Oriented calmness and sweeping process dynamics

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Abstract. Daniilidis and Drusviatskiy, in 2017, extended the celebrated KŁ-inequality from definable functions $f : \mathbb{R}^n \to \mathbb{R}$ to definable multivalued maps $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$, by establishing that the co-derivative mapping D^*S admits a desingularization around every critical value. As was the case in the gradient dynamics, this desingularization yields a uniform control of the lengths of all bounded orbits of the corresponding sweeping process $-\dot{\gamma}(t) \in N_{S(t)}(\gamma(t))$. In this paper, working outside the framework of o-minimal geometry, we characterize the existence of a desingularization for the coderivative in terms of the behaviour of the sweeping process orbits and the integrability of the talweg function. These results are close in spirit with the ones in Bolte et al, 2010, where characterizations for the desingularization of the (sub)gradient of functions had been obtained.

Key words Sweeping process, KŁ-inequality, desingularization.

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1 Introduction

It is well-known that every $\mathcal{C}^1$ smooth function $f : \mathbb{R}^n \to \mathbb{R}$ which is definable in some o-minimal structure has finitely many critical values. Kurdyka [11] showed that if $\bar{r} \in f(\mathbb{R}^n)$ is a critical value and $\mathcal{U}$ is a nonempty open bounded subset of $\mathbb{R}^n$, then there exist $\rho > 0$ and a continuous function $\psi : [\bar{r}, \bar{r} + \rho] \to [0, +\infty)$ and $\mathcal{C}^1$-smooth on $(\bar{r}, \bar{r} + \rho)$ satisfying

$$\|\nabla(\psi \circ f)(x)\| \geq 1, \qquad \text{for all } \ x \in \mathcal{U} \text{ such that } \ f(x) \in (\bar{r}, \bar{r} + \rho). \tag{1}$$

The above inequality generalizes the Łojasiewicz gradient inequality (established in [15] for the class of $\mathcal{C}^1$-subanalytic functions) to the class of functions *definable in some o-minimal structure* and is nowadays known as the Kurdyka-Łojasiewicz inequality (in short, KŁ-inequality). We refer the reader to [20] for precise definitions and properties of o-minimal functions.

Both the Łojasiewicz and the KŁ-inequality have been further extended to nonsmooth (subanalytic and respectively o-minimal) functions, see [1, 2]. These inequalities allow to control uniformly the lengths of the bounded (sub)gradient orbits, see [16, 11, 1]. A straightforward consequence of (1) is that the length of every bounded gradient curve $\dot{\gamma} = -\nabla f(\gamma)$ contained in $f^{-1}((\bar{r}, \bar{r}+\rho)) \cap \mathcal{U}$ is majorized by $\psi(\bar{r}+\rho) - \psi(\bar{r})$ (and therefore it is bounded). The same is true for the lengths of the piecewise gradient curves, that is, curves obtained by concatenating countably many gradient curves $\{\gamma_i\}_{i\geq 1}$, where $\gamma_i \subset f^{-1}([r_{i+1}, r_i))$ and $\{r_i\}_i$ is a strictly decreasing sequence in $(\bar{r}, \bar{r} + \rho)$ converging to $\bar{r}$. These curves may have countably many discontinuities.

Outside the framework of o-minimality the KŁ-inequality (1) may fail even for $\mathcal{C}^2$-smooth functions [3, Section 4.3] or for $\mathcal{C}^\infty$-smooth function with a unique critical value [18, p. 12].

Bolte, Daniilidis, Ley and Mazet in [3] considered the problem of characterizing the existence of a desingularization function ψ and the validity of (1) for an upper isolated critical value $\bar{r}$ of a *semiconvex coercive* function f defined in a Hilbert space. We recall that a function f is called semiconvex, if for every $\bar{x} \in \mathbb{R}^n$ there exist $M > 0$ and $\delta > 0$ such that the function $x \mapsto f(x) + M\|x - \bar{x}\|^2$ is convex on $B(\bar{x}, \delta)$. We also recall that f is called coercive, if it has bounded sublevel sets. (This latter assumption replaces the use of an open bounded set $\mathcal{U}$ in Kurdyka's result.) We reproduce below one of the main results of the aforementioned work, see [3, Theorem 20], for the special case where the function is smooth and defined in finite dimensions.

Theorem 1 (characterization of the KŁ-inequality)**.** *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a $\mathcal{C}^2$-smooth (or more generally $\mathcal{C}^1$-smooth semi-convex) coercive function and $\bar{r} \in f(\mathbb{R}^n)$ an upper isolated critical value. The following statements are equivalent:*

a) ***(KŁ-inequality)*** *There exist $\rho > 0$ and a continuous function $\psi : [\bar{r}, \bar{r} + \rho) \to [0, \infty)$, which is $\mathcal{C}^1$-smooth on $(\bar{r}, \bar{r} + \rho)$, such that*

$$\|\nabla(\psi \circ f)(x)\| \geq 1, \ \textit{for all } x \in f^{-1}((\bar{r}, \bar{r} + \rho)).$$

b) ***(Length control for gradient curves)*** *There exist $\rho > 0$ and a strictly increasing continuous function $\sigma : [\bar{r}, \bar{r} + \rho) \to [0, \infty)$ with $\sigma(\bar{r}) = 0$ such that*

$$\int_0^T \|\dot{\gamma}(t)\| dt \leq \sigma(f(\gamma(0))) - \lim_{t \to T} \sigma(f(\gamma(t))), \qquad \textit{(we may have } T = +\infty\textit{)}$$

for all gradient curves $\gamma : [0, T) \to \mathbb{R}^n$ with $\gamma([0, T)) \subset f^{-1}((\bar{r}, \bar{r} + \rho))$.

c) ***(Length bound for piecewise gradient curves)*** *There exist* $\rho, M > 0$ *such that*

$$\int_0^T \|\dot{\gamma}(t)\| dt \leq M,$$

for all piecewise gradient curves $\gamma : [0, T) \to \mathbb{R}^n$ *with* $\gamma([0, T)) \subset f^{-1}((\bar{r}, \bar{r} + \rho))$.

d) ***(Integrability condition)*** *There exists* $\rho > 0$ *such that the talweg function*

$$r \mapsto \sup_{x \in f^{-1}(r)} \frac{1}{\|\nabla f(x)\|}, \qquad r \in (\bar{r}, \bar{r} + \rho),$$

is finite-valued and belongs to $\mathcal{L}^1(\overline{r}, \overline{r} + \rho)$.

Let us now introduce the following dynamical system, known as *sweeping process*, determined by a multivalued function $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$, with graph

$$\mathcal{S} = \operatorname{gph}(S) := \{(t, x) \in \mathbb{R}^{n+1} : \ x \in S(t)\}$$

and effective domain

$$\operatorname{dom}(S) := \{t \in \mathbb{R} : \ S(t) \neq \emptyset\}.$$

Definition 2 (sweeping process dynamics)**.** Let $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ be a multivalued map and $I \subset \operatorname{dom}(S)$ be a nonempty interval of $\mathbb{R}$. We say that the absolutely continuous curve $\gamma : I \to \mathbb{R}^n$ is a solution (orbit) of the sweeping process defined by S if

$$\begin{cases} -\dot{\gamma}(t) \in N_{S(t)}(\gamma(t)), & \forall_{a.e.}\ t \in I, \\ \ \ \gamma(t) \in S(t), & \forall t \in I, \end{cases} \tag{2}$$

where $N_{S(t)}(\gamma(t))$ stands for the *limiting normal cone* of $S(t)$ at $\gamma(t)$ (see definition in Section 2), and "$\forall_{a.e.}\ t \in I$" means *almost everywhere* with respect to the Lebesgue measure of $\mathbb{R}$.

The above definition implicitely implies that $\operatorname{dom}(S)$ has nonempty interior and is often an interval (possibly unbounded). In particular, in our seeting (*c.f* Assumptions in Section 3.1) $\operatorname{dom}(S)$ will always be an interval (possibly unbounded).

Daniilidis and Drusvyatskiy have recently shown in [9] that every multivalued map $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ with definable graph admits a desingularization of its graphical coderivative $D^*S : \mathbb{R}^n \rightrightarrows \mathbb{R}$ around any critical value $t \in \mathbb{R}$. (Relevant definitions and a more precise statement are given in Section 2.3.) This result yields a uniform bound for the lengths of all bounded orbits of the sweeping process defined by S (see forthcoming Definition 2). The aforementioned results of [9] are also covering the results of Kurdyka in [11] by considering a sweeping process mapping S related to the sublevel sets of the smooth definable function f (*c.f.* Remark 9).

The main contributions of this work are the following:

- Without assuming o-minimality, we characterize the desingularization of the coderivative of a *smooth sweeping process* (Definition 10) by establishing an analogous result to Theorem 1. This is the main result of this work, which is resumed in Section 3.2.

- Since the evolution of the sweeping process is not reversible in time, we introduce in Definition 3 an asymmetric version of the modulus for the coderivative of a multivalued map S, $\|D^*S(t,x)\|^{+,\uparrow}$, that captures the orientation of the dynamics. (In [9], the prevailing assumption of o-minimality made it possible to work directly with the usual modulus.)
- We establish an asymmetric version of [19, Theorem 9.40] (sometimes known as the Mordukhovich Criterion) relating the asymmetric modulus of the coderivative to the *oriented calmenss* of the multivalued map (Proposition 22). We then obtain Theorem B (Section 3.3) which relates the desingularization of the coderivative with the length of discrete sequences given by the catching–up algorithm. (This algorithm can be perceived as the proximal algorithm over a function f whenever the multivalued map S is defined by the sublevel sets of f as in Remark 9.)

The outline of this manuscript is as follows: In Section 2, we fix our notation, we quote preliminary results of variational analysis required in the sequel. In Section 3, we fix our setting, explain our assumptions and state the two main results of this paper (Theorem A and Theorem B). The proofs of these results together with other auxiliary results will be given in Section 4.

2 Notation and Preliminaries

The notation used along this paper is standard and follows the lines of [19]. For any two nonempty sets $A,\ B \subset \mathbb{R}^n$, the excess of A over B is given by $\mathrm{ex}\,(A,B) := \sup\{d(x,B) :\ x \in A\}$, where $d(x,B) := \inf_{y\in B}\ \|x-y\|$, and the Hausdorff-Pompeiu distance between A and B is defined by $\mathrm{dist}\,(A,B) := \max\{\mathrm{ex}\,(A,B),\ \mathrm{ex}\,(B,A)\}$. Let $C \subseteq \mathbb{R}^n$ be a closed set. The set of projections of $x \in \mathbb{R}^n$ at C is defined by $\mathrm{Proj}_C(x) := \{y \in C : \|x-y\| = d(x,C)\}$ and the Fréchet normal cone to C at $x \in C$, denoted by $\hat{N}_C(x)$, is the set of vectors $v \in \mathbb{R}^n$ satisfying

$$\limsup_{\substack{y\in C\\ y\to x}} \frac{\langle v, y-x\rangle}{\|y-x\|} \le 0.$$

The limiting normal cone to C at x, denoted by $N_C(x)$, consists of all vectors $v \in \mathbb{R}^n$ such that there exists a sequence $(x_i)_i \subset C$ and $v_i \in \hat{N}_C(x_i)$ satisfying $x_i \to x$ and $v_i \to v$.

2.1 Orbits of a sweeping process dynamics

The sweeping process dynamics (2) can be formally satisfied by curves with possible discontinuities (the set of discontinuities has then to be of measure zero). For our purposes it is useful to consider the class of *piecewise absolutely continuous* curves, that is, curves $\gamma : I \to \mathbb{R}^n$ whose possible discontinuities are contained in a closed countable set D and being absolutely continuous on each interval of $I \setminus D$. This latter set is open, therefore it is a countable union of disjoint intervals J_i, and γ is required to be absolutely continuous on each J_i.

Notation ($\mathcal{AC}(S,I)$, $\mathcal{PAC}(S,I)$). We denote by $\mathcal{AC}(S,I)$ (respectively $\mathcal{PAC}(S,I)$) the set of absolutely continuous (respectively, piecewise absolutely continuous) orbits of the sweeping process S defined on the interval $I \subset \mathrm{dom}(S)$. The length of a (piecewise) absolutely continuous curve $\gamma : I \to \mathbb{R}^n$ is given by the formula

$$\ell(\gamma) := \int_I \|\dot{\gamma}(t)\| dt.$$

2.2 Coderivative, (oriented) modulus and (oriented) talweg.

Let $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ be a multivalued map with closed values.

Definition 3 (Coderivative)**.** The (limiting) coderivative of S at $(t, x) \in \mathcal{S}$ in $u \in \mathbb{R}^n$ is defined as follows:

$$D^*S(t,x)(u) := \{a \in \mathbb{R} : \ (a, -u) \in N_{\mathcal{S}}(t,x)\}.$$

Therefore $D^*S(t, x) : \mathbb{R}^n \rightrightarrows \mathbb{R}$ is a multivalued map and

$$(u, a) \in \operatorname{gph} D^*S(t,x) \quad \text{if and only if} \quad (a, -u) \in N_{\mathcal{S}}(t,x).$$

Since $\operatorname{gph} D^*S(t, x)$ is a cone, the map $D^*S(t, x)$ is positively homogeneous and we can define its modulus via the formula:

$$\|D^*S(t,x)\|^+ := \sup_{\|u\|\le 1} \Big\{|a| : \ a \in D^*S(t,x)(u)\Big\}.$$

Although the above definition of a modulus is classical and relates nicely to the Lipschitz continuity of S (*c.f.* [19, Theorem 9.40]), the symmetry of the absolute value of $\mathbb{R}$ (representing the time in our dynamics) does not fit to the non-reversible dynamics of the sweeping process. To remedy this, one needs to replace $|a|$ in the above formula by $a^+ := \max\{0, a\}$ which eventually gives rise to the following definition.

Definition 4 (Asymmetric modulus of coderivative)**.** For every $(t, x) \in \mathcal{S}$ we define the asymmetric modulus of the coderivative $D^*S(t, x)$ as follows:

$$\|D^*S(t,x)|^{+,\uparrow} = \sup_{\|u\|\le 1} \Big\{a^+ : \ a \in D^*S(t,x)(u)\Big\},$$

where we adopt the convention $\sup(\emptyset) = 0$.

The following example gives some insight about the difference between the two moduli.

Example 5. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^1$-smooth function and set

$$\begin{aligned} S_1(r) &= [f \le r] := \{x \in \mathbb{R}^n : \ f(x) \le r\}, \\ S_2(r) &= [f = r] := \{x \in \mathbb{R}^n : \ f(x) = r\}, \\ S_3(r) &= [f \ge r] := \{x \in \mathbb{R}^n : \ f(x) \ge r\}, \qquad \text{for all } r \in \mathbb{R}. \end{aligned}$$

These three formulas define the multivalued maps $S_1, S_2, S_3 : \mathbb{R} \rightrightarrows \mathbb{R}^n$ associated to f, in the sense that the graph $\mathcal{S}_i$ of S_i is respectively the epigraph, graph and hypograph of f for $i = 1, 2$ and 3 (up to a permutation of coordinates that brings the first coordinate of $\mathbb{R}^{n+1}$ to the last position). Let $x \in \mathbb{R}^n$ and $r \in \mathbb{R}$. If $f(x) < r$, then $x \in \operatorname{int}(S_1(r))$ and $N_{\mathcal{S}_1}(r, x) = \{0\}$, yielding $\|D^*S_1(r, x)\|^+ = \|D^*S_1(r, x)|^{+,\uparrow} = 0$. Analogously, if $f(x) > r$, we have that $\|D^*S_3(r, x)\|^+ = \|D^*S_3(r, x)|^{+,\uparrow} = 0$. On the other hand, since the normal space of $\operatorname{gph}(f)$ at $(x, f(x))$ is exactly $\mathbb{R}(\nabla f(x), -1)$, if $f(x) = r$, then $N_{\mathcal{S}_1}(r, x) = \mathbb{R}_+(-1, \nabla f(x))$, $N_{\mathcal{S}_2}(r, x) = \mathbb{R}(-1, \nabla f(x))$ and $N_{\mathcal{S}_3}(r, x) = -\mathbb{R}_+(-1, \nabla f(x))$. Thus,

$$\|D^*S_2(r,x)|^{+,\uparrow} = \|D^*S_3(r,x)|^{+,\uparrow} = \|D^*S_1(r,x)\|^+ = \frac{1}{\|\nabla f(x)\|}, \qquad \text{but} \qquad \|D^*S_1(r,x)|^{+,\uparrow} = 0.$$

We now define the *oriented talweg* function associated to the multivalued map $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$. This captures the worst case (larger value of the oriented modulus of the coderivative) on each set $S(t)$, $t \in \mathbb{R}$. This function will play an important role in our main result.

Definition 6 (oriented talweg)**.** The oriented talweg function of S denoted by $\varphi^{\uparrow}$ is defined as follows:

$$\varphi^{\uparrow}(t) = \sup_{x \in S(t)} \{\|D^*S(t,x)|^{+,\uparrow}\}, \quad \text{for all } t \in \mathrm{dom}(S).$$

Remark 7 (Asymmetric structures)**.** In [9] the usual talweg function φ has been considered, based on the (symmetric) modulus of the coderivative.

$$\varphi(t) = \sup_{x \in S(t)} \{\|D^*S(t,x)\|^{+}\}, \quad \text{for all } t \in \mathrm{dom}(S).$$

The difference between φ and $\varphi^{\uparrow}$ is that the modula $\|D^*S(t,x)\|^{+}$, $(t,x) \in \mathcal{S}$, are now replaced by their asymmetric versions $\|D^*S(t,x)|^{+,\uparrow}$. The reader might notice that $a^{+} := \max\{0,a\}$ is a typical asymmetric norm of $\mathbb{R}$ and $\|D^*S(t,x)|^{+,\uparrow}$ can be seen as a natural asymmetrization of the modulus $\|D^*S(t,x)\|^{+}$. The use of asymmetric objects seems to be a natural tool in nonsmooth dynamics as well as in operations research (orientable graphs). More details on asymmetric structures can be found in [4] and [10].

2.3 Desingularization of the coderivative (definable case).

We now recall the main result of [9]. If $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ is a multivalued map with a closed *bounded* graph $\mathcal{S}$, then assuming that $\mathcal{S}$ is definable in some o-minimal structure, for every $a \in \mathbb{R}$, there exist $\rho > 0$ and a strictly increasing, continuous function $\Psi \colon [0,\rho] \to \mathbb{R}$ that is $\mathcal{C}^1$-smooth on $(0,\rho)$, it satisfies $\Psi(0) = a$ and $\Psi'(r) > 0$ for all $r \in (0,\rho)$ and

$$\|D^*(S \circ \Psi)(r,x)\|^{+} \leq 1 \qquad \text{for all } r \in (0,\rho) \text{ and all } x \in S(\Psi(r)). \tag{3}$$

It is easily seen that Ψ is a homeomorphism between $[0,\rho]$ and $[a,b]$ where $b = \Psi(\rho)$ and a diffeomorphism between $(0,\rho)$ and (a,b). Inequality (3) has a particular interest when $a \in \mathbb{R}$ is a *critical value* of the coderivative D^*S of the sweeping process, that is,

$$\varphi(t) = \sup_{x \in S(t)} \|D^*S(t,x)\|^{+} = +\infty.$$

In this case we say that Ψ *desingularizes* the (modulus of the coderivative around the) critical value a. The assumption of o-minimality on S guarantees that the set of critical values is finite. In [9] it has further been established, as consequence of (3), that all bounded orbits of the sweeping process S have finite length and that the talweg function φ is integrable on $[a,b]$.

Let us notice that $\|D^*S(t,x)|^{+,\uparrow} \leq \|D^*S(t,x)\|^{+}$ (and consequently $\varphi^{\uparrow}(t) \leq \varphi(t)$) for all $t \in [a,b)$ and $x \in S(t)$. Therefore, we obtain the following.

Corollary 8 (Desingularization of oriented coderivative – definable case)**.** *If $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ is a multivalued map with a closed definable bounded graph, then for every $a \in \mathbb{R}$ (possibly critical for the oriented modulus) there exists $\rho > 0$ and $b > a$ such that:*

(i). there exists an increasing homeomorphism $\Psi\colon [0,\rho] \to [a,b]$ *which is* $\mathcal{C}^1$*-diffeomorphism on* $(0,\rho)$ *such that:*

$$\|D^*(S\circ\Psi)(r,x)|^{+,\uparrow} \le 1 \qquad \text{for all } r\in(0,\rho) \text{ and all } x\in S(\Psi(r)). \tag{4}$$

(ii). $\int_a^b \varphi^{\uparrow}(t)\,dt < \infty$ *(the oriented talweg function is integrable).*

Remark 9. [Relation with the KŁ–inequality] (i). The described desingularization of the coderivative can be seen as a generalization of the KŁ-inequality for $\mathcal{C}^1$-smooth definable functions (established by Kurdyka in [11]) in the following sense: let $f:\mathbb{R}^n\to\mathbb{R}$ be a $\mathcal{C}^1$-smooth coercive function which is definable in some o-minimal structure. Then, the multivalued function

$$\begin{cases} S_f:\mathbb{R}\rightrightarrows\mathbb{R}^n \\ S_f(t)=[f\le -t], \quad t\in\mathbb{R} \end{cases} \tag{5}$$

is o-minimal (it is definable in the same o-minimal structure as f) and the desingularization of its gradient described in (1) can be deduced from the desingularization of the coderivative of S and vice versa. We refer the reader to [9, Section 5.1] for more details.

(ii). In [9], the assumption that $\mathcal{S}$ is bounded has not been made and the supremum of the definition of $\varphi(t)$ was taken over $S(t)\cap\mathcal{U}$, where $\mathcal{U}\subset\mathbb{R}^n$ is a fixed open bounded set, which formally gives rise to a talweg function which depends on $\mathcal{U}$ (see [9, Definition 4.2]). Even if in Section 3 we deal with potentially unbounded sweeping processes, we do not need to make use of $\mathcal{U}$, thanks to the assumptions given in Section 3.1.

3 Characterization of desingularization of the coderivative

In this paper we are interested in sweeping process mappings S whose graph is not an o-minimal set, assuming, instead, its smoothness (see forthcoming definition in Subsection 3.1). This allows to establish a natural relation between orbit velocities and the oriented modulus of coderivative at the graph of the orbit (Lemma 18) and eventully the existence of a desingularizing function Ψ for the asymmetric modulus of the coderivative (*c.f.* Corollary 8), keeping technicalities in an acceptable level (*c.f.* Subsection 4.1). We give below our setting. A more general setting (Whitney stratifiable graph) will be briefly discussed in Remark 12.

3.1 Assumptions, setting

Let $S:\mathbb{R}\rightrightarrows\mathbb{R}^n$ be a multivalued map with closed graph $\mathcal{S}$. We shall assume that S is *locally bounded*, that is, for every $\bar t\in\mathbb{R}$ there exists $\delta>0$ and $M>0$ such that for all $t\in(\bar t-\delta,\bar t+\delta)$ and $x\in S(t)$ it holds $\|x\|\le M$.

We are ready to give the following definition.

Definition 10 (smooth sweeping process)**.** We say that S is a *smooth* sweeping process if either

– $\mathcal{S}$ is a closed connected $\mathcal{C}^1$-smooth submanifold of $\mathbb{R}^{n+1}$ of dimension at most n ; or

– $\mathcal{S}$ is a connected smooth manifold of full dimension with boundary $\partial\mathcal{S}$ and $\partial\mathcal{S}$ is a $\mathcal{C}^1$-smooth manifold of dimension n.

The above assumption is satisfied if S is a sweeping process associated to a $\mathcal{C}^1$-smooth function f (*c.f.* Example 5 or Remark 9). As a consequence of this assumption we have the following result, which compares the modulus of D^*S versus its asymmetric modulus.

Lemma 11. *Let $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ be a smooth sweeping process and $(t, x) \in \mathcal{S}$. If either*

$$(a)\ \mathcal{S} \text{ is a smooth manifold} \qquad or \qquad (b)\ \|D^*S(t,x)|^{+,\uparrow} > 0$$

then we have

$$\|D^*S(t,x)|^{+,\uparrow} = \|D^*S(t,x)\|^{+}.$$

Proof. If $\mathcal{S}$ is a smooth submanifold of $\mathbb{R}^{n+1}$, the requested equality holds true for every $(t, x) \in \mathcal{S}$ as a consequence of the fact that the limiting normal cone at any point coincides with the normal space of the manifold at the same point, which is a linear space. On the other hand, if $\mathcal{S}$ is a manifold of full dimension with boundary such that $\partial\mathcal{S}$ is also a smooth manifold, then the normal cone $N_{\mathcal{S}}(t, x)$ is either $\{0\}$ (in the interior of $\mathcal{S}$) or a ray generated by an outer pointing normal vector of $\mathcal{S}$ at (t, x). The conclusion follows. □

Connectedness of $\mathcal{S}$ yields that $\mathrm{dom}(S)$ is an interval (possibly unbounded). We shall use the following notation:

$$\mathrm{T} = \sup(\mathrm{dom}(S)). \quad (\text{Notice that } \mathrm{T} \in \mathbb{R} \cup \{+\infty\})$$

Assumptions. *We say that S satisfies the:*

$(A1)$ existence assumption *if for every $(t, x) \in \mathcal{S}$ with $\|D^*S(t,x)|^{+,\uparrow} < +\infty$, there exist $\delta_x > 0$ and at least one orbit $\gamma_x \in AC(S; [t, t + \delta_x))$ such that $\gamma_x(t) = x$.*

$(A2)$ upper regular assumption *at $\bar{t} \in \mathrm{dom}(S)$ with $\bar{t} < \mathrm{T}$, if there exists $\delta > 0$ such that $\varphi^{\uparrow}(t) < +\infty$ for all $t \in (\bar{t}, \bar{t} + \delta)$.*

Let us make some comments about the above assumptions:

Assumption (A1) ensures the existence of orbits issued from any non-critical point. This assumption is satisfied if the sweeping process is defined via (5) where f is a $\mathcal{C}^{1,1}$-smooth function, since in this case the existence of gradient orbits $\dot{\gamma} = -\nabla f(\gamma)$ is guaranteed, and these orbits are also orbits for the sweeping process S_f up to a suitable reparametrization, see Remark 9. Assumption (A1) is also fulfilled if S is a definable sweeping process, see [9, Section 6] or [12]. In the general case, classical existence results go back to the seminal work of J.J. Moreau [17] for convex-valued multifunctions which are Lipschitz continuous under the Hausdorff-Pompieu metric. Since then, several extensions have been obtained, see [5, 6, 14] and references therein.

Assumption (A2) is readily satisfied in the definable case, since the set of critical values is then finite. In the general case, this assumption is analogous to the hypothesis made in [3, Section 3.3] that the critical values of f are upper isolated (*c.f.* statement of Theorem 1).

Assumption (A2) will not appear explicitely in our main results. It is only required to define forthcoming set $\mathcal{T}$, see (6), which is of interest for our analysis.

3.2 Theorem A (characterizations via continuous dynamics)

We set:

$$\mathcal{T} := \{t \in \operatorname{dom}(S) : \text{(A2) is fulfilled at } t\}. \tag{6}$$

Observe that if $t \in \mathcal{T}$, then there is $\delta > 0$ such that $[t, t+\delta) \subset \mathcal{T}$.

We are now ready to state the main result of this work. The proof will be given in Section 4.2.

Theorem A. *Let $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ be a locally bounded smooth sweeping process that satisfies (A1). Let $a \in \mathcal{T}$ (typically a critical value for D^*S).*

The following assertions are equivalent:

*a) **(Desingularization of the coderivative)** There exist $b > a$, $\rho > 0$ and a homeomorphism $\Psi : [0, \rho] \to [a, b]$, which is a $\mathcal{C}^1$-diffeomorphism between $(0, \rho)$ and (a, b) with $\Psi'(r) > 0$ for every $r \in (0, \rho)$, such that:*

$$\|D^*(S \circ \Psi)(r, x)|^{+,\uparrow} \leq 1, \quad \text{for all } r \in (0, \rho), \text{ for all } x \in S(\Psi(r)).$$

*b) **(Uniform length control for the absolutely continuous orbits)** There exist $b > a$ and an increasing continuous function $\sigma : [a, b] \mapsto \mathbb{R}_+$ with $\sigma(a) = 0$ such that for every $a \leq t_1 < t_2 \leq b$ and $\gamma \in \mathcal{AC}(S, [t_1, t_2])$ we have:*

$$\ell(\gamma) \leq \sigma(t_2) - \sigma(t_1).$$

*c) **(Length bound for the piecewise absolutely continuous orbits)** There exist $b > a$ and $M > 0$ such that for every $\gamma \in \mathcal{PAC}(S, [a, b])$ we have:*

$$\ell(\gamma) \leq M.$$

*d) **(Integrability of the talweg)** There exists $b > a$ such that*

$$\int_a^b \varphi^{\uparrow}(t)dt < \infty.$$

Remark 12 (pertinence of smoothness assumption)**.** It is possible to replace the smoothness assumption (*c.f.* Definition 10) in Theorem A by the (weaker) assumption that $\mathcal{S}$ admits a *Whitney-(a) stratification* (see [2] *e.g.*), provided we reinforce (A1) as follows:

(A1)$'$ for every $(t, x) \in \mathcal{S}$ with $\|D^*S(t, x)|^{+,\uparrow} < +\infty$ there exists an orbit $\gamma_x \in AC(S; [t, t+\delta_x))$ with $\gamma_x(t) = x$ *and* the map $s \to \|D^*S(s, \gamma_x(s))|^{*,\uparrow}$ is right continuous at t.

The above technical assumption is needed to ensure the implication (c)⇒(d) of Theorem A (in our current setting, this is overcome thanks to the smoothness assumption).

This being said, the overall analysis in the new setting becomes much more technical, in particular in what concerns the results of Lemma 14, Lemma 15 and Lemma 18.

We will not pursue further this direction in this work.

3.3 Theorem B (characterizations via discrete dynamics)

We first need the following definition.

Definition 13 (piecewise catching-up sequence)**.** Let $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ be a multivalued map with closed values.

(i). A (finite or infinite) sequence $\{(t_i, x_i)\}_{i\geq 0} \subset \mathcal{S}$ is called a *catching-up sequence* for S if $\{t_i\}_{i\geq 0}$ is strictly increasing and

$$x_{i+1} \in \mathrm{Proj}_{S(t_{i+1})}(x_i), \quad \text{for } i \geq 0.$$

(ii). A (finite or infinite) sequence of the form

$$(t_0^0, Y_0^0), (t_1^0, Y_1^0), \ldots, (t_{k_0}^0, Y_{k_0}^0), (t_0^1, Y_0^1), (t_1^1, Y_1^1), \ldots, (t_{k_1}^1, Y_{k_1}^1), \ldots$$

is called a *piecewise catching-up sequence* for S if for every $j \geq 0$

$$\{(t_i^j, Y_i^j)\}_{i=0}^{k_j} \subset \mathcal{S} \ \text{ is a catching-up sequence for } S \ \text{ and } t_{k_j}^j = t_0^{j+1}.$$

Now we are ready to state our second result which complements Theorem A by adding two additional equivalences related to the behavior of (piecewise) catching-up sequences.

Theorem B. *Let $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ be a locally bounded smooth sweeping process that satisfies (A1) and let $a \in \mathcal{T}$. The statements (a)–(d) of Theorem A are also equivalent to the following:*

e) ***(Uniform control of catching-up sequences)*** *There exist $b > a$ and a continuous increasing function $\sigma : [a, b) \to [0, \infty)$, with $\sigma(a) = 0$, such that for every catching-up sequence $\{(t_i, x_i)\}_{i\geq 0} \subset \mathcal{S}$ with $\{t_i\}_{i\geq 0} \subset (a, b)$, and every $k \geq 1$ we have*

$$\sum_{i=0}^{k} \|x_{i+1} - x_i\| \leq \sigma(t_k) - \sigma(t_0). \tag{7}$$

f) ***(Length bound for piecewise catching-up sequences)*** *There exist $b > a$ and $C > 0$ such that for any piecewise catching-up sequence*

$$\left\{(t_i^j, Y_i^j) : \ j \geq 0, \ i \in \{0, \ldots, k_j\}\right\}$$

with

$$a < t_0^0 < t_1^0 < \ldots < t_{k_0}^0 = t_0^1 < t_1^1 < \ldots < b$$

we have:

$$\sum_{j\geq 0} \sum_{i=0}^{k_j} \|Y_{i+1}^j - Y_i^j\| \leq C.$$

4 Proofs

We now prove our main results, Theorem A (Subsection 4.2) and Theorem B (Subsection 4.4). To this end, we shall need some auxiliary results (Subsection 4.1) and a new notion of *oriented calmness* (Subsection 4.3).

4.1 Auxiliary results

The first result concerns continuity of the moduli maps. It is based on the fact that the normal space mapping of the smooth manifold is continuous (in the Grasmannian metric). The details are left to the reader.

Lemma 14 (continuity of the (oriented) modulus on $\partial\mathcal{S}$)**.** *Let $S:\mathbb{R}\rightrightarrows\mathbb{R}^n$ be a smooth sweeping process. Then, the functions*

$$(t,x)\mapsto \|D^*S(t,x)|^{+,\uparrow} \qquad \text{and} \qquad (t,x)\mapsto \|D^*S(t,x)\|^{+}$$

are continuous on $\partial\mathcal{S}$ for the usual topology on $\mathbb{R}\cup\{+\infty\}$.

The second result asserts continuity of the (oriented) talweg function.

Lemma 15. *Let $S:\mathbb{R}\rightrightarrows\mathbb{R}^n$ be a locally bounded smooth sweeping process. Assume that $\varphi^{\uparrow}$ is finite on (a,b). Then the talweg function $\varphi^{\uparrow}$ is continuous on (a,b).*

Proof. Since S is locally bounded and has a closed graph, for any compact interval $I\subset(a,b)$, $S(I)$ is a compact subset of $\mathbb{R}^n$. Since $\varphi^{\uparrow}$ is finite on (a,b), the element $(1,0)\in\mathbb{R}\times\mathbb{R}^n$ does not belong to $N_{\mathcal{S}}(t,x)$ for all $t\in(a,b)$ and $x\in S(t)$. Now, the continuity of the normal space mapping on $\partial\mathcal{S}$ with respect to the Grassmanian metric (or of the outer normal vector mapping if $\mathrm{int}(\mathcal{S})\neq\emptyset$) and the fact that for any $t\in\mathrm{dom}(S)$, there is $x\in S(t)$ and $u\in\mathbb{R}^d$, with $\|u\|=1$, such that $(\varphi^{\uparrow}(t),u)\in N_{\mathcal{S}}(t,x)$ readily imply the continuity of $\varphi^{\uparrow}$. □

Proposition 16 (diffeomorphic rescaling of time)**.** *Let $S:\mathbb{R}\rightrightarrows\mathbb{R}^n$ be a multivalued map and $\gamma\in\mathcal{AC}(S,(a,b))$. If $\Psi:(0,\rho)\to(a,b)$ is a $\mathcal{C}^1$-smooth diffeomorphism such that $\Psi'(r)>0$ for all $r\in(0,\rho)$, then $\widetilde{\gamma}=\gamma\circ\Psi$ is an orbit of the sweeping process defined by $\widetilde{S}:=S\circ\Psi$, that is, $\widetilde{\gamma}\in\mathcal{AC}(\widetilde{S},(0,\rho))$.*

Proof. It is straighforward that $\widetilde{\gamma}:=\gamma\circ\Psi$ is an absolutely continuous curve. Since Ψ is a bi-Lipschitz homeomorphism on each compact interval contained in $(0,\rho)$ we deduce that for any null subset A of (a,b) the set $\Psi^{-1}(A)$ is also null (with respect to the Lebesgue measure). If $\mathcal{I}$ be the points of differentiability of γ for which (2) holds, it follows that $\mathcal{J}:=\Psi^{-1}((a,b)\setminus\mathcal{I})$ is a null set and for every $r\in(0,\rho)\setminus\mathcal{J}$ it holds:

$$\widetilde{\gamma}'(r)=(\gamma\circ\Psi)'(r)=\gamma'(\Psi(r))\Psi'(r)\in N_{S(\Psi(r))}(\gamma(\Psi(r))),$$

yielding that $\widetilde{\gamma}$ is an orbit solution of the sweeping process defined by $\widetilde{S}:=S\circ\Psi$. □

In the sequel, given a curve $\gamma:I\to\mathbb{R}^n$ we define its lifting $\zeta:I\to\mathbb{R}^{n+1}$ by

$$\zeta(t):=(t,\gamma(t)),\qquad t\in I.$$

Proposition 17 (geometric facts)**.** *Let $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ be a smooth sweeping process. Fix $\bar{t} \in \operatorname{dom}(S)$, $t \neq \mathrm{T}$ and $\bar{x} \in S(\bar{t})$. Then:*

a) If there is $\delta > 0$ such that $\bar{x} \in S(t)$, for all $t \in (\bar{t}, \bar{t}+\delta)$, then $\alpha \leq 0$ for all $(\alpha, u) \in N_{\mathcal{S}}(\bar{t}, \bar{x})$.

b) If $\|D^*S(\bar{t}, \bar{x})|^{+,\uparrow} > 0$, then for any $\tau > \bar{t}$ and $\gamma \in \mathcal{AC}(S, [\bar{t}, \tau))$ with $\gamma(\bar{t}) = \bar{x}$, there exists $\delta > 0$ such that

$$\zeta(t) := (t, \gamma(t)) \in \partial\mathcal{S}, \qquad \text{for all } t \in [\bar{t}, \bar{t} + \delta).$$

c) If $\operatorname{int}(\mathcal{S})$ is nonempty and $N_{\mathcal{S}}(\bar{t}, \bar{x}) = \mathbb{R}_+(\alpha, u)$ with $\alpha < 0$, then there is $\delta > 0$ such that $\bar{x} \in S(t)$ for all $t \in [\bar{t}, \bar{t} + \delta)$.

Proof. (*a*). If $(\bar{t}, \bar{x}) \in \operatorname{int}(\mathcal{S})$ then $N_{\mathcal{S}}(\bar{t}, \bar{x}) = \{(0, 0)\}$ and the conclusion follows trivially. In the case when $(\bar{t}, \bar{x}) \in \partial\mathcal{S}$, since $\partial\mathcal{S}$ is a smooth manifold, the limiting normal cone $N_{\mathcal{S}}(\bar{t}, \bar{x})$ is equal to the Fréchet normal cone and is contained in the normal space of $\partial\mathcal{S}$ at $(\bar{t}, \bar{x})$. Therefore, for any $(\alpha, u) \in N_{\mathcal{S}}(\bar{t}, \bar{x})$ and $t \in (\bar{t}, \bar{t} + \delta)$, we have $(t, \bar{x}) \in \mathcal{S}$ and

$$\limsup_{t \searrow \bar{t}} \frac{\langle (\alpha, u), (t - \bar{t}, \bar{x} - \bar{x}) \rangle}{\|(t - \bar{t}, \bar{x} - \bar{x})\|} = \alpha \leq 0.$$

(*b*). Let $\tau > \bar{t}$ and $\gamma \in \mathcal{AC}(S, [\bar{t}, \tau))$ with $\gamma(\bar{t}) = \bar{x}$ and assume $\|D^*S(\bar{t}, \bar{x})|^{+,\uparrow} > 0$. Since $(t, y) \mapsto \|D^*S(t, y)|^{+,\uparrow}$ is continuous on $\partial\mathcal{S}$ (Lemma 14), there exists a neighborhood $\mathcal{V}$ of $(\bar{t}, \bar{x})$ such that for all $(t, y) \in \mathcal{V} \cap \partial\mathcal{S}$ we have $\|D^*S(t, y)|^{+,\uparrow} > 0$. Therefore, there is $\delta > 0$ such that $\|D^*S(\zeta(t))|^{+,\uparrow} > 0$ and consequently, $\zeta(t) \in \partial\mathcal{S}$ for all $t \in [\bar{t}, \bar{t} + \delta)$.

(*c*). It follows from our assumption that $\dim(\partial\mathcal{S}) = n$ and (α, u) is a nonzero outer normal vector of $\mathcal{S}$ at $(\bar{t}, \bar{x})$. Without loss of generality, let us assume that (α, u) is a unit vector. Since $\operatorname{int}(\mathcal{S}) \neq \emptyset$, we deduce that $(\bar{t}, \bar{x}) - \lambda(\alpha, u) \in \mathcal{S}$ for all $\lambda > 0$ sufficiently small. Let us assume, reasoning towards a contradiction, that there exists a decreasing sequence $\{t_k\}_k \subset \mathbb{R}$ converging to $\bar{t}$ such that $\bar{x} \notin S(t_k)$, for all $k \in \mathbb{N}$. Let us now take a decreasing sequence $\{\lambda_k\}_k \subseteq \mathbb{R}^+$ that converges to 0 and satisfies $(\bar{t}, \bar{x}) - \lambda_k(\alpha, u) \in \mathcal{S}$ for all k. Let $\boldsymbol{z}_k \in \mathbb{R}^{n+1}$ be the point of the intersection of $\partial\mathcal{S}$ with the line segment $[(t_k, \bar{x}), \tilde{\boldsymbol{z}}_k]$ that links the points $(t_k, \bar{x})$ and $\tilde{\boldsymbol{z}}_k := (t_k, \bar{x}) - \lambda_k(\alpha, u)$. We deduce easily that $\{\boldsymbol{z}_k\}_k \to (\bar{t}, \bar{x})$ as $k \to \infty$ and that any accumulation point $\boldsymbol{d}$ of the sequence of normalized vectors $(\boldsymbol{z}_k - (\bar{t}, \bar{x}))/\|\boldsymbol{z}_k - (\bar{t}, \bar{x})\|$ belongs to the Bouligand tangent cone of $\partial\mathcal{S}$. This latter coincides with the tangent space of $\mathcal{S}$ at the same point, therefore $\boldsymbol{d}$ is orthogonal to the normal vector (α, u). On the other hand we have

$$\left\langle \frac{(t_k, \bar{x}) - (\bar{t}, \bar{x})}{\|(t_k, \bar{x}) - (\bar{t}, \bar{x})\|}, (\alpha, u) \right\rangle = \langle (1, 0), (\alpha, u) \rangle = \alpha.$$

and

$$\left\langle \frac{\tilde{\boldsymbol{z}}_k - (\bar{t}, \bar{x})}{\|\tilde{\boldsymbol{z}}_k - (\bar{t}, \bar{x})\|}, (\alpha, u) \right\rangle = -\|(\alpha, u)\|^2 = -1.$$

which yields that

$$\langle \boldsymbol{d}, (\alpha, u) \rangle = \lim_{k \to \infty} \left\langle \frac{\boldsymbol{z}_k - (\bar{t}, \bar{x})}{\|\boldsymbol{z}_k - (\bar{t}, \bar{x})\|}, (\alpha, u) \right\rangle \leq \max\{-1, \alpha\} = \alpha < 0,$$

leading to a contradiction. $\square$

The following lemma is crucial in the proof of our main theorem since it relates the value of the coderivative with the velocity of the orbit of the sweeping process. The proof follows the scheme of the proof of [9, Theorem 4.1] where a similar result has been established for the usual modulus $\|D^*S(t,\gamma(t))\|^+$.

Lemma 18. *Let $S:\mathbb{R}\rightrightarrows\mathbb{R}^n$ be a smooth sweeping process and $\gamma\in\mathcal{AC}(S,[a,b))$. Then,*

$$\|\dot\gamma(t)\| = \|D^*S(t,\gamma(t))|^{+,\uparrow},$$

for all $t\in[a,b)$ such that $-\dot\gamma(t)\in N_{S(t)}(\gamma(t))$ and $\|D^*S(t,\gamma(t))|^{+,\uparrow}$ is finite.

Proof. Let $t\in[a,b)$ be a point of differentiability of γ such that $-\dot\gamma(t)\in N_{S(t)}(\gamma(t))$ and that $\|D^*S(t,\gamma(t))|^{+,\uparrow}$ is finite.

First case: $\dot\gamma(t)=0$.
If $\zeta(t) := (t,\gamma(t)) \in \operatorname{int}(\mathcal{S})$, the desired equality holds trivially, while if $\zeta(t)\in\partial\mathcal{S}$, then $\dot\zeta(t)=(1,0)$ belongs to the tangent space of $\partial\mathcal{S}$ at $\zeta(t)$. Since S is a smooth sweeping process, the normal cone $N_{\mathcal{S}}(\zeta(t))$ is contained in the normal space of $\partial\mathcal{S}$ at $\zeta(t)$. Therefore,

$$\langle(1,0),N_{\mathcal{S}}(\zeta(t))\rangle=\{0\}.$$

Hence, if $(\alpha,u)\in N_{\mathcal{S}}(\zeta(t))$, then $\alpha=0$. Thus, $\|D^*S(\zeta(t))|^{+,\uparrow}=0$.

Second case: $\dot\gamma(t)\neq 0$.
Then $\zeta(t)\in\partial\mathcal{S}$ and $\dot\zeta(t)$ belongs to the tangent space of $\partial\mathcal{S}$ at $\zeta(t)$. As in the first case, we obtain that

$$\langle(1,\dot\gamma(t)),N_{\mathcal{S}}(\zeta(t))\rangle=\{0\}.$$

Hence, for every $(\alpha,u)\in N_{\mathcal{S}}(\zeta(t))$ with $\|u\|=1$ we have $\alpha+\langle\dot\gamma(t),u\rangle=0$ and by the Cauchy-Schwartz inequality we deduce

$$\|\dot\gamma(t)\|\geq\|D^*S(\zeta(t)|^{+,\uparrow}.$$

Setting $H=\{t\}\times\mathbb{R}^n$ we have $\{t\}\times S(t)=H\cap\mathcal{S}$. Due to the fact that $-\dot\gamma(t)\in N_{S(t)}(\gamma(t))$, we have:

$$(1,-\dot\gamma(t))\in N_{\{t\}\times S(t)}(\zeta(t)).$$

In addition, since S is a smooth sweeping process, $\|D^*S(t,\gamma(t))|^{+,\uparrow}<\infty$ and $\dot\zeta(t)=(1,\dot\gamma(t))$ belongs to the tangent space of $\partial\mathcal{S}$ at $\zeta(t)$, we have that $(\alpha,0)\in N_{\mathcal{S}}(t,\gamma(t))$ only if $\alpha=0$. Hence, applying the calculus rule [19, Theorem 6.42], we get

$$N_{H\cap\mathcal{S}}(\zeta(t))\subset N_H(\zeta(t))+N_{\mathcal{S}}(\zeta(t))=\mathbb{R}\times\{0\}+N_{\mathcal{S}}(\zeta(t)).$$

Therefore, the inclusion $(\lambda,-\dot\gamma(t))\in N_{\mathcal{S}}(\zeta(t))$ holds for some $\lambda\in\mathbb{R}$. By orthogonality between normal and tangent vectors, we get that:

$$\langle(\lambda,-\dot\gamma(t)),(1,\dot\gamma(t))\rangle=0.$$

and thus $\lambda=\|\dot\gamma(t)\|^2$. After normalization, we obtain:

$$\left(\|\dot\gamma(t)\|,-\frac{\dot\gamma(t)}{\|\dot\gamma(t)\|}\right)\in N_{\mathcal{S}}(\zeta(t)),$$

which readily yields $\|D^*S(t,\gamma(t))|^{+,\uparrow} \geq \|\dot{\gamma}(t)\|$, as claimed. $\square$

Let us finally quote the following result, which is a restatement of [3, Proposition 27] and it can be proved in the same way as there.

Proposition 19 (concatenation)**.** *Let $b > a$ and Γ be a collection of absolutely continuous curves γ defined in some nontrivial interval $J \subset (a,b)$ with values in $\mathbb{R}^n$. Assume that for each $t \in (a,b)$ there exist $\varepsilon_t > 0$ and $\gamma_t \in \Gamma$ with $\mathrm{dom}\,(\gamma_t) = [t, t+\varepsilon_t)$. Then there exist a countable partition $\{I_n\}_{n\in\mathbb{N}}$ of (a,b) into intervals I_n of nonempty interior and a piecewise absolutely continuous curve $\gamma : (a,b) \to \mathbb{R}$ such that for each $n \in \mathbb{N}$, there is $\gamma^n \in \Gamma$ such that $\gamma = \gamma^n$ on I_n.*

We are now ready to prove our main result.

4.2 Proof of Theorem A

We prove $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a)$.

a) ⇒ b) : Let $\Psi : [0,\rho] \to [a,b]$ be given by (a). Let $\gamma \in \mathcal{AC}(S,[t_1,t_2))$ with $[t_1,t_2) \subset [a,b)$. Since Ψ is a $\mathcal{C}^1$-smooth function, $\partial\mathrm{gph}((S\circ\Psi)|_{(0,\rho)})$ is a smooth manifold. By Proposition 16, $\gamma\circ\Psi \in \mathcal{AC}(S\circ\Psi, \Psi^{-1}([t_1,t_2)))$. Applying Lemma 18, we deduce that

$$\left|\frac{d\,(\gamma\circ\Psi)}{dr}(r)\right| = \|D^*(S\circ\Psi)(r,\gamma(\Psi(r)))|^{+,\uparrow} \leq 1, \quad \forall_{a.e}\ r \in (a,d).$$

Since Ψ is increasing and smooth, by change of variables we obtain:

$$\begin{aligned}\int_{t_1}^{t_2} \|\dot{\gamma}(\tau)\|d\tau = \int_{\Psi^{-1}(t_1)}^{\Psi^{-1}(t_2)} \|\dot{\gamma}(\Psi(r))\|\,\dot{\Psi}(r)dr &= \int_{\Psi^{-1}(t_1)}^{\Psi^{-1}(t_2)} \left\|\frac{d\,(\gamma\circ\Psi)}{dr}(r)\right\| dr \\ &\leq \int_{\Psi^{-1}(t_1)}^{\Psi^{-1}(t_2)} dr = \Psi^{-1}(t_2) - \Psi^{-1}(t_1).\end{aligned}$$

Therefore (b) is satisfied by setting $\sigma := \Psi^{-1}$.

b) ⇒ c) : Since σ is an increasing function and $\sigma(a) = 0$, statement (c) follows by setting $M := \sigma(b)$.

c) ⇒ d) : Let $b > a$ and let $M > 0$ given by statement $c)$. Let $\varphi^{\uparrow} : (a,b) \to \mathbb{R}\cup\{+\infty\}$ be the oriented talweg function of S and let us assume, towards a contradiction, that for any $c \in (a,b)$ the function $\varphi^{\uparrow}$ is not integrable on (a,c). By Lemma 14, the function $(t,x) \mapsto \|D^*S(t,x)|^{+,\uparrow}$ is continuous on $\partial\mathcal{S}$. By assumption (A2), shrinking b if necessary, we may assume that $\varphi^{\uparrow}(t) < \infty$ for all $t \in (a,b)$. By Lemma 15, $\varphi^{\uparrow}$ is continuous on (a,b).
By Lemma 18, if J is a nontrivial interval of (a,b) then $\|\dot{\gamma}(t)\| = \|D^*S(t,\gamma(t))|^{+,\uparrow}$ for all $\gamma \in \mathcal{AC}(S,J)$ and almost all $t \in J$. Let $k \in \mathbb{N}$ and $t \in (a,b)$ and define a curve γ_t^k as follows:

- If $\varphi^{\uparrow}(t) = 0$, take $\gamma_t^k \in \mathcal{AC}(S,[t,\tau))$ be any curve such that $\tau - t < 1/k$.
- If $\varphi^{\uparrow}(t) > 0$, since the set $\partial\mathcal{S} \bigcap (\{t\}\times\mathbb{R}^n)$ is compact, there exists $x \in S(t)$ such that $\|D^*S(t,x)|^{+,\uparrow} = \varphi^{\uparrow}(t)$. Thanks to (A1) and Lemma 15, we can take $\gamma_t^k \in \mathcal{AC}(S,[t,\tau))$, for some $\tau > t$, such that $\gamma(t) = x$ and

$$\|\dot{\gamma}_t^k(s)\| > \left(\frac{k-1}{k}\right)\varphi^{\uparrow}(s), \quad \text{for almost every } s \in (t,\tau).$$

Gluing together, thanks to Proposition 19 (concatenation), we obtain $\gamma^k \in \mathcal{PAC}(S,(a,b))$ such that for almost every $t \in (a,b)$

$$\varphi^{\uparrow}(t) \geq \|\dot{\gamma}^k(t)\| \geq f_k(t) := \begin{cases} 0, & \text{if } t \in A_k \\ \left(\dfrac{k-1}{k}\right)\varphi^{\uparrow}(t), & \text{if } t \in (a,b) \setminus A_k. \end{cases}$$

where $A = \{t \in (a,b) : \ \varphi^{\uparrow}(t) = 0\}$ and $A_k = (a,b) \cap (A + [0,1/k])$ for all $k \in \mathbb{N}$.
The continuity of $\varphi^{\uparrow}$ yields that A is a closed set relatively to (a,b). Therefore, $A = \bigcap_{k\in\mathbb{N}} A_k$. Then, for all $t \in (a,b)$, $f_k(t) \nearrow \varphi^{\uparrow}(t)$ as k tends to infinity. Hence, by the Monotone Convergence Theorem, $(\int_a^b f_k)_k$ converges to $\int_a^b \varphi^{\uparrow}$, which is infinity. Thus, there is $K \in \mathbb{N}$ such that

$$\int_a^b \|\dot{\gamma}^K(t)\| dt \geq \int_a^b f_K(t) dt > M,$$

which contradicts statement (c) since $\gamma^K \in \mathcal{PAC}(S,(a,b))$.

d)* ⇒ *a) : Let us assume that the oriented talweg function $\varphi^{\uparrow}$ is integrable on $[a,b]$ for some $b > a$. As a consequence of assumptions (A2) and Lemma 15, shrinking b if necessary, we may assume that $\varphi^{\uparrow}$ is continuous on (a,b) and $\varphi^{\uparrow}(t) < \infty$ for all $t \in (a,b)$. Let $\overline{\varphi} := \max\{\varphi^{\uparrow}, 1\}$ which is an integrable continuous majorant of $\varphi^{\uparrow}$ and set

$$\theta(t) := \int_a^t \overline{\varphi}(s) ds, \quad \text{for } t \in [a,b].$$

Since $\overline{\varphi}$ is positive and integrable on $[a,b]$, we set $\rho := \theta(b)$ and define $\Psi : [0,\rho] \to [a,b]$ as the inverse function of θ, that is, $\Psi(r) = \theta^{-1}(r)$. Since $\theta'(t) = \overline{\varphi}(t) \in [1,+\infty)$, for every $t \in (a,b)$, it follows that Ψ is $\mathcal{C}^1$-smooth on $(0,\rho)$, with derivative

$$\Psi'(r) = \frac{1}{\overline{\varphi}(\Psi(r))} \leq 1, \quad \text{for all } r \in (0,\rho).$$

Thus, Ψ is a Lipschitz homeomorphism between $[0,\rho]$ and $[a,b]$. Finally, using the chain rule for coderivatives [19, Theorem 10.37], we deduce that

$$\|D^*(S \circ \Psi)(r,x)|^{+,\uparrow} \leq \frac{\|D^* S(\Psi(r),x)|^{+,\uparrow}}{\overline{\varphi}(\Psi(r))} \leq 1, \quad \text{for all } r \in (0,\rho).$$

The proof is complete. □

4.3 Oriented calmness

Before proceeding to the proof of Theorem B, we need to introduce the modulus of oriented calmness and establish a result analogous to the Mordukhovich criterium for the oriented modulus of the coderivative. Let us first recall that the *Lipschitzian graphical modulus* of $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ at t for x is defined by

$$\begin{aligned} \operatorname{Lip} S(t,x) := \inf\{\kappa > 0 |\ \exists \epsilon > 0,\ \delta > 0, \text{ such that} \\ S(t_2) \cap B(x,\delta) \subset S(t_1) + \kappa|t_2 - t_1|B, \quad \text{for all } t_1, t_2 \in (t-\epsilon, t+\epsilon)\}, \end{aligned}$$

where B stands for the open unit ball.

We recall that the multivalued function S has the Aubin property at t for x if and only if $\operatorname{Lip} S(t,x) < \infty$. More precisely, we have the following (see [19, Theorem 9.40]).

Theorem 20. *For every $(t,x) \in \mathcal{S}$ such that $\|D^*S(t,x)\|^+ < \infty$ it holds:*

$$\operatorname{Lip} S(t,x) = \|D^*S(t,x)\|^+.$$

Motivated by the above, we introduce the following graphical modulus.

Definition 21 (oriented calm modulus)**.** Let $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ be a multivalued map and $(t,x) \in \mathcal{S}$. The oriented calm graphical modulus, denoted by $\operatorname{calm}^{\uparrow} S$, at t for x is defined by

$$\begin{aligned}\operatorname{calm}^{\uparrow} S(t,x) := \inf\{\kappa > 0 |\ \exists \epsilon > 0,\ \delta > 0,\ \text{such that}\\ S(t) \cap B(x,\delta) \subset S(t_1) + \kappa|t_1 - t|B \text{ for all } t_1 \in (t, t+\epsilon)\}.\end{aligned}$$

Observe that, if S is a single-valued function and $\operatorname{calm}^{\uparrow} S(t,x) < \infty$, then S is calm at t to the right. More information on the notion of calmness for multivalued maps can be found in [13] and references therein. We are now ready to give the oriented version of Theorem 20.

Proposition 22 (oriented calm vs oriented modulus)**.** *Let $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ be a smooth sweeping process, $t \in \operatorname{dom}(S) \setminus \{\mathrm{T}\}$ and $x \in S(t)$ such that $\|D^*S(t,x)|^{+,\uparrow} < +\infty$. Then*

$$\operatorname{calm}^{\uparrow} S(t,x) = \|D^*S(t,x)|^{+,\uparrow}.$$

Proof. Let us first notice that $\operatorname{calm}^{\uparrow} S(t,x) \leq \operatorname{Lip} S(t,x)$. We consider two cases:

Case 1: $\|D^*S(t,x)|^{+,\uparrow} = 0$.

If $\|D^*S(t,x)\|^+ = 0$, then $\operatorname{calm}^{\uparrow} S(t,x) = 0$. If $\|D^*S(t,x)\|^+ > 0$, then, by Lemma 11 $\mathcal{S}$ is a manifold of full dimension with boundary $\partial\mathcal{S}$ which is a smooth manifold of dimension n. Let us assume by contradiction that $\operatorname{calm}^{\uparrow} S(t,x) > 0$. Then, for every $k \in \mathbb{N}$ such that $k^{-1} < \operatorname{calm}^{\uparrow} S(t,x)$, there exists $y_k \in S(t) \cap B(x, 1/k)$ such that

$$y_k \notin S(t_k) + \left(\frac{t_k' - t}{k}\right) B,\ \text{for some } t_k' \in (t,\ t + \frac{1}{k}).$$

Set $t_k := \inf\{r \in (t, t+\frac{1}{k}) :\ \ y_k \notin S(r)\}$. It is clear that $(t_k, y_k) \in \partial\mathcal{S}$ and that y_k is not right-locally stationary for S at t_k. Thus, by Proposition 17 (c), for every $k \in \mathbb{N}$ and $(\beta_k, v_k) \in N_{\mathcal{S}}(t_k, y_k)$, we have $\beta_k \geq 0$. Since $N_{\mathcal{S}}(t,x)$ is a ray and $\{(t_k, y_k)\}_k \to (t,x)$, the continuity of unit outer normal vectors of $\mathcal{S}$ on $\partial\mathcal{S}$ ensures that $\beta \geq 0$ whenever $(\beta, v) \in N_{\mathcal{S}}(t,x)$. This leads to the equality $\|D^*S(t,x)|^{+,\uparrow} = \|D^*S(t,x)\|^+$, which is a contradiction. Therefore, $\operatorname{calm}^{\uparrow} S(t,x) = 0$.

Case 2: $\|D^*S(t,x)|^{+,\uparrow} = \alpha > 0$.

In this case, we deduce from Lemma 11(b) that

$$\|D^*S(t,x)|^{+,\uparrow} = \|D^*S(t,x)\|^+ = \operatorname{Lip} S(t,x) \geq \operatorname{calm}^{\uparrow} S(t,x).$$

By compactness of the unit ball of $\mathbb{R}^n$, there exists $u \in \mathbb{R}^n$ with $\|u\| = 1$ such that $(\alpha, u) \in N_{\mathcal{S}}(t,x)$. Let $\{t_k\}_{k\geq 1} \subset \mathbb{R}$ be a decreasing sequence that converges to t. Let $\{y_k\}_{k\geq 1} \subset \mathbb{R}^n$ be a sequence that satisfies $y_k \in \mathrm{Proj}(x, S(t_k))$ for each $k \in \mathbb{N}$. By compactness of the unit sphere of $\mathbb{R}^{n+1}$, up to a subsequence we deduce that

$$\lim_{k\to\infty} \frac{(t_k - t, y_k - x)}{\|(t_k - t, y_k - x)\|} = (\beta, v),$$

where (β, v) belongs to the tangent space of $\mathcal{S}$ at (t, x) and $\beta \geq 0$. Since $\mathcal{S}$ is a smooth sweeping process, it follows that

$$(\alpha, u) \perp (\beta, v) \qquad \text{yielding} \qquad \langle u, v\rangle = -\alpha\beta.$$

Since $\mathrm{calm}^{\uparrow} S(t, x) \leq \|D^* S(t, x)|^{+,\uparrow} < +\infty$, β must be a strictly positive number. Therefore

$$\lim_{k\to\infty} \frac{\|y_k - x\|}{t_k - t} = \frac{\|v\|}{\beta} \geq \frac{|\langle u, v\rangle|}{\beta} = \alpha,$$

implying that

$$\mathrm{calm}^{\uparrow} S(t, x) \geq \alpha = \|D^* S(t, x)|^{+,\uparrow}.$$

The proof is complete. □

The next lemma will be used to relate the length of catching up sequences for S with the integral of the respective oriented talweg function.

Lemma 23 (controlling excess of $S(t_0)$)**.** *Let $S : \mathbb{R} \rightrightarrows \mathbb{R}^n$ be a locally bounded smooth sweeping process and let $[t_0, t_1] \subset \mathrm{dom}(S)$. Then*

$$\mathrm{ex}\,(S(t_0), S(t_1)) := \sup_{x\in S(t_0)} d(x, S(t_1)) \leq \left(\sup_{t\in[t_0,t_1]} \varphi^{\uparrow}(t)\right)(t_1 - t_0)$$

and

$$\mathrm{dist}(S(t_0), S(t_1)) \leq \left(\sup_{t\in[t_0,t_1]} \varphi(t)\right)(t_1 - t_0).$$

Proof. Let us first notice that

$$K := \sup_{t\in[t_0,t_1]} \varphi^{\uparrow}(t) \geq \|D^* S(t, x)|^{+,\uparrow} = \mathrm{calm}^{\uparrow} S(t, x), \quad \text{for all } t \in [t_0, t_1] \text{ and } x \in S(t)\,.$$

If $K = \infty$, there is nothing to prove. Let $K < +\infty$ and assume, towards a contradiction, that for some $\delta > 0$ we have

$$\mathrm{ex}\,(S(t_0), S(t_1)) > (K + \delta)(t_1 - t_0).$$

Let $\tau \in \mathbb{R}$ be defined by

$$\tau := \inf\,\{\, t \in [t_0, t_1] :\ \mathrm{ex}\,(S(t_0), S(t)) > (K + \delta)(t - t_0)\,\}\,.$$

By Proposition 22 and the definition of the graphical modulus $\mathrm{calm}^{\uparrow}$, for each $x \in S(t_0)$, there is $\varepsilon_x > 0$ and $\delta_x > 0$ such that

$$S(t_0) \cap B(x, \delta_x) \subset S(t) + (K + \frac{\delta}{2})|t - t_0|B, \quad \text{for all } t \in [t_0, t_0 + \varepsilon_x).$$

Let $\tilde{\varepsilon}_x > 0$ be the supremum of all $\varepsilon > 0$ such that:

$$x \in S(t) + (K + \frac{\delta}{2})|t - t_0|B, \text{ for all } t \in [t_0, t_0 + \varepsilon).$$

If $\tau = t_0$, then there exists a sequence $\{x_k\}_k \subset S(\tau)$ such that $\tilde{\varepsilon}_{x_k} < 1/k$, for all $k \geq 1$. Since $S(\tau)$ is compact, the sequence $\{x_k\}_k$ has some cluster point $\overline{x} \in S(\tau)$. By Proposition 22, there exist $\varepsilon_{\overline{x}} > 0$ and $\delta_{\overline{x}} > 0$ such that

$$S(t_0) \cap B(\overline{x}, \delta_{\overline{x}}) \subset S(t) + (K + \frac{\delta}{2})|t - t_0|B, \text{ for all } t \in [t_0, t_0 + \varepsilon_{\overline{x}}).$$

which contradicts the maximality of $\tilde{\varepsilon}_{x_k}$, for k large enough. This establishes that $t_0 < \tau$. Proceeding in the same way, we can actually show that $\tau \geq t_1$. Indeed, assuming $\tau < t_1$, and using the same argument as above (with t_0 in the place of τ) together with the triangle inequality we obtain a contradiction in a similar way. Therefore, for every $\delta > 0$ we have:

$$\text{ex}(S(t_0), S(t_1)) \leq (K + \delta)(t_1 - t_0),$$

which finishes the first assertion of the lemma.

For the second part, we follow the same procedure to estimate the reverse excess $\text{ex}\,(S(t_1), S(t_0))$, and conclude thanks to the fact that $\text{dist}(S(t_0), S(t_1)) = \max\{\text{ex}(S(t_0), S(t_1)), \text{ex}(S(t_1), S(t_0))\}$. The details are left to the reader. $\square$

Now, we proceed with the proof of our second main result.

4.4 Proof of Theorem B.

We recall from Section 4.2 the definition of $\mathcal{T}$ and fix $a \in \mathcal{T}$. We prove $(a) \Rightarrow (e) \Rightarrow (f) \Rightarrow (d)$.

a) ⇒ e) : Choose $b > a$ such that the statements (a)–(d) of Theorem A hold true, and $\varphi^{\uparrow}(t) < +\infty$ for all $t \in (a, b)$ (*c.f.* Assumption (A2)). We set

$$\sigma(t) = \int_a^t \varphi^{\uparrow}(s)ds, \quad t \in (a, b).$$

By (d) the above integral is well-defined and σ is continuous with $\sigma(a) = 0$. Let $\{(t_i, x_i)\}_{i \geq 0} \subset \mathcal{S}$ be any catching-up sequence for S with $I := [t_0, t_k] \subset (a, b)$. We shall prove that (7) holds for every $k \geq 1$. By Lemma 15, $\varphi^{\uparrow}$ is continuous (and finite), hence Riemann integrable there. Let $N > 0$. Let $\{s_j^i\}_{j=0}^{k_i}$ be a partition of the interval $[t_i, t_{i+1}]$, $i \in \{0, \ldots, k-1\}$, with width

$$\max_{j \in \{0, \ldots, k_i - 1\}} |s_{j+1}^i - s_j^i| < \frac{1}{N}, \quad \text{for all } i \in \{0, \ldots, k-1\}.$$

Note that for every $i \in \{0, \ldots, k-1\}$, we have $s_0^i = t_i$ and $s_{k_i}^i = t_{i+1}$. We set

$$z_0^i := x_i \in S(t_i) \qquad \text{and for each } \ j \in \{0, \ldots, k_i - 1\} \text{ we pick } z_{j+1}^i \in \text{Proj}_{S(s_{j+1})}(z_j^i).$$

Then using triangle inequality and the fact that

$$\|x_{i+1} - x_i\| = d(x_i, \underbrace{S(t_{i+1})}_{=S(s_{k_i}^i)})) \leq \|z_{k_i}^i - z_0^i\|.$$

we deduce from Lemma 23 that:

$$\|x_{i+1}-x_i\| \leq \sum_{j=0}^{k_i-1} \|z^i_{j+1}-z^i_j\| \leq \sum_{j=0}^{k_i-1} \left(\sup_{t\in[s^i_j,s^i_{j+1}]} \varphi^{\uparrow}(t) \right) (s_{j+1}-s_j)\,.$$

Taking the limit as $N\to\infty$ we obtain that

$$\|x_{i+1}-x_i\| \leq \int_{t_i}^{t_{i+1}} \varphi^{\uparrow}(s)ds$$

and consequently,

$$\sum_{i=0}^{k-1} \|x_{i+1}-x_i\| \leq \int_{t_0}^{t_k} \varphi^{\uparrow}(t)dt = \sigma(t_k)-\sigma(t_0).$$

e) ⇒ f) : It follows directly by taking $M=\sigma(b)$.

f) ⇒ d) : Let $b>a$ and $M>0$ be given by statement (f). By (A2), shrinking b if necessary, we may assume that $\varphi^{\uparrow}(t)<\infty$, for all $t\in(a,b)$. Note that for any compact interval $[c,d]\subset(a,b)$, the function $\varphi^{\uparrow}$ is continuous and finite on $[c,d]$, therefore Riemann integrable. We shall prove that its integral is bounded by M (independently of the values of c and d).

To this end, let $t_0\in[c,d]$ and $N\in\mathbb{N}$. By compactness, there exists $x\in S(t_0)$ such that $\|D^*S(t_0,x)|^{+,\uparrow}=\varphi^{\uparrow}(t_0)$. If $\varphi^{\uparrow}(t_0)<\frac{1}{N}$, we set $t_1:=\min\{t_0+\frac{1}{N},d\}$, $x_0=x$ and $y_0\in\mathrm{Proj}_{S(t_1)}(x_0)$. Observe that

$$\|x_0-y_0\| \geq 0 \geq (t_1-t_0)\left(\varphi^{\uparrow}(t_0)-\frac{1}{N}\right).$$

If $\varphi^{\uparrow}(t)\geq\frac{1}{N}$, by Proposition 22, since $\mathrm{calm}^{\uparrow}S(t,x)=\varphi^{\uparrow}(t)$, there are $x_0\in S(t_0)$ and $t_1\in(t_0,\min\{t_0+\frac{1}{N},b\})$ such that any $y_0\in\mathrm{Proj}_{S(t_1)}(x_0)$ satisfies

$$\|x_0-y_0\| \geq (t_1-t_0)\left(\varphi^{\uparrow}(t_0)-\frac{1}{N}\right).$$

Using transfinite induction we obtain an increasing net $\{t_\lambda\}_{\lambda\leq\Lambda}\subset[c,d]$ indexed over a countable ordinal Λ, such that $t_0=c$, $t_\Lambda=d$, $0<t_{\lambda+1}-t_\lambda\leq 1/N$ for all $\lambda<\Lambda$, and for any limit ordinal $\alpha\leq\Lambda$, $t_\alpha=\sup\{t_\lambda:\ \lambda<\alpha\}$. Also, we get a net $\{(x_\lambda,y_\lambda)\}_{\lambda\leq\Lambda}$ such that $x_\lambda\in S(t_\lambda)$, $y_\lambda\in\mathrm{Proj}_{S(t_{\lambda+1})}(x_\lambda)$ and

$$\|x_\lambda-y_\lambda\| \geq (t_{\lambda+1}-t_\lambda)\left(\varphi^{\uparrow}(t_\lambda)-\frac{1}{N}\right),\ \text{for all } \lambda<\Lambda.$$

For any finite subset $F\subset\Lambda$ we have

$$\sum_{\lambda\in F}\|x_\lambda-y_\lambda\| \geq \left(\sum_{\lambda\in F}(t_{\lambda+1}-t_\lambda)\varphi^{\uparrow}(t_\lambda)\right)-\frac{d-c}{N}.$$

Since $\{(t_\lambda,x_\lambda),\ (t_\lambda,y_\lambda):\ \lambda\in F\}$ is a subsequence of a piecewise catching-up sequence for S, from hypothesis $f)$ and taking the supremum over all finite subsets F of Λ we get

$$M \geq \sum_{\lambda<\Lambda}\|x_\lambda-y_\lambda\| \geq \left(\sum_{\lambda<\Lambda}(t_{\lambda+1}-t_\lambda)\varphi^{\uparrow}(t_\lambda)\right)-\frac{d-c}{N}.$$

Taking the limit as N goes to infinity we obtain:

$$M \geq \int_c^d \varphi^{\uparrow}(t)dt.$$

Since the above is independent of the interval $[c, d]$, we deduce that $\varphi^{\uparrow}$ is integrable on (a, b).□

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