Stability of periodic solutions obtained via the averaging method for nonsmooth Lipschitz systems

Adriana Buică & Aris Daniilidis

Abstract Existence and asymptotic stability of the periodic solutions of the Lipschitz system $x'(t) = \varepsilon F(t, x, \varepsilon)$ is hereby studied via the averaging method. The traditional C^1 dependence of $F(s, \cdot, \varepsilon)$ on z is relaxed to the mere strict differentiability of $F(s, \cdot, 0)$ at $z = z_0$ for $\varepsilon = 0$, giving room to potential applications for structured nonsmooth systems.

Key words Periodic solution, averaging method, nonsmooth Lipschitz system, Poincaré–Andronov mapping, fixed point.

AMS subject classification *Primary* 34C29, 34C25 ; *Secondary* 49J52

1 Introduction

We consider the following differential system

$$x'(t) = \varepsilon F(t, x, \varepsilon) \tag{1.1}$$

where $F : \mathbb{R} \times \Omega \times [0, 1] \to \mathbb{R}^n$ is a *continuous* function, T*-periodic* in the first variable and *locally Lipschitz* with respect to x (uniformly with respect to the other two variables). The set Ω is an open connected subset of $\mathbb{R}^n$. In this paper we are interested in the problem of existence and stability of T–periodic solutions of system (1.1). For sufficiently small values of the parameter ε, the system (1.1) is usually studied via the averaging method, see for example [13, 7, 6, 14, 8]. According to this method, we consider the function $f : \Omega \to \mathbb{R}^n$ defined by

$$f(z) = \int_0^T F(s, z, 0)ds. \tag{1.2}$$

Roughly speaking, the existence of a "non-degenerate" zero $z_0 \in f^{-1}(0)$ of the function f ensures, for all small values of $\varepsilon > 0$, the existence of a T–periodic solution of the system (1.1). A typical assumption is that $z \mapsto F(s, z, \varepsilon)$ is of class C^k $(k \geq 1)$ (see for instance [13, 7, 6]). In this case, the aforementioned non-degenerancy condition simply means that the determinant of the Jacobian of f at z_0 does not vanish, that is, $\det Jf(z_0) \neq 0$. Moreover, when JF has a continuous dependence on the parameter ε, a study of the asymptotic stability of the periodic solutions (depending on the eigenvalues of the Jacobian matrix of f at z_0) can be carried out: for example, if all the spectral pseudo-abscissa values of $Jf(z_0)$ are negative, the periodic solutions are asymptotically stable.

The result on the existence of periodic solutions via the averaging method can be extended also to nonsmooth systems (that is, F is merely continuous on z), like in [9, Theorem IV.13]. In such case, the condition $\det Jf(z_0) \neq 0$ is replaced by the more general assumption that the Brouwer degree of f is non–vanishing in some neighborhood of z_0. Since nonsmooth systems appear frequently in applications (simple operations as the max-operator or existence of constrains lead inevitably to a loss of differentiability) it is important to study the corresponding stability problem. In [15], it has been assumed that the nonsmooth system (1.1) admits the constant solution $x \equiv z_0$ and that F is (Lipschitz on z and) independent of ε. In that case, a study of stability of the constant solution has been carried out, via the global asymptotic stability of the averaged system $y' = \varepsilon f(y)$, see [15, Theorem 2]. In [10, Theorem 3.1] it has been proved that the same result holds when F is continuous and degree zero homogeneous. Both approaches make use of Lyapunov functions and converse Lyapunov theorems.

In this work we study the stability of the nontrivial periodic solutions of (1.1) and we show that the assumption that the function $z \mapsto F(s, z, \varepsilon)$ is C^1 (for all $s \in [0, T]$ and $\varepsilon \in [0, 1]$) can be relaxed to the mere strict differentiability of $z \mapsto F(s, z, 0)$ at $z = z_0$ for almost all $s \in [0, T]$. In this case we show in particular that f will be differentiable at z_0 (Proposition 3), hence the study of stability of the periodic solution can again be carried out via the eigenvalues (spectral pseudo-abscissa) of the Jacobian matrix of f at z_0 (Theorem 8).

The proof of our main result can be roughly summarized as follows: We study the Poincaré–Andronov operator, $P(\cdot, \varepsilon)$, of the system (1.1). This operator is locally Lipschitz, but not differentiable. Instead of the classical derivative we work with the generalized Jacobian (see [5, Section 2.6] and (3.9) below). We prove that, for $\varepsilon > 0$ sufficiently small, all the generalized Jacobian matrices at a fixed point z_ε of $P(\cdot, \varepsilon)$ are close to the Jacobian matrix of $I + \varepsilon f$ at z_0. This simple fact yields for example the asymptotic stability of the system (1.1) in the case that $Jf(z_0)$ is normal (thus diagonalizable in $\mathbb{C}$) with negative pseudo-abscissa values (Theorem 8) or the instability in case that $Jf(z_0)$ is normal and all its eigenvalues have positive real part (Theorem 10).

Let us finally mention, without entering in details, that our main results can potentially be applied to the study of stability of limit cycles of the nonsmooth system considered in [4]. As an illustration, we study a simplified planar system at the end of Section 3.

2 Notation and preliminaries

In this section we fix our notation and present some useful basic results. Throughout this work we shall deal with the system (1.1) and we shall assume that $z_0 \in \Omega$ is a zero of the function f given in (1.2). Unless otherwise stated, we shall always consider the Euclidean norm $||\cdot||$ on $\mathbb{R}^n$ and the corresponding operator norm on the space of $n \times n$ matrices. Let us fix a ball $B(z_0, r_0) \subset \Omega$ centered at z_0 with radius $r_0 > 0$. Then for each $z \in B(z_0, r_0)$ we denote by $x(\cdot, z, \varepsilon) : [0, t_{(z,\varepsilon)}) \to \mathbb{R}^n$ the solution of (1.1) with initial point $x(0, z, \varepsilon) = z$. Using the local existence and uniqueness theorem (see [7, Section 1.3], for example) we deduce that there exists $\varepsilon_0 \in (0, 1]$ (sufficiently small) such that $t_{(z,\varepsilon)} > T$ for all $z \in B(z_0, r_0)$ and all $\varepsilon \in [0, \varepsilon_0]$.

We define the Poincaré–Andronov operator as follows:

$$P : B(z_0, r_0) \times [0, \varepsilon_0] \to \mathbb{R}^n \qquad \text{with} \qquad P(z, \varepsilon) := x(T, z, \varepsilon). \tag{2.3}$$

We recall that a solution $x(\cdot, z, \varepsilon)$ of (1.1) satisfying $x(0, z, \varepsilon) = x(T, z, \varepsilon)$ is T*–periodic*. Thus, z_ε is a fixed point of $P(\cdot, \varepsilon)$ if, and only if, $x(\cdot, z_\varepsilon, \varepsilon)$ is a T–periodic solution of (1.1).

Definition 1 (asymptotic stability). (i) A fixed point z^* of the operator $P(\cdot, \varepsilon)$ is called *stable* if for each $\eta > 0$ there exists $\delta > 0$ such that $||z - z^*|| < \delta$ implies $||P^n(z, \varepsilon) - z^*|| < \eta$ for all $n \geq 0$. A stable fixed point for which there exists $\delta > 0$ such that for all $z \in B(z^*, \delta)$

$$\lim_{n \to +\infty} ||P^n(z, \varepsilon) - z^*|| = 0,$$

is called *asymptotically stable.*

(ii) A periodic solution $x(\cdot, z^*, \varepsilon)$ of the system (1.1) is called *stable* (in the sense of Lyapunov) if for each $\eta > 0$ there exists $\delta > 0$ such that $||z - z^*|| < \delta$ implies $||x(t, z, \varepsilon) - x(t, z^*, \varepsilon)|| < \eta$ for all $t \geq 0$. A stable periodic solution for which there exists $\delta > 0$ such that for all $z \in B(z^*, \delta)$

$$\lim_{t \to +\infty} ||x(t, z, \varepsilon) - x(t, z^*, \varepsilon)|| = 0,$$

is called *asymptotically stable.*

The following result shows that the study of asymptotical stability of the periodic solutions of (1.1) can be reduced to the study of the Poincaré–Andronov operator (see [6], for example).

Theorem 2 (periodic solutions vs Poincaré-Andronov operator). *For fixed $\varepsilon > 0$, z_ε is an asymptotically stable fixed point of $P(\cdot, \varepsilon)$ if, and only if, $x(\cdot, z_\varepsilon, \varepsilon)$ is an asymptotically stable periodic solution of* (1.1).

In view of this theorem and the above definitions, it is easy to see that if there exists $\delta > 0$ and $0 < \rho < 1$ such that

$$||P(z_1, \varepsilon) - P(z_2, \varepsilon)|| \leq \rho ||z_1 - z_2|| \text{ for all } z_1, z_2 \in B(z^*, \delta)$$

then the fixed point z^* of $P(\cdot, \varepsilon)$ will be asymptotically stable, thus the same holds for the periodic solution $x(\cdot, z^*, \varepsilon)$ of (1.1) as well.

Using the continuous dependence of the solution with respect to initial condition $z \in B(z_0, r_0)$ and parameter $\varepsilon \in [0, \varepsilon_0]$ (see [7, Section 1.3], for example), we deduce that there exists a compact set K with $B(z_0, r_0) \subset K \subset \Omega$ such that for all $(t, z, \varepsilon) \in [0, T] \times B(z_0, r_0) \times [0, \varepsilon_0]$

$$x(t, z, \varepsilon) \in K. \tag{2.4}$$

Let $M > 0$ be such that for all $t \in [0, T]$, $x \in K$ and $\varepsilon \in [0, \varepsilon_0]$

$$||F(t, x, \varepsilon)|| \leq M. \tag{2.5}$$

Using compactness of K we also deduce the existence of a uniform Lipschitz constant $L > 0$ of F such that for all $t \in [0, T]$ and $\varepsilon \in [0, \varepsilon_0]$

$$||F(t, x, \varepsilon) - F(t, y, \varepsilon)|| \leq L ||x - y||, \quad \text{for all } x, y \in K. \tag{2.6}$$

Let us finally mention the equivalent to (1.1) integral equation for $x(\cdot, z, \varepsilon)$:

$$x(t, z, \varepsilon) = z + \varepsilon \int_0^t F(s, x(s, z, \varepsilon), \varepsilon) ds \tag{2.7}$$

for all $t \in [0, T]$, $z \in B(z_0, r_0)$ and $\varepsilon \in [0, \varepsilon_0]$. Combining (2.6) with (2.7) and applying the classical Gronwall Lemma, we obtain for each $t \in [0, T]$ and $\varepsilon \in [0, \varepsilon_0]$ that

$$||x(t, z, \varepsilon) - x(t, w, \varepsilon)|| \leq e^{\varepsilon L t} ||z - w||, \quad \text{for all } z, w \in B(z_0, r_0). \tag{2.8}$$

3 Main results

We shall now study the stability of the periodic solutions of (1.1) (or equivalently of (2.7)). In this work the C^1 assumption on F is relaxed as follows:

$(\mathcal{H}_1)$ $\quad F(s,\cdot,0)$ is strictly differentiable at z_0, for almost all $s \in [0,T]$.

In other words, for almost all $s \in [0,T]$, there exists $A(s) := JF(s,z_0,0)$ such that

$$\lim_{z\to z_0\, u\to 0} ||u||^{-1}\, ||F(s,z+u,0) - F(s,z,0) - A(s)u|| = 0.$$

Note that the existence of the Jacobian JF is no more required for points (s,z,ε) with either $\varepsilon > 0$ or $z \neq z_0$. For given (fixed) values of $s \in [0,T]$ and $\varepsilon \in [0,\varepsilon_0]$, we shall consider instead, the generalized Jacobian $\partial F(s,z,\varepsilon)$, which is defined as the closed convex envelope of all possible limits of Jacobians of points (s,z_n,ε) of differentiability of $F(s,\cdot,\varepsilon)$ as $z_n \to z$ ([5, Definition 2.6.1]). An even more general description can be found in [12, Theorem 9.62]. More precisely, if $D_F(s,\varepsilon)$ denotes the points of differentiability of $F(s,\cdot,\varepsilon)$, it follows by the Rademacher theorem that $\mathbb{R}^n \setminus D_F(s,\varepsilon)$ is of Lebesgue measure zero. Then given any null subset N of $\mathbb{R}^n$ the following formula holds:

$$\partial F(s,z,\varepsilon) \;=\; \overline{\mathrm{co}}\left\{ B_\varepsilon \;=\; \lim_{z_n\to z} JF(s,z_n,\varepsilon); \quad z_n \in D_F(s,\varepsilon)\setminus N \right\}. \tag{3.9}$$

We shall also need make the following mild assumption, which can be seen as a relaxation of the continuous dependence of the Jacobian JF on the parameter $\varepsilon \in [0,\varepsilon_0]$ in the C^1 case.

$(\mathcal{H}_2)$ For all $\eta > 0$, there exists $\delta > 0$ such that for all $z \in B(z_0,\delta)$, $\varepsilon \in [0,\delta]$
. and $s \in [0,T]$: $\quad \partial F(s,z,\varepsilon) \subset \partial F(s,z_0,0) + B(0,\eta)$

Before we proceed, let us observe that $(\mathcal{H}_1)$ has the following useful consequence (the interested reader might want to compare this result with [5, Theorem 7.4.1]):

Proposition 3 (strict differentiability of f). *Under the assumption $(\mathcal{H}_1)$, the function f defined in (1.2) is strictly differentiable at z_0 with derivative*

$$Jf(z_0) = \int_0^T A(s)ds.$$

Proof. Set $A(s) := JF(s,z_0,0)$ for all $s \in [0,T]$ where the Jacobian exists and note that the set $\{A(s) : s \in [0,T] \text{ a.e.}\}$ is bounded by the Lipschitz constant L of F. Thus $A = \int_0^T A(s)\,ds$ is well–defined. We need to prove the equality

$$\lim_{z\to z_0\, u\to 0} ||u||^{-1}||f(z+u) - f(z) - Au|| = 0.$$

To this end, consider the sequences $z_n \to z_0$, $u_n \to 0$. The conclusion follows using the inequality

$$\frac{||f(z_n+u_n) - f(z_n) - Au_n||}{||u_n||} \leq \int_0^T \frac{||F(s,z_n+u_n,0) - F(s,z_n,0) - A(s)u_n||}{||u_n||}\,ds,$$

assumption $(\mathcal{H}_1)$ and the Lebesgue dominated convergence theorem. $\square$

Let us introduce for every $\varepsilon \geq 0$ the functions $g_\varepsilon, h_\varepsilon : B(z_0, r_0) \to \mathbb{R}^n$ defined by

$$g_\varepsilon(z) = \int_0^T F(s, x(s, z, \varepsilon), \varepsilon) ds, \tag{3.10}$$

and

$$h_\varepsilon(z) = \int_0^T \left[F(s, x(s, z, \varepsilon), \varepsilon) - F(s, z, 0) \right] ds. \tag{3.11}$$

Observe that $h_\varepsilon(z) = g_\varepsilon(z) - f(z)$ and $g_0(z) = f(z)$.

Proposition 4 (Lipschitz estimation of h_ε around z_0). *Under the assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, for every $\eta > 0$ there exists $\delta > 0$ such that for all $\varepsilon \leq \delta$, the function h_ε is Lipschitz on $B(z_0, \delta)$ with a Lipschitz constant at most equal to η, that is*

$$||h_\varepsilon(w) - h_\varepsilon(z)|| \;\leq\; \eta \, ||w - z|| \;\; \text{for all } w, z \in B(z_0, \delta).$$

Proof. Set

$$M_1 = \, L \, T \, e^{LT} \int_0^T ||A(s)|| \, ds \qquad \text{and} \qquad M_2 = \, T \, (1 + e^{LT}) \tag{3.12}$$

and fix any $\eta > 0$. Using $(\mathcal{H}_2)$, choose $0 < \delta_1 \leq r_0$ such that for all $z \in B(z_0, \delta_1)$ all $\varepsilon \in (0, \delta_1)$ and all $s \in [0, T]$ the following relation holds:

$$\partial F(s, z, \varepsilon) \subset \partial F(s, z_0, 0) + B(0, \frac{\eta}{2M_2}). \tag{3.13}$$

Let

$$\delta = \min \left\{ \varepsilon_0, \frac{\delta_1}{2}, \frac{\delta_1}{2MT}, \frac{\eta}{2M_1} \right\} \tag{3.14}$$

and fix any $0 < \varepsilon \leq \delta$. By (2.7) we deduce that

$$||x(s, z, \varepsilon) - z|| \leq \varepsilon \int_0^s ||F(t, x(t, z, \varepsilon), \varepsilon)\, || \, dt \leq \varepsilon M T \leq \delta_1/2,$$

thus for every $z \in B(z_0, \delta_1/2)$ and $s \in [0, T]$ we get

$$x(s, z, \varepsilon) \in B(z_0, \delta_1). \tag{3.15}$$

Let now $z, w \in B(z_0, \delta_1/2)$, with $z \neq w$. To simplify notation we shall write in the sequel x instead of $x(s, z, \varepsilon)$ and y instead of $x(s, w, \varepsilon)$. Applying [5, Proposition 2.6.5] (generalized mean value theorem) we obtain $\xi^*(s) \in \operatorname{co} \partial F\,(s, [x, y], \varepsilon)$, $\rho^*(s) \in \operatorname{co} \partial F\,(s, [z, w], 0)$ such that

$$F(s, y, \varepsilon) - F(s, x, \varepsilon) = \langle \xi^*(s), y - x \rangle \tag{3.16}$$

and

$$F(s, w, 0) - F(s, z, 0) = \langle \rho^*(s), w - z \rangle. \tag{3.17}$$

Since $z, w, x, y \in B(z_0, \delta_1)$, using (3.13) and the convexity of the norm we deduce that for almost all $s \in [0, T]$,

$$\xi^*(s), \rho^*(s) \in B(A(s), \frac{\eta}{2M_2}). \tag{3.18}$$

Note that

$$||h_\varepsilon(w) - h_\varepsilon(z)|| \leq \int_0^T ||\, F(s, y, \varepsilon) - F(s, x, \varepsilon) - (F(s, w, 0) - F(s, z, 0))\, || \, ds$$

thus in view of (3.16), (3.17) and (3.18)

$$||h_\varepsilon(w) - h_\varepsilon(z)|| \leq \int_0^T || \langle A(s), y - x - (w - z)\rangle || \, ds \, + \, \frac{\eta}{2M_2} \int_0^T (||y - x|| + ||w - z||) \, ds \quad (3.19)$$

In view of (2.8), we get $||y-x|| \leq e^{\varepsilon L s}||z-w||$, thus since $\varepsilon \leq 1$ and $s \leq T$ it follows from (3.12) that

$$\frac{\eta}{2M_2} \int_0^T (||y - x|| + ||w - z||) ds \leq (\eta/2)||w - z||. \quad (3.20)$$

On the other hand, since

$$\int_0^s ||F(t, y, \varepsilon) - F(t, x, \varepsilon)||dt \leq L \int_0^s ||y - x||dt \leq L||w - z|| \int_0^s e^{\varepsilon L t} dt \leq LTe^{LT}||w - z||,$$

it follows from (2.7) and (3.14) that

$$\int_0^T ||\langle A(s), y - x - (w - z)\rangle| \, |ds \leq \varepsilon \left(LTe^{LT} \int_0^T |A(s)|ds \right) ||w - z|| \leq (\eta/2)||w - z||.$$

Combining this last relation with (3.19) and (3.20) the assertion follows. □

Using Proposition 4 we obtain the following result.

Proposition 5 (persistence of generalized Jacobians around z_0). *Under the assumptions ($\mathcal{H}_1$) and ($\mathcal{H}_2$), for every $\eta > 0$ there exists $\delta > 0$ such that for all $\varepsilon \leq \delta$, all $z \in B(z_0, \delta)$ and all generalized Jacobians $\Gamma_\varepsilon \in \partial g_\varepsilon(z)$ we have*

$$||\Gamma_\varepsilon - A|| \leq \eta, \quad (3.21)$$

where $A = Jf(z_0)$ (cf. Proposition 3).

Proof. Note that $g_\varepsilon(z) = f(z) + h_\varepsilon(z)$ and that all three functions are locally Lipschitz, thus differentiable almost everywhere. Let us denote by D_f (respectively, D_g, D_h) the points of differentiability of f (respectively, g_ε, h_ε), and $N = \mathbb{R}^n \setminus (D_f \cap D_h)$. Note that N is a Lebesgue null set, thus the generalized Jacobian of g_ε is given by the formula:

$$\partial g_\varepsilon(z) \; = \; \overline{\text{co}} \left\{ B_\varepsilon = \lim_{z_n \to z} (Jf(z_n) + Jh_\varepsilon(z_n)); \; z_n \in D_f \cap D_h \right\}.$$

Fix $\eta > 0$ and let B_ε be defined as in the above formula. Then applying Proposition 4 for $\eta_1 = \eta/2$ we obtain $\delta_1 > 0$ such that for all $0 < \varepsilon \leq \delta_1$, the function $h_\varepsilon(\cdot)$ is Lipschitz with constant at most $\eta/2$ on $B(z_0, \delta_1)$. This yields $||Jh_\varepsilon(w)|| \leq \eta/2$ for all $\varepsilon \leq \delta_1$ and all $w \in D_h \cap B(z_0, \delta_1)$. Since ∂f is upper semicontinuous and $\partial f(z_0) = \{A\}$, there exists $\delta_2 > 0$ such that $\partial f(w) \subset B(A, \eta/4)$ for all $w \in B(z_0, \delta_2)$. Let $\delta = \min\{\delta_1, \delta_2\}$ and fix $z \in B(z_0, \delta/2)$. Let $n_1 > 0$ be such that for all $n \geq n_1$ we have $||\, B_\varepsilon - (Jf(z_n) + Jh_\varepsilon(z_n)) \,|| \leq \eta/4$ and $z_n \in B(z, \delta/2)$. In particular, since $z_n \in B(z_0, \delta_1)$ for $n \geq n_1$ we obtain

$$||Jh_\varepsilon(z_n)|| \leq \eta/2 \qquad \text{and} \qquad ||Jf(z_n) - A|| \leq \eta/4.$$

This yields

$$||B_\varepsilon - A|| \leq ||B_\varepsilon - (Jf(z_n) + Jh_\varepsilon(z_n))\,|| + ||Jf(z_n) - A|| + ||Jh_\varepsilon(z_n)|| \leq \eta,$$

thus for all $\varepsilon \leq \delta$ and all $z \in B(z_0, \delta)$ we get $||B_\varepsilon - A|| \leq \eta$ and the result follows from the convexity of the ball $B(A, \eta)$. □

We further denote by spec (Γ) the (complex) eigenvalues of the matrix Γ. Using the continuity of the spectral mapping we obtain directly the following corollary.

Corollary 6 (Spectral stability of generalized Jacobians). *Fix any $\eta > 0$ and assume* spec $(A) = \{\lambda_1, \ldots, \lambda_k\}$ *for $A = Jf(z_0)$. Then there exists $\delta_0 > 0$ such that for all $\varepsilon \in [0, \delta_0]$, $z \in B(z_0, \delta_0)$ and $\Gamma_\varepsilon \in \partial g_\varepsilon(z)$, and for all $i \in \{1, \ldots, k\}$*

$$\operatorname{spec}(\Gamma_\varepsilon) \bigcap B(\lambda_i, \eta) \neq \emptyset \qquad \textit{and} \qquad \operatorname{spec}(\Gamma_\varepsilon) \subset \bigcup_{i=1}^{k} B(\lambda_i, \eta)$$

Let us discuss an application of the above result. Let us consider the set of zeros of the mapping $(z, \varepsilon) \mapsto g_\varepsilon(z)$, or equivalently, the set of fixed points of the Poincaré-Andronov operator P defined in (2.3), that is,

$$\begin{aligned} Z &= \{(z_\varepsilon, \varepsilon) \in B(z_0, \delta) \times [0, \varepsilon_0] \ : \ g_\varepsilon(z_\varepsilon) = 0\} \\ &= \{(z_\varepsilon, \varepsilon) \in B(z_0, \delta) \times [0, \varepsilon_0] \ : \ P(z_\varepsilon, \varepsilon) = z_\varepsilon\}. \end{aligned}$$

In the C^1 case, (existence and) asymptotic stability of the periodic solutions is ensured if the Jacobian of f has negative spectral pseudo-abscissa values. In fact, the C^1 assumption can be replaced by the assumption that for each $\varepsilon > 0$ the Poincaré–Andronov operator $P(\cdot, \varepsilon)$ has a fixed point z_ε and is differentiable there. Indeed, in such a case, shrinking ε if necessary, we deduce from the above corollary that the matrix $JP(z_\varepsilon, \varepsilon)$ will have all its eigenvalues inside the unit disk of the complex plain. Let us extend this result to the nonsmooth case.

Theorem 7 (existence and asymptotic stability of periodic solutions - I). *Let $z_0 \in f^{-1}(0)$ (f is defined in (1.2)) and let ($\mathcal{H}_1$), ($\mathcal{H}_2$) hold true. Assume* $\det(Jf(z_0)) \neq 0$ *and that for some equivalent norm $|\cdot|$ of $\mathbb{R}^n$ (that does not depend on ε) :*

$$|\,(z + \varepsilon f(z)) - (w + \varepsilon f(w))\,| \ \leq \ (1 - \omega\varepsilon)\,|z - w|, \tag{3.22}$$

for all z, w around z_0, where $\omega > 0$. Then for every $\varepsilon > 0$ sufficiently small the system (1.1) has a unique and asymptotically stable periodic solution near z_0 with initial value z_ε satisfying $\lim\limits_{\varepsilon \to 0} z_\varepsilon \ = \ z_0$.

Proof. Let us observe that the assumption $z_0 \in f^{-1}(0)$ simply means that $(z_0, 0) \in Z$, while the fact that $\det(A) \neq 0$ yields that z_0 is isolated in $f^{-1}(0)$ and its topological index with respect to f does not vanish. Thus from the theory of topological degree we deduce (see also [3]) that for all $\varepsilon > 0$ small enough there exists z_ε with $(z_\varepsilon, \varepsilon) \in Z$ and $\lim\limits_{\varepsilon \to 0} z_\varepsilon \ = \ z_0$.

Fix $\varepsilon > 0$ and apply Proposition 4 (which remains true for the equivalent norm $|\cdot|$) for $\eta \leq \omega/2$. We conclude that the $|\cdot|$-norm of any generalized Jacobian $\partial h_\varepsilon(z)$ around z_0 is bounded by $\omega/2$. Since $P(\cdot, \varepsilon) = I + \varepsilon f + \varepsilon h_\varepsilon$, it follows easily that

$$|P(z, \varepsilon) - P(w, \varepsilon)| \leq (1 - (\omega/2)\varepsilon)|z - w|,$$

which for $\varepsilon > 0$ sufficiently small guarantees uniqueness and asymptotic stability of the fixed point z_ε. □

In what follows, we denote by $r(\Gamma)$ the spectral radius of the matrix Γ, that is,

$$r(\Gamma) = \max\,\{\,|\lambda| : \lambda \in \operatorname{spec}(\Gamma)\}\,.$$

Theorem 8 (existence and asymptotic stability of periodic solutions - II). *Let $z_0 \in f^{-1}(0)$ and assume $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$. Then the conclusion of Theorem 7 remains true provided that the matrix $A = Jf(z_0)$ is normal (thus diagonalizable in $\mathbb{C}$) and*

$$\alpha = \max\{\mathrm{Re}(\lambda) : \lambda \in \mathrm{spec}\,(A)\} < 0\,. \tag{3.23}$$

Proof. Since the eigenvalues of the matrix $I + \varepsilon A$ are of the form $1 + \varepsilon\lambda$ for $\lambda \in \mathrm{spec}\,(A)$, it follows that

$$r(I + \varepsilon A) < \sqrt{1 - 2|\alpha|\,\varepsilon + ||A||^2\varepsilon^2}.$$

On the other hand, the assumption that A is normal implies that the matrix $I + \varepsilon A$ has n-orthonormal eigenvectors in $\mathbb{C}^n$, which guarantees that

$$\max\{\,||\,(I + \varepsilon A)z\,||_{\mathbb{C}} \,:\, z \in \mathbb{C}^n, ||z|| = 1\,\} \;=\; r(I + \varepsilon A)\,,$$

thus

$$||I + \varepsilon A|| = r(I + \varepsilon A).$$

Let $\delta_0 \le |\alpha|/||A||^2$ and shrink it further if necessary to ensure that for all $\varepsilon \in (0, \delta_0]$ we have the estimate:

$$||\,I \,+\, \varepsilon A\,|| \;<\; 1 \;-\; (|\alpha| - 2^{-1}\,||A||^2\,\varepsilon)\,\varepsilon < 1 - (|\alpha|/2)\,\varepsilon.$$

From the strictly differentiability of f at z_0 we deduce that, for all z, w around z_0,

$$||\,f(z) - f(w) - A(z - w)\,|| \;\le\; (|\alpha|/4)\,||z - w||\,.$$

Then

$$\begin{aligned} ||(z + \varepsilon f(z)) - (w + \varepsilon f(w))|| \;&\le\; \varepsilon||f(z) - f(w) - A(z - w)|| + ||I + \varepsilon A||\,||z - w|| \\ &\le\; (1 - (|\alpha|/4)\,\varepsilon)\;||z - w||\,. \end{aligned}$$

Hence (3.22) is valid around z_0, thus the conclusion follows from Theorem 7. □

Remark 9. In the above theorem, the assumption that A is normal has been used to ensure that $||I + \varepsilon A|| = r(I + \varepsilon A)$ for all $\varepsilon > 0$. In fact, for any matrix A satisfying (3.23) and any $r(I+\varepsilon A) < \rho < 1$, one can always define an equivalent norm $|\cdot|$ in $\mathbb{R}^n$ for which the corresponding operator norm satisfies $|I + \varepsilon A| < \rho$. However, this norm depends on ε and eventually becomes large as $\varepsilon \to 0$.

The following result concerns instability of the periodic solutions.

Theorem 10 (Unstable periodic solutions). *Let z_0 be a zero of the function f defined in* (1.2) *and assume that $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ hold true and that the matrix $A = Jf(z_0)$ is normal and*

$$\beta = \min\{\mathrm{Re}(\lambda) : \lambda \in \mathrm{spec}\,(A)\} > 0\,. \tag{3.24}$$

Then for all $\varepsilon > 0$ sufficiently small, (1.1) *has an unstable periodic solution with initial value near z_0.*

Proof. The existence of (at least one) periodic solution with initial point near z_0 follows in the same way as in Theorem 7. Let us apply Corollary 6 for $\eta \le \beta/2$. Then there exists $\delta_0 > 0$ such that for all $\varepsilon \in (0, \delta_0]$ and $z \in B(z_0, \delta_0)$, every generalized Jacobian matrix $\Gamma_\varepsilon \in \partial g_\varepsilon(z)$ satisfies $||A - \Gamma_\varepsilon|| \le \beta/2$. It follows that

$$\min_{||x||=1} \{ \, ||(I + \varepsilon\Gamma_\varepsilon)(x)|| : x \in \mathbb{R}^n \} \ge \min_{||z||=1} \{ \, ||(I + \varepsilon\Gamma_\varepsilon)(z)|| : z \in \mathbb{C}^n \} \ge 1 + \frac{\varepsilon\,\beta}{2} \, ,$$

which guarantees the invertibility of the mapping $P(\cdot, \varepsilon)$ around any $z \in B(z_0, \delta_0)$, see [5, Section 7.1] for details. Moreover, the inverse function $Q(\cdot, \varepsilon)$ is Lipschitz with constant at most $(1 + \varepsilon\,\beta/2)^{-1}$. Thus, if $z_\varepsilon \in B(z_0, \delta_0)$ is the initial point of a periodic solution of (1.1), then for every $z \in B(z_0, \delta_0)$ we have

$$||z - z_\varepsilon|| = ||Q(P(z, \varepsilon), \varepsilon) - z_\varepsilon|| \le (1 + \frac{\varepsilon\,\beta}{2})^{-1} ||P(z, \varepsilon) - z_\varepsilon|| \, .$$

This shows that z_ε is a repelling fixed point of $P(\cdot, \varepsilon)$ and the conclusion follows. □

Let us give an application to the existence and stability of limit cycles of a (nonsmooth) planar system, which is studied by essentially the same technique, after transformation to polar coordinates.

Example. Consider the following planar system

$$\begin{aligned} \dot{x_1} &= -x_2 + \varepsilon\, x_1(1 - 2\varphi(x_1^2 + x_2^2)) \\ \dot{x_2} &= x_1, \end{aligned} \tag{3.25}$$

where $\varepsilon > 0$ is a small parameter and $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is defined as follows:

$$\varphi(x) = \begin{cases} x, & \text{for } x \in [0, 1], \\ 1, & \text{for } x \in (1, \infty). \end{cases} \tag{3.26}$$

Passing to polar coordinates $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, differentiating the relations $r^2 = x_1^2 + x_2^2$ and $\theta = \arctan(x_2/x_1)$ and eliminating the parameter t we get from (3.25) a one–dimensional equation of the form (1.1), that is,

$$\dot{r}(\theta) \;=\; \varepsilon\, F(\theta, r, \varepsilon),$$

where

$$F(\theta, r, \varepsilon) = \frac{r\cos^2\theta \;(1 - 2\varphi(r^2))}{1 - \varepsilon 2^{-1}\sin 2\theta(1 - 2\varphi(r^2))}.$$

Then $F : \mathbb{R} \times (0, \infty) \times [0, 1] \to \mathbb{R}$ is continuous, 2π–periodic with respect to θ, and locally Lipschitz with respect to r. Moreover it satisfies $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ (note that $F(\theta, \cdot, 0)$ is strictly differentiable at every $r > 0$ provided $r \neq 1$. Then (1.2) gives

$$f(r) = \int_0^{2\pi} F(\theta, r, 0)\, d\theta \;=\; \int_0^{2\pi} r\cos\theta \;(1 - 2\varphi(r^2))\, dr \;=\; \pi\, r(1 - 2\varphi(r^2)).$$

It is easily seen that f has a unique zero at $r_0 \in (0, \infty)$ and $f'(r_0) < 0$ (cf. Proposition 3). We conclude that (3.25) has a unique limit cycle that is asymptotically stable. □

Let us conclude with the following remark. Nonsmoothness appears naturally in most concrete problems, and leads to the development of the so-called *nonsmooth analysis* (see for example [1] and references therein, as well as the classical textbooks [5], [12] and [11]). Here, instead of the classical C^1-assumption, F is assumed to be strictly differentiable only at points of the form $(s, z_0, 0)$, which enlarges the domain of applicability of the theory: in fact, nonsmoothness seldom occurs in a random manner, but instead it is often well-structured: this is the case for instance in problems involving semialgebraic or subanalytic structures, see [2] for example.

Acknowledgement. This work has been realized during a visit of the first author at the CRM and the Autonomous University of Barcelona. The first author wishes to thank her hosts for hospitality.

References

[1] Aussel, D., Daniilidis, A. & Thibault, L., Subsmooth sets: functional characterizations and related concepts, *Trans. Amer. Math. Soc.* **357** (2005), 1275–1301.

[2] Bolte, J., Daniilidis, A. & Lewis, A., The Łojasiewicz inequality for nonsmooth subanalytic functions with applications to subgradient dynamical systems, *SIAM J. Optim.* **17** (2006), 1205–1223.

[3] Buică, A. and Llibre, J., Averaging methods for finding periodic orbits via Brouwer degree, *Bull. Sci. Math.* **128** (2004), 7–22.

[4] Buică, A. and Llibre, J., Bifurcation of limit cycles from a 4–dimensional center in control systems, *International Journal of Bifurcation and Chaos* **15** (2005), 2653–2662.

[5] Clarke, F., *Optimization and nonsmooth analysis,* A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1983.

[6] Guckenheimer, J. and Holmes, P., *Nonlinear oscillations, dynamical systems, and bifurcation of vector fields*, Springer–Verlag New York, Heidelberg, Berlin, 1983.

[7] Hale, J., *Ordinary Differential Equations*, Robert E. Krieger Publishing Co., 1980.

[8] Hartono and van der Burgh, A.H.P., Higher–order averaging: periodic solutions, linear systems and an application, *Nonlinear Analysis* **52** (2003), 1727–1744.

[9] Mawhin, J., *Topological degree methods in nonlinear boundary value problems*, CBMS Regional Conference Series in Mathematics, **40**. American Mathematical Society, Providence, R.I., 1979.

[10] M'Closkey, R. T., An averaging theorem for time–periodic degree zero homogeneous differential equations, *Systems Control Lett.* **32** (1999), 179–183.

[11] Mordukhovich, B., *Variational analysis and generalized differentiation.* (Vol. I & II) Grundlehren der Mathematischen Wissenschaften **331**, Springer-Verlag, Berlin, 2006.

[12] Rockafellar, T. and Wets, R., *Variational analysis*, Grundlehren der Mathematischen Wissenschaften **317**, Springer-Verlag, Berlin, 1998.

[13] Sanders, J. and Verhulst, F., *Averaging methods in nonlinear dynamical systems*, Appl. Math. Sci. **59**, Springer, 1985.

[14] SCHECTER, S., Stable manifolds in the method of averaging, *Trans. Amer. Math. Soc.* **308** (1988), 159–176.

[15] TEEL, A. R., PEUTEMAN, J. AND AEYELS, D., Semi–global practical asymptotic stability and averaging, *Systems Control Lett.* **37** (1999), 329–334.

Adriana Buică

Department of Applied Mathematics
Babeş-Bolyai University,
RO-400084 Cluj-Napoca, Romania

E-mail: abuica@math.ubbcluj.ro
http://math.ubbcluj.ro/~abuica/

Research supported by the “Agence universitaire de la Francophonie” (France)

Aris Daniilidis

Departament de Matemàtiques
Universitat Autònoma de Barcelona
E-08193 Bellaterra, Spain

E-mail: arisd@mat.uab.es
http://mat.uab.es/~arisd

Research supported by the MEC Grant No. MTM2005-08572-C03-03 (Spain).