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# Modern Optimization Modelling Techniques

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ISBN 978-3-0348-0290-1      ISBN 978-3-0348-0291-8 (eBook)  
DOI 10.1007/978-3-0348-0291-8  
Springer Basel Heidelberg New York Dordrecht London

Library of Congress Control Number: 2012944421

Mathematics Subject Classification (2010): Primary: 90-XX, 91-XX; Secondary: 12Y05, 14P10, 65K10, 65K15, 90B20, 90C22, 90C26, 90C30, 90C33, 90C40, 91A10, 91A13, 91A26, 91B51

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Printed on acid-free paper

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*Jean B. Lasserre*

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*Francisco Facchinei*

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# Preface

During the period July 20–24, 2009, the research group on Optimization of the Autonomous University of Barcelona organized an advanced course at the CRM, with the aim of promoting research in the area of optimization in all of its components: theory, algorithms, and practical problems. This volume is a unified version of the material presented in the course.

The advanced course was entitled *Optimization: Theory, Methods, and Applications*. The courses and the written material were accordingly divided into these three main parts. The theoretical part of the book is a self-contained course on the general moment problem and its relations with semidefinite programming, presented by Jean B. Lasserre, senior researcher at the CNRS (France), world-leading specialist of the domain and author of a recent research monograph on this topic (Imperial College Press, 2009). The second part is dedicated to the problem of determination of Nash equilibria from an algorithmic viewpoint. This part is presented by Francisco Facchinei, professor at the University of Roma “La Sapienza”, established researcher and co-author of an extended monograph on this topic (Springer, two volumes). The third part is a study of congestion models for traffic networks. This part develops modern optimization techniques for finding traffic equilibria based on stochastic optimization and game theory. It has been presented by Roberto Cominetti, professor at the University of Chile, who has been working for several years on congestion models of the traffic of the municipality of Santiago de Chile.

This advanced course was an i-MATH activity (ref. 2009 MIGS-C4-0212), which was also supported by the Spanish Ministry of Science and Innovation (Complementary Actions, ref. MTM2008-04356E). We wish to thank the CRM direction and administrative staff for the logistic support, and our three main lecturers for the excellent course and the quality of the material presented. We also thank our colleagues Emilio Carrizosa (Sevilla), Laureano Escudero (Rey Juan Carlos), Claude Lemaréchal (INRIA Rhône-Alpes), and Justo Puerto (Sevilla), who agreed to deliver invited talks complementary to the courses, as well as the 70 participants of the event. Our special thanks to Sabine Burgdorf (Konstanz), Vianney Perchet (Paris), Philipp Renner (Zürich), Marco Rocco (Bergamo), and Guillaume Vigerat (Paris), who accepted to review carefully several parts of this material.

Bellaterra, February 2011

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