N}$ and a sequence $\{z_n\}_n \subset B(y, \varepsilon)$ such that $s_{f_{k_n}}(z_{k_n}) < M$ for all $n \in \mathbb{N}$. It follows that $s_f(z) \leq M$ for any cluster point $z \in \overline{B}(y, \varepsilon)$ of $\{z_{k_n}\}_n$ leading to a contradiction.

We now set

$$M_n := \inf_{z \in B(y,\varepsilon)} s_{f_n}(z), \quad n \geq 1,$$

and observe that $M_n \to +\infty$ as $n \to \infty$. With this in mind, let us show that $f_l(y) = +\infty$.

We proceed by contradiction: assume that $f_l(y) \in \mathbb{R}$, that is, for some sequence $\{y_n\}_n \subset B(y, \varepsilon)$ converging to y we have $\liminf\limits_{n\to\infty} f_n(y_n) < \infty$. Then for n sufficiently large (say $n \geq N$), let $\gamma_n : [0, \infty) \to \mathbb{R}^d$ be the steepest descend curve of the convex function f_n starting at y_n, that is,

$$\dot{\gamma}_n \in -\partial f_n(\gamma_n) \qquad \text{and} \qquad \gamma_n(0) = y_n.$$

Let further $\{t_n\}_n \subset (0, \infty)$ be the *least escape-time* sequence defined by

$$t_n = \inf\{t > 0 : \gamma_n(t) \in \mathbb{R}^d \setminus B(y, \varepsilon)\}.$$

In other words, $t_n > 0$ is the first instant where the steepest descent curve γ_n escapes from the ball $B(y, \varepsilon)$. Thus, $\gamma_n(t_n) \in \partial B(y, \varepsilon)$ for all $n \geq N$. Since $\|\dot{\gamma}_n(\tau)\| =$

$s_{f_n}(\gamma_n(\tau))$ (*c.f.* [5, Theorem 17.2.2]) and since the length of the curve γ_n in $[0, t_n]$ is larger than the distance $\operatorname{dist}(y_n, \partial B(y, \varepsilon)) = \varepsilon - \|y_n - y\|$ of the initial point $\gamma_n(0) = y_n$ to the boundary, we can write

$$f_n(\gamma_n(t_n)) = f_n(y_n) - \int_0^{t_n} s_{f_n}(\gamma_n(\tau)) \, \|\dot{\gamma}_n(\tau)\| \, d\tau \le f_n(y_n) - (\varepsilon - \|y_n - y\|) \, M_n,$$

concluding that $\liminf_{n\to\infty} f_n(\gamma_n(t_n)) = -\infty$. However, convexity of f_n at $x_n \in \operatorname{dom} \partial f_n$ yields that for all $n \in \mathbb{N}$ we have:

$$f_n(\cdot) \ge g_n(\cdot) := f_n(x_n) - s_{f_n}(x_n) \, \| \cdot - x_n\|,$$

and consequently,

$$\begin{aligned} \liminf_{n\to\infty} f_n(\gamma_n(t_n)) &\ge \liminf_{n\to\infty} g_n(\gamma_n(t_n)) \\ &\ge \lim_{n\to\infty} f_n(x_n) - \limsup_{n\to\infty} \left\{ s_{f_n}(x_n) \, (\|x_n - y\| + \|y - \gamma_n(t_n)\|) \right\} \\ &\ge f(x) - \left(\sup_{n\in\mathbb{N}} s_{f_n}(x_n) \right) (\|x - y\| + \varepsilon) > -\infty, \end{aligned}$$

which is a contradiction.

Therefore, $\operatorname{dom} f_l \subset \operatorname{cl}(\operatorname{dom} s_f)$. Since $f_l \le f_u$, the proof is complete. □

4 From slope convergence to epigraphical convergence

In this section we establish the difficult part of our main result, which states that up to a normalization condition, slope epigraphical convergence yields epigraphical convergence of the functions. This will be done in two stages: in Sect. 4.1 we show that $f_u \le f$ while in Sect. 4.2 we will control the gap between f_u and f_l, then use (1.12) to deduce our result.

4.1 Domination of the upper epigraphical limit

We start with the following known proposition:

Proposition 4.1 *Let* $f_n : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, $n \in \mathbb{N}$, *be convex functions. Then*

$$f_u = \text{e}-\text{limsup}\, f_n$$

is a convex lsc function.

Proof It follows directly from [32, Theorem 7.4a & Theorem 7.17]. □

Let us also recall (Remark 1.3) that the function $f_l = \text{e}-\text{liminf}\, f_n$ is also lsc.

Lemma 4.2 *Let $f, \{f_n\}_n : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be proper convex lsc functions. Assume that $\{s_{f_n}\}_n$ epigraphically converges to s_f and that $f_u(\bar{x}) \in \mathbb{R}$ for some $\bar{x} \in \operatorname{dom} s_f$. Then*

$$s_{f_u} \le s_f.$$

Proof In view of Lemma 3.5 we have $\operatorname{dom} s_f \subset \operatorname{dom} f_u$. Let $y \in \operatorname{dom} s_f$ and let $\{y_n\}_n$ be such that

$$(y_n, s_{f_n}(y_n)) \to (y, s_f(y)).$$

The sequence $\{y_n^*\}_n := \{\partial^\circ f_n(y_n)\}_n$ is then bounded. By Lemma 3.4 we have that

$$f_u(y) = \limsup_{n\to\infty} f_n(y_n).$$

Passing to a subsequence, we may assume that for some $y^* \in \mathbb{R}^d$

$$\lim_{n\to\infty} (y_{k_n}, y_{k_n}^*, f_{k_n}(y_{k_n})) = (y, y^*, f_u(y)).$$

Since $s_{f_n}(y_n) \to s_f(y)$, it follows easily that $\|y^*\| = s_f(y)$. Furthermore, for any $z \in \mathbb{R}^d$ and any sequence $\{z_n\}_n$ converging to z we have

$$\begin{aligned}\limsup_{n\to\infty} f_n(z_n) &\ge \limsup_{n\to\infty} f_{k_n}(z_{k_n})\\ &\ge \limsup_{n\to\infty} \left\{ f_{k_n}(y_{k_n}) + \langle y_{k_n}^*, z_{k_n} - y_{k_n}\rangle \right\}\\ &= f_u(y) + \langle y^*, z - y\rangle.\end{aligned}$$

Since $\{z_n\}_n$ is an arbitrary sequence, we deduce $f_u(z) \ge f_u(y) + \langle y^*, z - y\rangle$. Since z is arbitrary, we obtain that $y^* \in \partial f_u(y)$. Thus, $s_{f_u}(y) \le \|y^*\| = s_f(y)$.

If $y \notin \operatorname{dom} s_f$, the inequality $s_{f_u}(y) \le s_f(y) \equiv +\infty$ is obvious. The proof is complete. □

The above lemma will be used in the following result.

Proposition 4.3 *Let $f, \{f_n\}_n : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be proper convex lsc functions. Assume that $\{s_{f_n}\}_n$ converges epigraphically to s_f and that $\inf f = \inf f_u \in \mathbb{R}$. Then*

$$f_u \le f.$$

Proof Let $\bar{x} \in \operatorname{dom} s_f$ and let us show that $f_u(\bar{x}) \in \mathbb{R}$. Indeed, one obviously has $f_u(\bar{x}) \ge \inf f_u > -\infty$. Reasoning towards a contradiction, let us assume that $f_u(\bar{x}) = +\infty$. Then for any sequence $\{x_n\}_n$ converging to $\bar{x}$, we have that $\limsup_{n\to\infty} f_n(x_n) = +\infty$. We may choose $\{x_n\}_n$ so that $(x_n, s_{f_n}(x_n)) \to (\bar{x}, s_f(\bar{x}))$, ensuring in particular that $\{s_{f_n}(x_n)\}_n$ is bounded. Then for every $y \in \mathbb{R}^d$ and every sequence $\{y_n\}_n \subset \mathbb{R}^d$ such that $y_n \to y$ we have:

$$\begin{aligned}\limsup_{n\to\infty} f_n(y) &\geq \limsup_{n\to\infty} \left\{ f_n(x_n) - s_{f_n}(x_n)\, \|x_n - y_n\| \right\} \\ &= \limsup_{n\to\infty} f_n(x_n) - s_f(\bar{x})\, \|\bar{x} - y\| = +\infty.\end{aligned}$$

This yields that $f_u \equiv +\infty$, which is a contradiction.

Therefore, $f_u(\bar{x}) \in \mathbb{R}$ and we can apply Lemma 4.2 to get that $s_{f_u} \leq s_f$. The conclusion follows from Theorem 1.2 (comparison principle). □

Forthcoming Lemma 4.5 provides a criterium for a limit of steepest descent curves (of convex functions converging epigraphically to a limit function) to be a steepest descent curve of the limit function. This is an intermediate result, which will be further refined in Lemma 4.7 and eventually lead to Proposition 4.8 (domination of f_u by f).

We shall first need the following result.

Proposition 4.4 *Let $\{\gamma_n\}_n$ be a sequence of Lipschitz curves from $[0, +\infty)$ to $\mathbb{R}^d$. Assume that the sequence $\{\gamma_n(0)\}_n$ is bounded and that all Lipschitz constants of the curves $\{\gamma_n\}_n$ are bounded by a constant $K > 0$, that is, $\mathrm{Lip}(\gamma_n) \leq K$ for all $n \geq 1$. Then, there exists an increasing sequence $\{k(n)\}_n$ such that $\{\gamma_{k(n)}\}_n$ converges uniformly on compact sets to a Lipschitz curve $\gamma : [0, +\infty) \to \mathbb{R}^d$ and the sequence of its tangents $\{\dot{\gamma}_{k(n)}|_{[0,T]}\}_n$ converges weakly to $\dot{\gamma}|_{[0,T]}$ in $\mathcal{L}^2([0, T]; \mathbb{R}^d)$, for any $T > 0$.*

Proof Let us first assume that the curves $\{\gamma_n\}_n$ are defined on $[0, 1]$. Since $\{\gamma_n(0)\}_n$ is relatively compact and $\mathrm{Lip}(\gamma_n) \leq K$ for all $n \in \mathbb{N}$, we can apply Arzelà–Ascoli theorem to get a subsequence $\{\gamma_{k(n)}\}_n$ which converges uniformly to some continuous curve γ on $[0, 1]$. It follows easily that γ is Lipschitz with $\mathrm{Lip}(\gamma_n) \leq K$. Since $\{\dot{\gamma}_{k(n)}\}_n$ is bounded on $\mathcal{L}^2([0, T]; \mathbb{R}^d)$ (in fact $\|\dot{\gamma}_{k(n)}\|_{\mathcal{L}^2} \leq K$), by the Eberlein–Šmulian theorem, there exists a subsequence $\{k'(k(n))\}_n$ which we denote by $\{\bar{k}(n)\}_n$ (that is, $\bar{k} = k' \circ k$), such that $\{\dot{\gamma}_{\bar{k}(n)}\}_n$ converges weakly to $\nu : [0, 1] \to \mathbb{R}^d$. Notice that, for each $n \geq 1$ we have that

$$\gamma_{\bar{k}(n)}(t) = \gamma_{\bar{k}(n)}(0) + \int_0^t \dot{\gamma}_{\bar{k}(n)}(s)ds, \text{ for all } t \in [0, 1].$$

Taking the limit as $n \to +\infty$, we obtain

$$\gamma(t) = \gamma(0) + \int_0^t \nu(s)ds, \text{ for all } t \in [0, 1].$$

Therefore, $\dot{\gamma}(t) = \nu(t)$, for all $t \in [0, 1]$ and the assertion follows.

The general case follows easily: if the curves $\{\gamma_n\}_n$ are defined on $[0, +\infty)$, we fix $T > 0$ and proceed as before for the restricted curves $\{\gamma_n|_{[0,T]}\}_n$. The result follows via a standard diagonal argument.

We are now ready to prove our lemma.

Lemma 4.5 *Let* $f, \{f_n\}_n : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ *be proper convex lsc functions such that*

$$s_{f_n} \xrightarrow{e} s_f \quad \textit{and} \quad \inf f > -\infty.$$

Assume that there is a sequence $(x_n, x_n^*, f_n(x_n)) \in \triangle f_n$, $n \geq 1$ *such that*

$$\lim_{n\to\infty} (x_n, x_n^*, f_n(x_n)) = (\bar{x}, \bar{x}^*, f(\bar{x})) \in \triangle f.$$

For every $n \geq 1$, *let* $\gamma_n : [0, +\infty) \to \mathbb{R}^d$ *denote the steepest descent curve of* f_n *starting from the point* $x_n \in \operatorname{dom} f_n$, *and let us assume that:*

(i). $\{\gamma_n\}_n$ *converges to some Lipschitz curve* ν *uniformly on compact sets; and*

(ii). *for all* $T > 0$, *the tangents* $\{\dot{\gamma}_n|_{[0,T]}\}_n$ *converge to* $\dot{\nu}|_{[0,T]}$ *weakly on* $\mathcal{L}^2([0, T], \mathbb{R}^d)$.

Then ν *is a steepest descent curve for the function* f_u.

Proof Thanks to Lemma 3.5, we know that $\operatorname{dom} s_f \subset \operatorname{dom} f_u \subset \operatorname{cl}(\operatorname{dom} s_f)$, which yields

$$\operatorname{ri}(\operatorname{dom} s_f) = \operatorname{ri}(\operatorname{dom} f_u).$$

Since γ_n is a steepest descent curve of f_n emanating from x_n, for every $t > 0$ and $n \in \mathbb{N}$ we have:

$$f_n(\gamma_n(t)) \leq f_n(x_n) \quad \text{and} \quad s_{f_n}(\gamma_n(t)) \leq s_{f_n}(x_n).$$

It follows easily from our hypothesis that the sequence $\{s_{f_n}(\gamma_n(t))\}_n$ is bounded. Since $\gamma_n(t) \to \nu(t)$, Lemma 3.4 entails that

$$f_u(\nu(t)) = \limsup_{n\to\infty} f_n(\gamma_n(t)) \leq \limsup_{n\to\infty} f_n(x_n) = f(\bar{x}) < +\infty.$$

Thus, for every $t > 0$, we have $\nu(t) \in \operatorname{dom} f_u$.

Let $y \in \operatorname{ri}(\operatorname{dom} f_u)$ and let $(y_n, y_n^*) \in \partial f_n$ be such that $\{y_n\}_n$ converges to y and the sequence $\{\|y_n^*\|\}_n = \{s_{f_n}(y_n)\}_n$ converges to $s_f(y)$. Passing to a subsequence $\{(y_{k_n}, y_{k_n}^*)\}_n$, we obtain

$$f_u(y) = \lim_{n\to\infty} f_{k_n}(y_{k_n}) \quad \text{and} \quad \lim_{n\to\infty} y_{k_n}^* = y^* \quad \text{for some } y^* \in \mathbb{R}^d.$$

Using the same argument as in the proof of Lemma 4.2 we deduce that $y^* \in \partial f_u(y)$. Then, for any bounded Borel set $A \subset [0, +\infty)$ and any $n \in \mathbb{N}$ we have

$$0 \leq \int_A \left\langle y_{k_n}^* + \dot{\gamma}_{k_n}(t),\ y_{k_n} - \gamma_{k_n}(t) \right\rangle dt.$$

Taking the limit as $n \to \infty$ we obtain (thanks to our assumption) that

$$0 \le \int_A \left\langle y^* + \dot{\nu}(t),\ y - \nu(t) \right\rangle dt.$$

Since $A \subset [0, +\infty)$ is an arbitrary bounded Borel set, we deduce that $\langle y^* + \dot{\nu}(t), y - \nu(t) \rangle \ge 0$ for a.e. $t \in [0, +\infty)$. Since $y \in \mathrm{ri}(\mathrm{dom}\ f_u)$ is arbitrary, we can take a sequence $\{(z_n, z_n^*)\}_n \subset \partial f_u$ such that $\|z_n^*\| = s_f(z_n)$, for all $n \in \mathbb{N}$, and $\{z_n\}_n$ is dense in $\mathrm{ri}(\mathrm{dom}\ f_u)$, obtaining

$$0 \le \langle z_n^* + \dot{\nu}(t), z_n - \nu(t) \rangle,\ \forall_{a.e.} t \in [0, +\infty),\ \forall n \in \mathbb{N}.$$

Thus, applying Proposition 3.3, we deduce that

$$\dot{\nu}(t) \in -\partial f_u(\nu(t)),\ \forall_{a.e.} t \in [0, +\infty).$$

The proof is complete. □

Before we proceed, let us recall an important technical result ensuring that absolutely continuous curves verifying an integrability condition for the slope of a convex function must be infimizing. This result is essentially known, but we include a proof for completeness, since the precise statement that we use below in not directly available in the literature. A strengthened version (which is also contained in the proposition below) can be obtained if the curve is a steepest descent curve of another function, see [27, Lemma 3.1]. A discretized version has been used in [33].

Proposition 4.6 (infimizing curves by integrability of slope) *Let $g : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a proper convex lsc function, and let $\gamma : [0, +\infty) \mapsto \mathbb{R}^d$ be an absolutely continuous curve. The following assertions hold:*

(i). If γ satisfies that

$$\liminf s_g(\gamma(t)) = 0 \qquad \text{and} \qquad \int_0^{+\infty} s_g(\gamma(t))\, \|\dot{\gamma}(t)\|\, dt < +\infty, \tag{4.1}$$

then

$$\liminf_{t \to +\infty} g(\gamma(t)) = \inf g.$$

(ii). Let f be another proper convex lsc function such that

$$s_f(x) \ge s_g(x), \quad \text{for all } x \in \mathbb{R}^d \qquad \text{and} \qquad \inf f > -\infty.$$

If $\gamma : [0, +\infty) \mapsto \mathbb{R}^d$ is the steepest descent curve for f starting at a point $\bar{x} \in \mathrm{dom}\, f$, then

$$\lim_{t \to +\infty} g(\gamma_{\bar{x}}(t)) = \inf\ g.$$

Moreover, if $f(\bar{x}) = g(\bar{x})$, *then* $\inf g \geq \inf f$.

Proof (i). Integrability of the slope implies that $\gamma(t) \in \operatorname{dom} \partial g$ for almost every $t \geq 0$. Thus, for almost every $t \geq 0$, we can define $h(t) = \partial^\circ f(\gamma(t))$ so that $\|h(t)\| = s_f(\gamma(t))$. Then via the standard subdifferential calculus (chain rule) for convex functions (see, e.g., [5, Proposition 17.2.5]) we deduce:

$$g(\gamma(t)) - g(\gamma(0)) = \int_0^t \langle h(\tau), \dot{\gamma}(\tau)\rangle d\tau \leq \int_0^{+\infty} s_g(\gamma(t))\|\dot{\gamma}(t)\|dt < +\infty.$$

Thus, $\liminf_{t\to+\infty} g(\gamma(t)) \leq \limsup_{t\to+\infty} g(\gamma(t)) < +\infty$.

Claim. There exists an increasing sequence $\{t_n\}_{n\in\mathbb{N}}$ such that

$$t_n \nearrow +\infty \qquad \text{and} \qquad s_g(\gamma(t_n)) = \min_{t\in[t_0,t_n]} s_g(\gamma(t)). \tag{4.2}$$

By assumption we have that $\liminf_{t\to+\infty} s_g(\gamma(t)) = 0$. If $s_g(\gamma(t)) = 0$ recurrently as $t \to +\infty$, then we choose $\{t_n\}_n$ to be any increasing sequence with $t_n \to +\infty$ and $s_g(\gamma(t_n)) = 0$ for every $n \in \mathbb{N}$. Otherwise, we fix $t_0 \geq \sup\{t \in [0, +\infty)\, s_g(\gamma(t)) = 0\} + 1$ and we define:

$$M_n := \operatorname*{argmin}_{t\in[t_0,t_0+n]} s_g(\gamma(\cdot)) \qquad \text{and} \qquad t_n = \max M_n.$$

Since s_g is lsc by convexity of g (see, e.g., [1]), the sequence $\{t_n\}_n$ is well defined and (4.2) holds.

Now, take any $v \in \operatorname{dom} g$ and $T_\varepsilon \geq t_0$ large enough such that

$$\int_{T_\varepsilon}^{+\infty} s_g(\gamma(t))\|\dot{\gamma}(t)\|dt \leq \varepsilon.$$

Using convexity, Cauchy-Schwarz inequality and (4.2) we deduce that for all $t_n > T_\varepsilon$ we have:

$$\begin{aligned} g(\gamma(t_n)) &\leq g(v) + \langle h(t_n), \gamma(t_n) - v\rangle \\ &\leq g(v) + |\langle h(t_n), v\rangle| + |\langle h(t_n), \gamma(t_\varepsilon)\rangle| + \int_{T_\varepsilon}^{t_n} |\langle h(t_n), \dot{\gamma}(s)\rangle| ds \\ &\leq g(v) + |\langle h(t_n), v\rangle| + |\langle h(t_n), \gamma(t_\varepsilon)\rangle| + \int_{T_\varepsilon}^{t_n} s_g(\gamma(t_n))\,\|\dot{\gamma}(s)\|ds \\ &\leq g(v) + s_g(\gamma(t_n))\|v\| + s_g(\gamma(t_n))\|\gamma(t_\varepsilon)\| + \int_{T_\varepsilon}^{t_n} s_g(\gamma(s))\|\dot{\gamma}(s)\|ds \\ &\xrightarrow{n\to\infty} g(v) + \int_{T_\varepsilon}^{+\infty} s_g(\gamma(s))\|\dot{\gamma}(s)\|ds \leq g(v) + \varepsilon. \end{aligned}$$

Thus, for every $\varepsilon > 0$ and every $v \in \operatorname{dom} g$, we have that

$$\liminf_{t\to+\infty} g(\gamma(t)) \le \liminf_{n\to\infty} g(\gamma(t_n)) \le g(v) + \varepsilon.$$

Thus, $\liminf_{t\to+\infty} g(\gamma(t)) = \inf g$, finishing the proof of this part.

(ii). The first conclusion of the second part is given by [27, Lemma 3.1] and the proof is very similar of the latter development, but using that $s_f(\gamma(t))$ is nonincreasing as $t \to +\infty$. For the last part, it is enough to write

$$\begin{aligned}
\inf g - g(\bar{x}) &= \liminf_{t\to\infty} \int_0^t \frac{d}{dt}[g\circ\gamma](\tau)\,d\tau = \liminf_{t\to\infty} \int_0^t \langle \partial^\circ g(\gamma(\tau)), \dot{\gamma}(\tau)\rangle\,d\tau \\
&\underbrace{\geq}_{Cauchy-Schwarz} -\limsup_{t\to\infty} \int_0^t s_g(\gamma(\tau))\|\dot{\gamma}(\tau)\|\,d\tau \underbrace{\geq}_{s_f\ge s_g} -\lim_{t\to\infty} \int_0^t s_f(\gamma(\tau))^2\,d\tau \\
&= \lim_{t\to\infty} \int_0^t \langle \partial^\circ f(\gamma(\tau)), \dot{\gamma}(\tau)\rangle\,d\tau = \lim_{t\to\infty} \int_0^t \frac{d}{dt}[f\circ\gamma](\tau))\,d\tau = \inf f - f(\bar{x}).
\end{aligned}$$

The result follows. □

We are now ready to obtain an enhanced version of Lemma 4.5.

Lemma 4.7 *Under the same assumptions as in Lemma 4.5 we conclude:*

$$\inf f_u = \inf f \qquad \text{and} \qquad f_u \le f.$$

Proof By hypothesis $\inf f > -\infty$. Moreover, by Lemma 3.4 we obtain $f_u(\bar{x}) = f(\bar{x})$, while by Lemma 4.5 (and following notation therein) the limit curve $\nu = \lim_{n\to\infty}\gamma_n$ is a steepest descent curve for (the convex function) f_u starting at $\bar{x} = \nu(0)$, that is,

$$\forall_{\text{a.e.}}\, t\in[0,+\infty):\ \dot{\nu}(t) = -\,\partial^\circ f_u(\nu(t)),\quad \|\dot{\nu}(t)\| = s_{f_u}(\nu(t)) \quad \text{and} \quad f_u(\nu(t)) \to \inf f_u. \tag{4.3}$$

Let us also recall from Lemma 4.2 that

$$s_f(x) \ge s_{f_u}(x), \qquad \text{for all } x \in \mathbb{R}^d. \tag{4.4}$$

By Proposition 4.6.(ii) it holds:

$$\inf f_u \ge \inf f > -\infty. \tag{4.5}$$

Let us set $M = \sup\{\|x_n^*\| : n \in \mathbb{N}\} = \sup\{s_{f_n}(x_n) : n \in \mathbb{N}\}$ and notice that for all $t \ge 0$ and $n \ge 1$ we have $s_{f_n}(\gamma_n(t)) \le s_{f_n}(x_n) \le M$. We deduce easily from Lemma 3.4 that

$$f_u(\nu(t)) = \limsup_{n\to+\infty} f_n(\gamma_n(t)), \quad \text{for all } t \ge 0.$$

By Fatou's Lemma and (4.4) we have

$$\begin{aligned} f_u(\nu(t)) &= \limsup_{n\to\infty} f_n(\gamma_n(t)) = \limsup_{n\to\infty} \left\{ f_n(\gamma_n(0)) - \int_0^t s_{f_n}(\gamma_n(\tau))^2\, d\tau \right\} \\ &= f(\bar{x}) - \liminf_{n\to\infty} \int_0^t s_{f_n}(\gamma_n(\tau))^2\, d\tau \leq f(\bar{x}) - \int_0^t \liminf_{n\to\infty} s_{f_n}(\gamma_n(\tau))^2\, d\tau \\ &\leq f(\bar{x}) - \int_0^t s_f(\nu(\tau))^2\, d\tau \leq f(\nu(t)). \end{aligned} \tag{4.6}$$

Therefore, we deduce:

$$\int_0^t s_f(\nu(\tau))^2\, d\tau \leq f(\bar{x}) - \inf f_u < +\infty, \qquad \text{for every } t \geq 0$$

and

$$\inf f_u = \inf (f_u \circ \nu) \leq \liminf_{t\to\infty} (f \circ \nu).$$

Applying Proposition 4.6.(i) to $g = f$ and $\gamma = \nu$, and noting that

$$\liminf s_{f_u}(\nu(t)) \leq \liminf s_f(\nu(t)) = 0,$$

we get that $\liminf_{t\to\infty} (f \circ \nu) = \inf f$. We conclude that

$$\inf f_u = \inf f \in \mathbb{R}.$$

The result follows by applying Proposition 4.3. □

We finish this subsection with the following proposition that, together with Proposition 4.3, provides a partial result towards our main theorem: If (ii) or (iii) of Theorem 1.6 hold, then $f_u \leq f$.

Proposition 4.8 *Let $f, \{f_n\}_n : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be proper convex lsc functions such that*

$$s_{f_n} \xrightarrow{e} s_f \quad \text{and} \quad \inf f > -\infty.$$

Assume that there is a sequence

$$(x_n, x_n^*, f_n(x_n)) \in \triangle f_n \quad \text{and} \quad \lim_{n\to\infty} (x_n, x_n^*, f_n(x_n)) = (\bar{x}, \bar{x}^*, f(\bar{x})) \in \triangle f.$$

Then

$$\inf f_u = \inf f \quad \text{and} \quad f_u \leq f.$$

Proof Let γ_n be the steepest descent curve of f_n starting at $x_n = \gamma_n(0)$. Set

$$M = \sup_{n\geq 1} \|x_n^*\|$$

so that

$$s_{f_n}(\gamma_n(t)) \leq s_{f_n}(\gamma_n(0)) = \|x_n^*\| \leq M \qquad \text{for all } t \geq 0 \text{ and } n \geq 1.$$

By a standard application of Arzelà-Ascoli theorem, for every strictly increasing sequence $\{k_1(n)\}_n$ there exists a subsequence $\{(k_2 \circ k_1)(n)\}_n$ that we simply denote by $\{k_n\}_n$ such that $\{\gamma_{k_n}\}_n$ uniformly converges to some Lipschitz curve on $[0, T]$, for every $T > 0$ (as in the statement of Lemma 4.5). Up to a new subsequence, which we keep denoting as before, $\{\gamma_{k_n}\}_n$ converges to a Lipschitz curve ν uniformly on bounded sets and $\{\dot{\gamma}_{k_n}|_{[0,T]}\}_n$ converges weakly to $\dot{\nu}|_{[0,T]}$ in $\mathcal{L}^2([0, T], \mathbb{R}^d)$. Let

$$f_{u,k_n} := \text{e}-\limsup f_{k_n}, \quad \text{for all } n \geq 1.$$

Thanks to Lemma 4.7, we have $f_{u,k_n} \leq f$. Since this holds true for any sequence $\{k_n\}_n$ such that $\{\gamma_{k_n}\}_n$ converges (as in the statement of Lemma 4.7), we can claim that $f_u \leq f$.

Indeed, for any $y \in \operatorname{dom} s_f$, there is a sequence $\{y_n\}_n$ such that $(y_n, s_{f_n}(y_n)) \to (y, s_f(y))$. By Lemma 3.4, there exists a subsequence $\{k_1(n)\}_n$ such that $f_u(x) = \lim_n f_{k_1(n)}(y_{k_1(n)})$. By Arzelà-Ascoli theorem, there exists a sub-subsequence $\{k_2(k_1(n)\}\}_n$ such that for $k = k_2 \circ k_1$ the sequence of the steepest descend curves $\{\gamma_{k_n}\}_n$ converges to a curve ν (as in the statement of Lemma 4.7) and we get $f_{u,k_n} \leq f$. Therefore, thanks to Lemma 3.4, we infer that

$$f_u(y) = f_{u,k_n}(y) \leq f(y).$$

Since y is an arbitrary vector in $\operatorname{dom} s_f$, we obtain $f_u \leq f$ on $\operatorname{dom} s_f$. Now, recalling Proposition 4.1, f and f_u are convex lsc functions and it is enough to apply Proposition 3.2 and Lemma 3.5 to conclude that $f_u \leq f$ on $\mathbb{R}^d$. □

4.2 Controlling the gap between upper and lower epigraphical limits

Let us first recall the following important result from [12, Lemma 2.4].

Proposition 4.9 *Let $f_n : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be convex lsc functions such that there exists a sequence $(x_n, x_n^*, f_n(x_n)) \in \triangle f_n$, $n \geq 1$, such that*

$$\lim_{n\to\infty} (x_n, x_n^*, f_n(x_n)) = (\bar{x}, \bar{x}^*, \alpha) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}.$$

Then,

$$\bar{x}^* \in \partial f_u(\bar{x}) \cap \partial f_l(\bar{x}) \qquad \textit{and} \qquad \alpha = f_l(\bar{x}) = f_u(\bar{x}).$$

Proof By Lemma 3.4, we have $f_l(\bar{x}) = f_u(\bar{x}) = \lim_n f_n(\bar{x})$. Then, for any $y \in \mathbb{R}^d$ and any sequence $\{y_n\}_n \subset \mathbb{R}^d$ converging to y, we have

$$f_n(y_n) \geq f_n(x_n) + \langle x_n^*, y_n - x_n \rangle, \qquad \text{for all } n \geq 1.$$

The desired conclusion follows by taking lim sup and lim inf to the above expression. □

The following result states that epigraphical convergence of the sequence of slope functions guarantees the local Lipschitz continuity of the lower epigraphical limit function f_l under a mild condition.

Proposition 4.10 *Let $f, \{f_n\}_n : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be proper convex lsc functions. Assume that $\{s_{f_n}\}_n$ epigraphically converges to s_f. Assume further that there is a sequence $\{x_n\}_n \subset \mathbb{R}^d$ converging to $\bar{x}$ such that $\{s_{f_n}(x_n)\}_n$ is bounded and $\{f_n(x_n)\}_n$ converges. Then, f_l is locally Lipschitz on* $\mathrm{ri}(\mathrm{dom}\ f_l)$.

Proof Since f is convex and lsc, we know by Proposition 3.1 that $\mathrm{ri}(\mathrm{dom}\, s_f)$ is a convex set. Thus, thanks to Lemma 3.5, we have that $\mathrm{ri}(\mathrm{dom}\ f_l) = \mathrm{ri}(\mathrm{dom}\, s_f)$. Let $y, z \in \mathrm{dom}\, s_f$ and let $\{y_n\}_n, \{z_n\}_n \subset \mathbb{R}^d$ be two sequences convergent to y and z, such that $\{s_{f_n}(y_n)\}_n$ and $(s_{f_n}(z_n))_n$ converge to $s_f(y)$ and $s_f(z)$, respectively. By Lemma 3.4, we get that $f_l(y) = \liminf_n f_n(y_n)$ and $f_l(z) = \liminf_n\ f_n(z_n)$. Take a subsequence $(k_n)_k$ such that $f_{k_n}(y_{k_n}) \to f_l(y)$. Then,

$$\begin{aligned} f_l(y) - f_l(z) &= \liminf_{n\to\infty}\ f_n(y_n) - \liminf_{n\to\infty}\ f_n(z_n) \geq \lim_{n\to\infty}\ f_{k_n}(y_{k_n}) - \liminf_{n\to\infty}\ f_{k_n}(z_{k_n}) \\ &\geq \liminf_{n\to\infty} \left(f_{k_n}(y_{k_n}) - f_{k_n}(z_{k_n})\right) \geq \liminf_{n\to\infty} \left\{-s_{f_{k_n}}(z_{k_n})\, \|y_{k_n} - z_{k_n}\|\right\} \\ &= -s_f(z)\, \|y - z\|. \end{aligned}$$

Since s_f is locally bounded on $\mathrm{ri}(\mathrm{dom}\, s_f) = \mathrm{ri}(\mathrm{dom}\ f_l)$, we get that f_l is locally Lipschitz on $\mathrm{ri}(\mathrm{dom}\ f_l)$. This finishes the proof. □

Remark 4.11 Observe that, under the same assumptions, the proof of Proposition 4.10 shows that that $s_{f_l} \leq s_f$ on $\mathrm{dom}\, s_f$.

We finish this subsection with the next proposition showing that, by taking a suitable subsequence, we can eliminate the gap between lower and upper epigraphical limits.

Proposition 4.12 *Let $f, \{f_n\}_n : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be proper convex lsc functions. Assume that $\{s_{f_n}\}_n$ epigraphically converges to s_f and that there exists some sequence $(x_n, x_n^*, f_n(x_n)) \in \triangle f_n$ that converges to $(\bar{x}, \bar{x}^*, \alpha) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$. Then,*

$$f_l(\bar{x}) = f_u(\bar{x}) = \alpha,$$

and for some strictly increasing sequence $\{k_n\}_n$ we have

$$f_{l,k_n} = f_{u,k_n}.$$

Proof Let us first observe that thanks to Proposition 4.9 we have $f_l(x) = f_u(x) = \alpha$. Let now

$$\mathcal{D} = \{z_i\}_{i=1}^{\infty} \subset \mathrm{ri}(\mathrm{dom}\, s_f)$$

be a dense countable set. For $i = 1$, let $\{z_{1,n}\}_n \longrightarrow z_1$ be such that

$$s_{f_n}(z_{1,n}) \longrightarrow s_f(z_1) \quad \text{and} \quad f_l(z_1) = \liminf_{n\to\infty} f_n(z_{1,n}).$$

Take a subsequence $\{k_1(n)\}_n$ such that:

$$f_l(z_1) = \lim_{n\to\infty} f_{k_1(n)}(z_{1,k_1(n)}) \quad \text{and} \quad \partial^\circ f_{k_1(n)}(z_{1,k_1(n)}) \longrightarrow \partial^\circ f(z_1) := z_1^*.$$

For $i = 2$, consider a sequence $\{z_{2,n}\}_n \longrightarrow z_2$ such that

$$s_{f_n}(z_{2,n}) \longrightarrow s_f(z_2).$$

Observe that since $s_{f_{k_1(n)}} \xrightarrow{e} s_f$, Lemma 3.5 applies and we deduce that $\mathrm{dom}\, s_f \subset \mathrm{dom}\, f_{l,k_1(n)}$. In particular, $f_{l,k_1(n)}(z_2) \in \mathbb{R}$.

Replacing $\{z_{2,n}\}_n$ by its subsequence $\{z_{2,k_1(n)}\}_n$ we still have $s_{f_{k_1(n)}}(z_{i,k_1(n)}) \longrightarrow s_f(z_i)$, $i \in \{1, 2\}$. Then taking a sub-subsequence $\{k_2(k_1(n))\}_n$ we can ensure that

$$\partial^\circ f_{(k_2 o k_1)(n)}(z_{2,(k_2 o k_1)(n)}) \longrightarrow \partial^\circ f(z_2) := z_2^* \text{ and } \lim_{n\to\infty} f_{(k_2 o k_1)(n)}(z_{2,(k_2 o k_1)(n)}) \text{ exists in } \mathbb{R}.$$

We set $\bar{k}_2 := k_2 \circ k_1$. Using induction, for every $m > 1$, we obtain a subsequence $\bar{k}_m = k_m \circ \ldots \circ k_1$ such that for all $i \in \{1, \ldots m\}$ we have:

$$\{z_{i,\bar{k}_i(n)}\}_n \longrightarrow z_i \qquad f_l(z_i) = \lim_{n\to\infty} f_{\bar{k}_{i-1}}(z_{i,\bar{k}_i(n)})$$
$$\text{and} \qquad \partial^\circ f_{\bar{k}_i(n)}(z_{1,\bar{k}_i(n)}) \longrightarrow \partial^\circ f(z_i) := z_i^*.$$

A standard diagonal argument ensures that for every $i \in \mathbb{N}$ the sequence $\{\bar{k}_n(n)\}_{n\geq i}$ is subsequence of $\{\bar{k}_i(n)\}_{n\geq i}$. Therefore, thanks to Lemma 3.4 and the construction, we obtain:

$$f_{l,\bar{k}_n(n)}(z_i) = \lim_{n\to\infty} f_{\bar{k}_n(n)}(z_{i,\bar{k}_n(n)}) = f_{u,\bar{k}_n(n)}(z_i), \quad \forall z_i \in \mathcal{D}.$$

Since $f_{u,n(k)}$ is convex and lsc, using Proposition 4.10 and Lemma 3.5 we deduce that

$$f_{l,\bar{k}_n(n)} = f_{u,\bar{k}_n(n)} \quad \text{on } \mathrm{ri}(\mathrm{dom}\, s_f).$$

Thanks to Proposition 4.9,

$$z_i^* \in \partial f_{u,\bar{k}_n(n)}(z_i) \cap \partial f_{l,\bar{k}_n(n)}(z_i).$$

Let us now define a function $L : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ by

$$L(z) := \sup_{i\in\mathbb{N}} \left\{ f_{u,\bar{k}_n(n)}(z_i) + \langle z_i^*, z - z_i \rangle \right\}, \quad \text{for all } z \in \mathbb{R}^d.$$

It is straightforward from the definition that L is a lsc convex function and $L \le \min \left\{ f_{u,\bar{k}_n(n)},\ f_{l,\bar{k}_n(n)} \right\}$ on the whole space. Notice further that $L = f_{u,\bar{k}_n(n)}$ on $\mathrm{ri}(\mathrm{dom}\, s_f)$. Then, since f_u is convex (by Proposition 4.1) and $\mathrm{ri}(\mathrm{dom}\, s_f)$ is a convex set (by Proposition 3.1), we can apply Proposition 3.2 to get $L = f_{u,\bar{k}_n(n)}$ on $\mathrm{cl}(\mathrm{dom}(s_f))$, yielding

$$f_{l,\bar{k}_n(n)} \ge f_{u,\bar{k}_n(n)} \quad \text{on } \mathrm{cl}(\mathrm{dom}(s_f)).$$

Finally, thanks to Lemma 3.5, we conclude that $f_{l,\bar{k}_n(n)} = f_{u,\bar{k}_n(n)}$ on $\mathbb{R}^d$. □

5 Main result, final comments and perspectives

We are now ready to establish the implications (ii)⇒(i) and (iii)⇒(i) of our main result (Theorem 1.6).

Theorem 5.1 *Let $f, \{f_n\}_n : \mathbb{R}^d \to \mathbb{R}\cup\{+\infty\}$ be proper convex lsc functions such that* $\inf f \in \mathbb{R}$. *Assume that $\{s_{f_n}\}_n$ epigraphically converges to s_f and for some sequence $(x_n, x_n^*, f_n(x_n)) \in \triangle f_n$, $n \ge 1$ we have:*

$$\lim_{n\to\infty} (x_n, x_n^*, f_n(x_n)) = (\bar{x}, \bar{x}^*, f(\bar{x})) \in \triangle f.$$

Then $f_n \xrightarrow{e} f$.

Proof We only need to show that $f \le f_l$ since Proposition 4.8 ensures that $f_u \le f$.
Let $y \in \mathrm{dom}\, f_l$. We claim that there exists a sequence $\{y_n\}_n \subset \mathbb{R}^d$ such that

$$s_{f_n}(y_n) \longrightarrow s_f(y) \qquad \text{and} \qquad f_l(y) = \liminf_{n\to\infty} f_n(y_n). \tag{5.1}$$

Indeed, we distinguish two cases:

- *Case 1*: $s_f(y) < +\infty$.

In this case, since $s_{f_n} \xrightarrow{e} s_f$, we can choose $\{y_n\}_n$ such that $s_f(y) = \lim_n s_{f_n}(y_n)$ and apply Lemma 3.4 to deduce that $f_l(y) = \liminf_n f_n(y_n)$.

- *Case 2*: $s_f(y) = +\infty$.

In this case, every sequence $\{y_n\}$ that converges to y should verify that $\lim_{n\to\infty} s_{f_n}(y_n) = +\infty$. Among these sequences, we chose one such that $f_l(y) = \liminf_{n\to\infty} f_n(y_n)$ (*c.f.* Remark 1.3). Therefore (5.1) holds and the claim is proved.

Let now $\{k_1(n)\}_n$ be a strictly increasing subsequence such that

$$f_l(y) = \liminf_{n\to\infty} f_n(y_n) = \lim_{n\to\infty} f_{k_1(n)}(y_{k_1(n)}).$$

Applying Proposition 4.12 to the sequence $\{f_{k_1(n)}\}_{n\geq 1}$ we get a subsequence $k = k_2 \circ k_1$ of $\{k_1(n)\}_n$, such that $f_{u,k_n} = f_{l,k_n} =: g$. Observe that f_{k_n} epi-converges to g and that g is proper convex lsc. Thus, Theorem 2.1 ensures that $\{s_{f_{k(n)}}\}_n$ epigraphically converges to s_g. Therefore $s_g = s_f$. Applying Proposition 4.8 to $\{f_{k_n}\}_n$ and f, we deduce that $\inf g = \inf f_{u,k_n} = \inf f \in \mathbb{R}$. Thus, we can apply [27, Corollary 3.1] (or apply twice Theorem 1.2) to deduce that $f = g$. In particular,

$$f(y) = g(y) = f_{l,k_n}(y) \leq \lim_{n\to\infty} f_{k_n}(y_{k_n}) = f_l(y).$$

Since y is arbitrary, we deduce $f \leq f_l$. The proof is complete. □

Theorem 5.2 *Let $f, \{f_n\}_n : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be proper convex lsc functions with* $\inf f \in \mathbb{R}$. *Assume that $\{s_{f_n}\}_n$ epigraphically converges to s_f and*

$$\inf f_u = \inf f = \inf f_l \in \mathbb{R}.$$

Then $\{f_n\}_n$ epigraphically converges to f.

Proof We follow the arguments of the proof of Theorem 5.1 with slight modifications. Indeed, it suffices to show $f \leq f_l$ since Proposition 4.3 ensures that $f_u \leq f$. To this end, we only need to show that $f(y) \leq f_l(y)$ for all $y \in \operatorname{dom} f_l$. Fix such $y \in \operatorname{dom} f_l$ and choose again a sequence $\{y_n\}_n \subset \mathbb{R}^d$ such that $s_{f_n}(y_n) \to s_f(y)$ and $f_l(y) = \liminf_n f_n(y_n)$. Take $\{k_n\}_n$ be an increasing subsequence such that

$$f_l(y) = \liminf_{n\to\infty} f_n(y_n) = \lim_{n\to\infty} f_{k_n}(y_{k_n}).$$

The main difference, with respect to the proof of Theorem 5.1, is that in order to apply Proposition 4.12 to $\{f_{k_n}\}_n$ we need to ensure the existence of a sequence $(x_{k_n}, x^*_{k_n}, f_{k_n}(x_{k_n})) \in \triangle f_{k_n}$ that converges (up to a subsequence) to some point (x, x^*, α). Since f is proper, there exists at least one point $x \in \operatorname{dom} s_f$ and sequence $\{x_m\}_m$ converging to x such that $s_{f_{k_m}}(x_{k_m})$ converges to $s_f(x)$. Take $x^*_m := \partial^\circ f_{k(m)}(x_m)$. Using Lemma 3.4, the hypothesis that $\inf f_l = \inf f$, and the fact that $f_u \leq f$, we can write

$$\begin{aligned}\inf f \leq f_l(x) \leq f_{l,k(m)}(x) &= \liminf_{m\to\infty} f_{k(m)}(x_m) \\ &\leq \limsup_{m\to\infty} f_{k(m)}(x_m) = f_{u,k(m)}(x) \leq f_u(x) \leq f(x).\end{aligned}$$

Thus, $\{f_{k(m)}(x_m)\}_m$ is a bounded sequence. We deduce that $\{(x_m, f_{k(m)}(x_m)), x^*_m)\}_m$ is also a bounded sequence, thus it converges, up to a second subsequence. Therefore, we can apply Proposition 4.12 to the sequence of functions $\{f_{k(m)}\}_m$. The rest of the proof follows exactly the lines of Theorem 5.1. □

Remark 5.3 Due to the fact that in our main result, Theorem 1.6, the limit function f is bounded from below, we can slightly generalize it by replacing (NC) with the following weaker condition:

$\widetilde{(NC)}$ There exist $x \in \operatorname{dom} \partial f$ and a sequence $\{x_n\}_n \subset \mathbb{R}^d$ such that:

$$\lim_{n\to+\infty} (x_n, f_n(x_n)) = (x, f(x)) \quad \text{and} \quad \{s_{f_n}(x_n)\}_n \quad \text{is bounded.}$$

Open problems: This work is motivated by the celebrated Attouch theorem (Theorem 1.4), the determination result of slopes [27], and the sensitivity result of [14]. All of these results are valid in Hilbert spaces, while the first two are also valid in Banach spaces (see [4, 12] and [33]). Therefore, a natural question is whether Theorem 1.6 (our main result) is true in Hilbert spaces, or more generally, in reflexive Banach spaces (or even in general Banach spaces). While there is no obvious obstruction for this extension, the present work relies heavily on local compactness of the space, for many of its intermediate results and consequently any potential extension should rather rely in a completely different approach.

In [7], a quantified version of Attouch theorem has been obtained. Indeed, the authors introduced the epi-distance topology and showed (*c.f.* [7, Theorem 4.2]) that in finite dimensional spaces, it corresponds to the topology of epi-convergence. Then in [7, Theorem 5.2], they showed how the (truncated) graph distance of the subdifferentials of two convex lsc functions is controlled by their (truncated) epi-distance. The mentioned result suggests that a quantified slope-version of Attouch theorem, that is, a slope generalization of [7, Theorem 5.2] is worth to be investigated.

A much more ambitious project would be to extend the result to pure metric spaces, without vector structure, according to the spirit of the determination results [15, 17]. One might focus on the notions of convexity that have been coined for metric spaces (see, e.g., [1]). This is a more challenging task, but the perspective of obtaining a metric version of Attouch theorem with its insight on variational deviations is tempting and should be explored in the future.

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